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Proving general sentences in mathematics and justification of mathematical statements by learners of mathematics. The anthropomathematical studies

Abstract: The article contains theoretical considerations and the description of the didactic research which I carried out. Theoretical reflections relate to one of the most important components of mathematical knowledge and skills of students, such as the proving, explaining and justification, as necessary elements included in the mathematical methodology. Theoretical considerations also include the establishment of research-theoretical approach to exploring learning-teaching mathematics process, called anthropomathematics. The article describes the results of the didactic research which the main objective was diagnosis relating to the question: what criteria decide that a mathematical statement can be considered (by students) to be true? Their main part of mentioned research was presented in my monograph entitled: The methodology of mathematics as a subject of anthropomathematical studies (Żeromska, 2013).

1 Introduction

The methodology of mathematics is the most characteristic feature that distinguishes mathematics from other disciplines. It is the methodology of mathe-

Key words: anthropomathematics, methodology of mathematics, proving, justification, argumentation, the role of concrete.

matics[1] which determines the functioning of the existing and the creation of new elements of mathematical knowledge. For the purposes of teaching school mathematics, it is important to define the meaning of the term quite precisely so that it is possible to implement the elements of the methodology of mathematics to the general mathematical education of the entire society.

The purpose of this article is among others (however brief and fragmentary) the introduction into extremely extensive scientific issues associated with the structure and method of operation as well as the development of the mathematical knowledge. Selected views of teachers of mathematics, relating to this subject matter, will be presented as well.

Mathematics is a certain theory or a set of such interrelated theories, regardless of the subjective views on the origins and nature of this field of science[2]. Theory is defined by the mathematical logic as every consistent set of sentences formulated in the language of this theory. Consistency of a set of sentences means that in the set there is no such sentence (written in the language of the theory under consideration) that under this theory one can prove both the sentence and its negation.

The problem of proving and justifying statements relating to mathematical objects as well as the relation between them has been an important issue in the philosophy of mathematics, metamathematics and the mathematics itself for a long time. The traditionally basic idea of building mathematical knowledge (already presented by Euclid) is to accept (without logical justification) a finite number of basic concepts (the so-called original) and a finite number of sentences defining the relations between these concepts (the so-called axioms), accepted as "obvious", i.e. true (Tarski, 1994). Thus, the issue of proving in this approach is resolved by specifying a set of "obvious truths" and deriving all other statements from this set using the undisputed principles of logic[3].

[1]The methodology of mathematics is a subject of interest to a branch of knowledge studying mathematics using its own, mathematical means, i.e. metamathematics. Metamathematics, as the research field of the foundations of mathematics, was distinguished in the late nineteenth and early twentieth centuries. Its formation is associated with the so-called Hilbert's program, which aimed to provide a proof of the consistency of mathematics with finite means (Woleński, 1984). The term metamathematics means all consideration of formalized systems using mathematical means. This field of science is, therefore, preoccupied with an analysis of the structure and property theories as well as mathematical methods (Tarski, 1994). This study does not pretend to the complementarity of description of all the issues related to the position of the methodology of mathematics in metamathematical terms.

[2]Description of the structure of knowledge and axiomatic-deductive method we can find in the famous work on Euclidean geometry. However, it has to be kept in mind that Euclid had some gaps in the system of axioms and that his proofs did not meet today's standards (Semadeni, 2004).

[3]Mathematicians, however, do not agree on what constitutes a single and indisputable set

Euclidean geometry, which was the first example of a formal deductive system, has become a model for all such systems. Geometry was a great testing ground for logical thinking, and learning geometry was treated, which was right or wrong, as a basis for the student to practice in such thinking (Davis, Hersh, 1994, p. 18).

Recognition of axioms and a suitable inference means that all the sentences derived from these axioms (or other sentences already recognized in the theory) by the accepted way of inference should be considered to be correct. This type of inference in the mathematical theory is called deduction, and the whole manner of building this theory is called the axiomatic-deductive one (Tocki, 2000)[4]. Such an approach to the subject clearly identifies the mathematical method with inference of a deductive type, and such a view is represented by the majority of mathematicians. They aim at a strict observance of deduction as a primary mode of operation within mathematics; they want to see mathematics as "the science of deduction", thus they make some (sometimes even arbitrary) assumptions and examine what results from them, and each conclusion of the assumptions - if it is to be accepted as justified – must be justified by a proof.

In some opinions, a proof is the essence of mathematical procedure, without a proof mathematics does not exist. For others it is a nonsense, in mathematics there are many types of procedures (Davies, Hersh, 1994, p. 131).

For a sentence to be called a mathematical theorem, there must be met certain formal conditions imposed on the means of determining its correctness, thus a proof must be carried out, which is often a difficult task, sometimes it even seems to be unattainable, despite a widespread conviction of the veracity of the hypothesis which is the subject of this proof[5]. The degree of formalization of such a mathematical proof is questionable as well. A formal proof must be expressed in a formal language, based on formal axioms, and reasoning in such a proof must be based on formal rules of inference. Of course, the proof can be formalized in varying degrees, which is affected by a number of factors, including the mathematical experience level of the audience of this proof. And here let us remember that:

the true measure of the length of a proof must take into account not only

of axioms, nor even as to what should be considered acceptable rules of inference. In the past, other views on the subject were represented by logicians (e.g. Russell), other by formalists (e.g., Hilbert), and still others intuitionists (e.g. Brouwer) (after: Prus-Wiśniowska, 1995).

[4]It turns out, however, that the issue is not so obvious if we consider the already mentioned earlier views of I. Lakatos of a quasi-empirical nature of the methods, which sometimes are used in mathematics (Lakatos, 1995)

[5]For example, the Riemann hypothesis, the Poincare hypothesis, the Euler hypothesis.

the number of lines of argumentation, but also the number of necessary pages of mathematics prior to these lines (Dunham, 1994, p. 125).

The term of a proof in mathematics is one of its essential concepts, the meaning of the correctness of a proof is crucial. One of the terms of a proof in mathematics is as follows:

a formalized proof of the expression p on the basis of a set of statements X is the next transformation of the expressions belonging to X in accordance with the laws of transformation and inference, so that as a result of these operations the expression p was obtained (Turnau, 1984).

This formulation uses the term "inference", which is quite often used, also in everyday language. In the narrower sense, inference means matching the consequences for unquestionable sentences, already recognized as true. In a broader sense, it is a thought process where, on the basis of the sentences which have already been accepted, a new sentence is obtained which has previously been unrecognized, and its confidence does not have to be undisputed, it may only be partially confirmed. Therefore, all inferences may be divided into reliable and unreliable ones and they serve merely as "substantiation". According to a popular belief, deductive reasoning (a mathematical proof) is considered to be reliable, and among different kinds of unreliable inferences, inductive inference and reasoning by analogy can be distinguished.

Pogorzelski and Słupecki (1970) begin their analysis of the concept of a proof by noting that the word "proof" must be inherently accompanied by the word "premise", as well as an implicit assumption of its truth. The premises of a proof may be the axioms of a certain theory, its definitions (or substitutions) and the substitution of logical laws. The authors give the following definition of a mathematical proof:

a proof of the expression α on the basis of a set of expressions X is a finite sequence of expressions $\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n$ ($1 \leqslant k \leqslant n$), where $\alpha_n = \alpha$ the terms $\alpha_1, \ldots, \alpha_k$ are premises. For each of the i, such that: $k < i \leqslant n$, the expression α_i results from the previous terms of the sequence. The premises of a proof may be those (and only those) expressions, which are elements of the set X or their substitutions of logical laws (Pogorzelski, Słupecki, 1970, p. 73).

Performing a mathematical proof, in accordance with such authoritarian views, means that it is necessary to break away from the empirical and heuristic reasoning. Each step of such a proof must clearly result from the preceding ones or be an accepted axiom or a previously proven theorem; the reasoning that does not meet this condition is not a proof. The last step of a proof should be a sentence that is subject to the process of proving, which thus becomes a new theorem of a given theory (according to the definition of the term theorem provided by A. Mostowski[6]).

It is a potentially unassailable logic of proofs which leads to a widespread understanding of mathematics as a domain of absolute truths, and truths independent of man, according to some thinkers.

Karl Popper's view (after: Duda, 1995), who argues that no proofs nor other confirmations play in mathematical research procedures as important role as they are commonly attributed to, questions this infallibility. If the theory has successfully passed through successive attempts to refute it, that only means that we should continue such efforts. It does not mean, however, that any of the elements of our system should be regarded as fixed, we can not treat it as absolutely true[7]

A common idea of a mathematical proof as a finite and excellent structure, which is a flash of genius, does not correspond to the everyday mathematical reality. Mathematicians are people, after all, proving is a human activity, and thus it bears all its features (Turnau, 2001, p. 29).

This quotation emphasizes a human factor in this characteristic mathematical creativity. Even if the world around us has an objective, independent of a human, feature called mathematicity, then performing proofs and verifying their correctness have their origins exclusively in sublime human needs. This means that we should treat ready mathematical proofs as artifacts of a particular form, demonstrated and directed to a specific social environment in order to achieve a common assessment of the correctness in this environment at any given time. The proofs are therefore explanations with a particular structure of expression, communication-oriented in a given social environment, compatible with a strictly defined set of rules adopted in this environment (Balacheff, 1987).

It should be emphasized, that not just deduction but intuition has sometimes decisive role in the social recognition of the truth; deduction does not have

[6]According to A. Mostowski (1948) the expression T is called a theorem in the mathematical theory Q, if there is a finite sequence of predicates in this theory T, T_1, T_2, $\dots$ T_n satisfying the following conditions:

1 T_n is identical with T,

2 For $i \leqslant n$, the expression T_i is either an axiom of the theory Q, or is derived from the tautology of the propositional calculus by the substitution for the propositional variables the predicates of the theory Q, or it is the substitution of a definitional axiom or axiom of extensionality, or, finally, T_i is formed from one or two predicates T_j, T_k preceding T_i in the sequence T, T_1, $T_2 \dots T_n$ by one of the operations: substitution, detachment, removing the quantifiers, attaching quantifiers or generalizing.

[7]The principle formulated by Karl Popper is called the principle of falsificationism and is considered by many researchers as a criterion of scientific method (Duda, 1995). There are more similar views among philosophers and mathematicians.

such a power (Fischbein, 1982). It happens that some people, despite getting acquainted with the formal proof of some fact, intuitively do not accept the veracity into their consciousness. For example, first-year students of mathematics (if they are brave enough) say that a formal proof does not convince them that there is the same amount of the natural numbers as the even numbers and the same as all rational numbers. Even the great mathematician, Cantor wrote in a letter to another great mathematician, Dedekind wondering if the unit square could be mapped into a line of unit length with a 1-1 correspondence.

I see it, but I do not believe it (after: Turnau, 2001).

These types of observations give rise to a reflection that although deduction and formal reasoning claim to an indisputable and authoritarian determination of absolute truths, and among philosophers of mathematics a rigorous opinion prevails that the primary objective of axiomatic-deductive recognition of mathematical knowledge is precisely its separation from the intuitive or empirical content[8] (Tocki, 2000), it is a human intuition which is of a primary importance when analyzing, communicating and accepting the truth of mathematical statements.

These issues are particularly important for the problems related to the teaching- learning of mathematics. For educational purposes, we should remember that the most important feature which should be kept even by the most formalized proof, is its explanatory nature (Hanna, 1989; Hersh, 1993). It may happen, that a formally correct and "elegant" proof adds nothing to the understanding of the theorem, and may even be completely unconvincing (Hanna, 1989).

Review of the literature in the field of didactics of mathematics shows a continuous interest of the researchers in the subject oriented to the reasoning of the type of a proof in school mathematics. We are observing a steady increase in the number of research works aimed at the teaching and learning of a proof, regarding both theory and its practical applications in school. This subject is still up-to-date, which is expressed in still ongoing shifts of the topics related to this area in the plans and programs of teaching mathematics. As it can be guessed, these changes have two divergent directions. One trend is emphasis on intuitive reasoning without proofs in the teaching of mathematics. The educators and teachers proclaiming this view argue that due to the difficult conditions of teaching and learning of mathematics in the class-lesson system, a choice must be made between shaping students' skills of proving

[8]The crowning achievement of this approach is practicing mathematics using the axiomatic method, exclusively. Bringing this method to the limit is pure formalization (see: the Bourbakis group), which changes the practice of mathematics into a game of symbols untainted by any interpretation.

and justifying, and developing their competence in solving mathematical problems, which policymakers consider to be more useful in human life. Such a view, if it is radical, treats proofs and proving as a hindrance to mathematical understanding, and not the way to this understanding (Hanna, 2000). Even fairly extreme views exist; it is regretted, for example, that:

proofs are horrors required by scientific journals (...) do we really need a proof at all? Especially in schools?(...) Why can't we – the vast majority of us – just assume that something on earth is intuitive and very probably true, or just obvious? (Barnard after: Hanna, 2000).

The opposite trend of changing the "image" of mathematical methods in teaching is the emphasis on justification and proofs as a necessary and indispensable element of mathematical education. It is recognized that proving, and even learning ready proofs, is not deprived of an important educational value. For example, the National Council of Teachers of Mathematics (USA), in a document of 2000 entitled "Principles and Standards for School Mathematics" (as opposed to the previous document of 1989, where the concept of a proof virtually disappeared from the curriculum), recommends performing the reasoning and proofs as part of the math curriculum at all levels of this education. A fragment of this document states that:

students should be able to:

- *recognize reasoning and a proof as fundamental aspects of mathematics;*
- *investigate mathematical conjectures and put forward hypotheses;*
- *develop and evaluate mathematical arguments and proofs;*
- *select and use various types of reasoning and proving methods* (Hanna, 2000).

These are very clear indications for the authors of curricula and for the teachers who implement these programs.

Many scientific works in the field of didactics of mathematics show that, unfortunately, the teaching of mathematics relating to a mathematical proof seems to be ineffective in many countries, regardless of how it is described in their educational documents and how this teaching is organized. The difficulties that learners of mathematics have in that matter are of a diverse nature. It is stated, for example, that pupils and students:

- do not know the definitions required for proving of the concepts, they are not able to use them;
- they have ideas (also intuitive ones) of mathematical objects which are insufficient to perform proofs with their use;
- they are not able (or not willing) to generate and use their own examples;

- they are not able to understand the overall structure of a proof;
- they do not understand and are not able to use mathematical language and symbols;
- they do not know how to start proofs[9].

Among these reasons, an important aspect is not referred to, which regards learning difficulties associated with frequently false and negative perception of the role and origin of proofs in practicing mathematics, shaped in the course of learning, and various prejudices and negative beliefs arising therefrom. One of the doubts that a learner may have is referred to in this quote:

So, mathematics is a subject, where there are proofs. (...) During a typical university lecture on advanced mathematics, particularly conducted by a lecturer with "clear" interests, consists solely of definitions, theorems, proofs, definitions, theorems, proofs – written down solemnly and monotonously. Why is this happening? If, as it is claimed, the proof is a justification and certification, then one may think that the moment it was accepted by a competent group of scientists, the rest of the scientific world should welcome it with satisfaction and move on. Why do mathematicians and their students recognize the proving of the Pythagorean theorem one more time as valuable, etc.? (Davis, Hersh, 1994, p. 133).

Studies in philosophy and the history of mathematics show that there were, and there still are, conflicting opinions on the role of a proof in mathematics. That is the reason why it is also reflected in the deliberations of the scientists dealing with the problems of mathematical education. For example, N. Balacheff (1987, 1990) distinguishes two aspects of the proof, relating them to the function of the proof in various systems of social interaction. We can distinguish (after the author) different frames of reference. The first one refers to a proof in the broad context of its functioning in a community of interested people (e.g., in a community of mathematicians). Then we can note that the proofs of mathematical general sentences can be looked at through the prism of their *cognitive aspect of proof* and *the social aspect of proof*. Both aspects are substantially different in many features. A cognitive look at the role and structure of mathematical proofs makes us notice that they do not rely merely on a simple verbalization of the properties of mathematical objects and the relations between them. At this level of examining mathematical proofs, a particular state of the knowledge used is required, which should be well organized in theory and presented using appropriate language as well as the recognition of deductive rules and previously accepted facts. In this way, the formal side

[9]On the basis of the studies by R. Moore (1994).

of a mathematical proof is primarily emphasized. Such a cognitive look at the proof reduces its importance to the development of the internal structure of the mathematical knowledge, and if we want to see a human factor, then it is limited exclusively to the use of proofs as a means of communication between mathematicians. In addition to the cognitive aspect, Balacheff makes us look at the role of a mathematical proof through the prism of social reference. In social terms, the proof serves men to persuade others of the truth of statements of the mathematical nature, thus it is a tool and means of ascertaining the fact of social recognition of the truth of a given statement. In this approach, the communication function comes forward (not necessarily between professional mathematicians) aimed at understanding the content and social justification of a mathematical proof. A proof, in this sense, is a means of "rhetoric" aimed at acknowledging the fact of social recognition of the truth of the statement, its formal correctness recedes into the background.

The social aspect of the proof and proving is also perceived by other authors, in particular, by some mathematicians. In the paper "On proof and progress in mathematics", W. Thurston (1994) analyzes the question as to how mathematicians can help improve the understanding of mathematics and, in general, it outlines the perspective of the issues relating to the ways of understanding and thinking about mathematics. He admits that there is a whole list of different ways of thinking about the concept of mathematical concept and the ways through which the process of creating this concept in the human mind goes. He emphasizes that this list is much more extensive than the specification of various logical definitions of this object. These resources – according to Thurston – must not be underestimated, it can be a source of knowledge on the manners of mathematical human thinking. Therefore, as emphasized by Thurston, the transfer of understanding from one person to another, to whom the data is presented, is not automatic. It requires the use of various methods and "tricks", as different as differently a mind of an individual human being can be organized. For those methods and "tricks" to be applied, one must previously get acquainted with them well. Therefore, it is necessary to analyze the understanding of mathematics, it is important to take into account: who understood it, what and when (Thurston, 1994, p.165). The author emphasizes the role of communicating the results of the proving process; in his opinion, this method of communication (e.g. the degree of accuracy and formalization of a proof) is determined by the community which the communication relates to.

A view that a mathematical proof is a means of social communication, and its important role is to enrich the intuitive understanding of the fact, mentioned in the theorem, has important implications to determine the role which the proof can play in a given cognitive process of an individual (see: Davis, Hersh, 1994; Schoenfeld, 1985; Hanna, 1989).

2 The role of a concrete in the mathematical study

Let us begin by reminding that, historically, the justification was based on the approximate conformity of the obtained results with empiricism, and on the receipt of a manner of reaching the desired result – on the authority of the record[10]. Thus, the source of mathematical knowledge were observation, experience, induction, analogy, i.e. natural methods in fact. With such an attitude, of course, there could be no incidence of deduction and proving, or of the axioms and theorems. The role of the authority in the recognition of authenticity was equally important. Such were the beginnings of recognition of the truth of statements in mathematics. It is important to remember that, especially in the school teaching, where it is postulated to respect the principle of inductive-genetic development of the mathematical knowledge.

In later years of the development of mathematics, the role of empiricism and authority in recognition of the truth of sentences in the mathematical theory has not been so exposed, but that does not mean that this type of drawing conclusions has been abandoned. Mathematical studies, even those of the great authorities, have their origins in empiricism (which is sometimes carefully hidden later on), and only the ready products of scientific work are likely to be brought to daylight. Then the community of professional mathematicians accepts the correctness of the presented reasoning and decides that from now on the proof of some hypothesis shall be regarded as correct. Sometimes it happens that reasoning, which is commonly considered to be a proof of a certain theorem, turns out to be incorrect or incomplete. The history of the development of mathematics knows such cases[11] .

Drawing general conclusions from the observation of specific cases plays a special role in students' reasoning. Let us think what might underlie the student's belief that to recognize the truth of a general statement - relating to a certain class of mathematical objects – it is enough to check if this statement meets one or several specific objects of this class. The reasons for this status quo are complex. Firstly, they may have their origins in the epistemology of

[10]This is evidenced by the phrases "do likewise", "you behaved reasonably", etc. often found in the Egyptian and Babylonian sources (Duda, 1995).

[11]For example (among many others), a solution to the problem of "four colors" of 1879, accepted as correct, in 1890 was found to contain an error. Another example is the discovery by B. Russell of a paradox in the concept of axiomatization of arithmetic formulated by G. Frege (after: Tarski, 1994).

mathematical cognition. R. Duda turns attention to this issue, particularly to the question of a concrete as an epistemological obstacle:

The transition from this stage [of reasoning on a concrete] *to the one in which there are already occurring* (...) *"supplements" in the form of general explanations and guidelines of conduct* (...)*, required* [in the history of the development of mathematics] *a very difficult epistemological obstacle which most reasonably should be called an obstacle of a "concrete" to be overcome* (Duda, 1995, p. 5).

R. Duda continues:

As a chef does not need strict definitions of the products which are being used, their status or the operations to be performed on them (cf. a command "take a pinch of salt", "stir-fry until golden brown", etc.) and such an attempt the chef would probably take as too ridiculous and unnecessary, so for the ancient mathematician intuitive, acquired from his master, understanding of the concepts such as triangle, circle, area, cylinder, cone, volume, etc. was enough. He "knew" that it was a triangle or a cylinder, as chef knows what it is a leg or flour (ibid.).

Referring his findings to the teaching of mathematics, R. Duda states:

Pedagogical experience teaches that the minds of many people, especially of the youth, easier acquire the rules obtained against specific examples, clearly indicating the steps to be taken and their order. And what is more, listening to a teenage boy presenting a theory to his colleague, we find that he follows in the footsteps of his ancient teachers and commandeers: add, multiply, etc. Description of an action makes its way to a head, and the explanations contained in a scientific language are perceived as a hindrance and not as an aid (ibid., 1995, p. 6).

Adopting such a point of view, it should be understood that verification against an example is for students the most natural form of checking general properties of objects in mathematics. Such a method of verification lies in the roots of a human mathematical development. Recognition of the naturalness of such a method of student's reasoning should form the basis for the evaluation of this reasoning and also the initial base for teaching activities, whose purpose will be to show relatively frequent unreliability of thinking based on induction.

In addition to epistemologically conditioned naturalness of reasoning based on the analysis of specific objects, we should also bear in mind the very complicated relation between the methods of drawing conclusions by a man in his everyday life and inference within mathematics, i.e. the relation between common knowledge and scientific knowledge of man. Such a grasp of the situation is particularly important from the point of view of anthropomathematics. Obviously, an assumption of mutual influences of the two types of human knowledge is natural and necessary. However, we must remember that the way of thinking, having its origins in a daily life, carried (mostly unconsciously) into mathematics, can cause the creation of the patterns of thinking which are not always compatible with the correct mathematical thinking.

Such a situation is described by, inter alia, G. Polya (1954, 1964, 1975), who emphasizes the special role of intuition, induction, "reasonable guessing", noticing analogies, and other informal heuristic procedures in practicing mathematics. Polya has shown in his books a certain general pattern of thinking that author called *a fundamental inductive pattern* or shorter, *an induction pattern*: each subsequent positive result of the verification of a given hypothesis makes it more credible.

Naturally, it is not out intention to convince the reader that the recognition of the truth of general conclusions on the basis of confirmation for several special cases shall be considered as a mathematically satisfactory way of justifying this truth. But I would like to emphasize the naturalness and universality of human thinking based on the intuitively adopted assumption of the regularity or constancy of the phenomena around us. With respect to mathematics, human perception of its harmony, regularity and constancy of certain relations is very natural. Sometimes it is an unconscious message accompanying the activities performed within mathematics, at a high level of this activity. Of course, an experienced mathematician shall not be satisfied only with an intuitive feeling, he or she knows the rules of recognition of the truth in mathematics, as well as experienced situations in which the apparent regularity and constancy failed. However, the mathematician knows that such circumstances are not very numerous or frequent, and a vast majority of the hypotheses confirmed by their occurrence in certain special cases, all in all, also turns out to be true.

The answer, of course, the learner may be provided with, inter alia, through experiences that show the failure of the inductive method of reasoning. We shall analyze an appropriate example (Dunham, 1994, p. 150) and consider the following hypothesis:

Factoring a positive integer n into a polynomial

$$f(n) = n^7 - 28n^6 + 322n^5 - 1960n^4 + 6769n^3 - 13132n^2 + 13069n - 5040 \quad (*)$$

as a result we always get the same integer n. It means that $f(n) = n$ for any integer n.

The reader shall admit that the verified hypothesis is quite "artificial" in its form, and its origin defies intuition. However, here as well, as it was before, the obvious starting point for assessing the truth of the hypothesis (for a naturally acting man) is verifying it for the first few natural numbers.

n	$f(n)$
1	$1-28+322-1960+6769-13132+13069-5040=1$
2	$2^7-28\left(2^6\right)+322\left(2^5\right)-1960\left(2^4\right)+6769\left(2^3\right)-13132\left(2^2\right)+13069\,(2)-5040=2$
3	$3^7-28\left(3^6\right)+322\left(3^5\right)-1960\left(3^4\right)+6769\left(3^3\right)-13132\left(3^2\right)+13069\,(3)-5040=3$

Table 1.

We may continue patiently the verification.

n	$f(n)$
4	$4^7-28\left(4^6\right)+322\left(4^5\right)-1960\left(4^4\right)+6769\left(4^3\right)-13132\left(4^2\right)+13069\,(4)-5040=4$
5	$5^7-28\left(5^6\right)+322\left(5^5\right)-1960\left(5^4\right)+6769\left(5^3\right)-13132\left(5^2\right)+13069\,(5)-5040=5$
6	$6^7-28\left(6^6\right)+322\left(6^5\right)-1960\left(6^4\right)+6769\left(6^3\right)-13132\left(6^2\right)+13069\,(6)-5040=6$
7	$7^7-28\left(7^6\right)+322\left(7^5\right)-1960\left(7^4\right)+6769\left(7^3\right)-13132\left(7^2\right)+13069\,(7)-5040=7$

Table 2.

Checking as many as seven cases seems to have reassured us (writes Dunham) that the hypothesis is true for any number n, however, the substitution $n=8$ leads to the following observation:

n	$f(n)$
8	$8^7-28\left(8^6\right)+322\left(8^5\right)-1960\left(8^4\right)+6769\left(8^3\right)-13132\left(8^2\right)+13069\,(8)-5040=5048$

Table 3.

Thus, our hypothesis falls through, although its truth (after seven trials!) was highly probable.

This situation was predictable for the author of the analyzed hypothesis, who obtained the form of the polynomial subject to the analysis, after multiplying and organizing terms of the following expression:

$$f(n)=n+[(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)]. \qquad (^{**})$$

It can be noticed that for $n=1,\ldots 7$, the expression in square brackets will zero, which gives us the postulated equality: $f(n)=n$.On the other hand, already for $n=8$ we get $\ldots f(8)=8+[7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1]=8+7!$. Similarly, it will continue $f(9)=9+8!$, $f(10)=10+9!$, etc. It appears that the inductive method of justification of the analyzed hypothesis proved to be unreliable. It was caused by a certain deceit of its author. Indeed, if the person conducting the verification knew the analyzed equality in the form (**), they could reason relying not only on its calculations, as it was in the case of (*). Then, probably some induction trials would suggest a fact of zeroing of a part of the expression in square brackets.

It is arguable if examples of this type are natural, or even ethical, although without any doubts they satisfy the cognitive function. They show the fallibility of inductive inference. But it is happening (at least in the case described above) on unnatural grounds, detached from the sphere of human belief and intuition, because what sort of intuition we can have while looking at the equality (*)?

3 Anthropomathematical point of view

The theoretical meaning of the term *anthropomathematics*[12] assumes that from the anthropomathematical view, we perceive mathematics as an intellectual activity (not "ready knowledge") of man, we treat it as an object and the result of the cognitive process[13], inherently associated with man – an actively learning individual, recognizing – and emphasizing at the same time – that occurrence of the cognitive process in specific social conditions. As researchers of mathematical education, we are clearly interested in a widely understood interdependence between mathematics and human learners[14]. After all, this is the nature of the educational processes.

Thus, the anthropomathematical approach focuses on the phenomena accompanying the *cognitive process,* in which the *cognitive subject* is man (*anthropos*) and the *cognitive object* is mathematics (*mathematika*)[15]. The most important assumption that I accept, highlighting and defining the described research area, is that in the area of anthropomathematical interest lie all phenomena strongly associated with the nature of mathematics as a cognitive

[12]The term is used by the participants of the seminar on Teaching of Mathematics at Charles University in Prague, which is conducted by prof. Milan Hejný. The meaning of this term was also explained in other works (Żeromska, 2009a, 2009b).

[13]The use of the term *cognitive process* obliges to take into account the fact that we are talking about acquiring information from the outside world: from the perception of that information, through its processing, categorizing and remembering, as well as using information and communication resources, until it is maintained in the non-volatile memory (Nęcka, Orzechowski, Szymura, 2006).

[14]At the philosophical level, there are many ways of understanding what mathematics is, and many ways of understanding what it means to be man. These capabilities have serious consequences for the way of understanding practice as well as practicing this scientific field (Brown, 2008).

[15]The word *mathematika* comes from the Greek, where the term *mathema* means learning, knowledge and cognition, while another term *mathesis* means to learn through meditation (footnote added by the author). For the ancient thinkers, mathematics was therefore equated with science, cognition and reasoning.

subject, but they do not exist autonomously and independently of the cognitive entity, which is man[16]. So I assume that there is as much mathematics, as it is the subject of intellectual cognitive activities of man.

Another basic assumption, defining the anthropomathematical, theoretical-research approach is that the process of teaching-learning of mathematics frequently occurs in some social context (it rarely is a spontaneous process of individual learning). In this sense, mathematics is a social science and this feature is strongly emphasized by anthropomathematical recognition. This social aspect of the teaching-learning process of mathematics must be perceived also as having an important and sometimes decisive role of a teacher, whose knowledge and attitudes to certain parts of mathematics has an impact on the formation of these categories in the mind of a student. An equally important assumption of the anthropomathematical approach is to take into account mutual relationships of common knowledge, which is used by man in the environment which is not associated with mathematics, and scientific knowledge within mathematics. Particularly important is the influence of everyday patterns of thinking and acting on the creation and the use of such patterns in the mathematical activity of man.

Let us specify the anthropomathematical research approach on the example of a process of school teaching-learning of mathematics. Clearly, we find that this process bears all the resemblance of the cognitive process that runs in a certain social context. The *subject* of this cognitive process is a student, and the *object* is mathematics (in its school transposition)[17]. And the whole process takes place in a specific social system, which is (usually) a classroom . This process does not proceed in a plane which is merely substantive, relating to the matter of the cognitive subject. The research within the teaching of mathematics increasingly emphasizes the importance of emotional-motivational sphere, having a huge, and sometimes a decisive, influence on the course and results of the cognitive process, which is the process of teaching-learning mathematics (Kilpatrick, 1992; McLeod, 1992; Leron and Hazzan, 1997, Mason et al., 1998; Czajkowska, 2002; Żeromska, 2004). What is important – and additional – in this described cognitive process, is the fact of an active and significant influence of a teacher who inspires this process most often (sometimes even forces) and controls it. Here again, we emphasize the social nature of the process, highlighting two planes of the social context: student – classroom

[16]The issues related to the problems of the mutual relations of "pure" and "applied" mathematics do not belong, for example, to such a specific area.

[17]The cognitive process described in that way comprises a number of phenomena included in it, from the perception of information, through its processing, to incorporating new information into the existing system of the knowledge of the cognitive subject.

and student – teacher.

We are therefore dealing with a cognitive process, in which the *subject* does not function autonomously, it is under a large – sometimes dominant – influence of another *co-subject* of this process, the teacher, integrally associated with this process. It is a certain peculiarity of this approach. This means a need to include two components of the subjective part in the analyzed cognitive process, between which there are numerous complex interactions. Such an assumption has its consequences in taking into account a large, sometimes decisive, influence of a teacher and his or her knowledge system on the results of the studied cognitive process. The situation is further complicated by the fact that both *co-subjects* – a teacher and a student – are often not fully aware of their mutual influence on the results of mathematical knowledge, especially in the context of the non-substantial spheres of this cognition.

A similarly complex character has the *object* of the analyzed cognitive process in its school anthropomathematical terms. The immediate cognitive object is, of course, mathematics. But it is not a mere transfer of the essence of this scientific discipline. This is a certain didactic transposition of mathematics, conditioned by a variety of factors: from institutional through personal ones, both from the student's and the teacher's side.

To sum up, I advance the thesis that mathematics in anthropomathematical terms belongs to humanities[18] and social sciences[19], and that experience as well as knowledge which do not belong to the realm of mathematics, with their source in everyday life, have a substantial impact on the formation of mathematical knowledge of man. Such knowledge can be a source of numerous, often unconscious, beliefs, having a strong impact on the thinking of man used outside mathematics, as well as within it.

This view is new to the point that it merges different research strands in the teaching of mathematics: the study on the didactic transposition of mathematical knowledge, research considerations on the social dimension of mathematics (highlighting the perception of mathematics as a social product), and the study focused on the issues relating to cognitive learning of an individual, taking into account a significant influence of a variety of non-substantive factors of these processes.

Establishing my own research position requires the specification of a number of assumptions. I assume that:

[18]Humanities in the sense that they deal with the matters concerning man and his activity in a particular field of human activity.

[19]Social, as they relate to a series of phenomena that most frequently occur in certain social systems and have an impact on the personality of an individual or a social group.

- One of the basic objectives of conducting a didactic research on the teaching and learning of mathematics is a deep, insightful and multilateral understanding of the conditions and the course of this process.

- Anthropomathematical depiction of educational mathematics should reflect the aim to adapt the teaching of mathematics to the new conditions, among which the ongoing social and cultural changes should be highlighted, supported by a widespread use of information technology. The prevalence of using these means has a huge impact on the way young people perceive the world. In the teaching of mathematics, it is not possible not to notice and neglect that.

- It is essential to make efforts regarding a long-term process of changing the popular opinion and ideas about mathematics by changing, among others, its image conveyed through teaching. It is important what methods should be applied so that this subject has become more accessible to most students. The teaching should be deprived of authoritative knowledge transfer and the pursuit of portraying this field as a domain of absolute truths. It is necessary to try to find a common ground for mathematicians and educators as to the nature and expected outcomes of the general education of mathematics.

- The process of teaching and learning mathematics should be considered in a broad context, including the role of a teacher, his or her knowledge and belief in the results of the considered process.

- The existence of a teacher and their functioning in the teaching-learning process of conscious and unconscious mathematical teaching knowledge organized by him or her, should be considered and strongly emphasized, as well as of the knowledge which refers to the psychological and educational components of this process.

- Argumentation and proving, in anthropomathematical terms, fully are a humanistic social process. They have their origins in a particular human need – a natural or provoked one – related to convincing others of the rightness of some general opinion. The effects of the verification process of mathematical statements under natural conditions of teaching and learning of mathematics are communicated to other participants in this process. They are verified by other students, a teacher or a textbook.

- Both developing skills of argumentation and proving, as well as the assessment of student's achievement in this field, must take into account the humanistic and social aspects of the problem. We should note that:

a. A contribution of intuition, empiricism and inductive inferences in the search for argumentation to recognize the truth of mathematical theses is both natural and important. Performance of formal proofs may be of secondary importance.
b. A concrete and concretization in students' mathematical reasoning are of the first-class and key importance[20].
c. The meaning of the terms: proving, argumentation, or justification, etc. (common in maths lessons), conferred by a student, is affected by the patterns of thinking characteristic of the contexts of every day life[21].

Anthropomathematical approach to the didactic studies should relate to these issues both conceptually and empirically. Such an approach requires appropriate changes and re-evaluations of the methodological approach to conducting anthropomathematical research, and takes into account wide possibilities of interpretation of the results.

4 Description of research, the results and conclusions

The research which I carried out were to provide answers to the following questions:

- What criteria, according to school and university students' opinion, decide that a mathematical statement can be considered to be true[22]?
- What is the role of an example in justifying reasoning?

In this study, the method of written product analysis was used (Łobocki, 2000) of the respondents solving appropriately selected math problems. The analysis was supplemented with the data obtained from unstructured interviews

[20]According to the revealed epistemological obstacle, named by R. Duda "the obstacle of a concrete" (1995).

[21]Psychologists distinguish two cognitive systems: System I and System II. In the thinking processes of everyday life, System I participates (acting spontaneously, quickly, comprehensively). System II operates analytically, rationally, is controlled and is relatively slow. Mathematical contexts require thinking, which is the result of System II. The greater part of life, a student uses the thinking processes characteristic of System I and this way of thinking he or she also applies to mathematical contexts, and this may be the reason of applying undesirable patterns of reasoning in mathematics (Vinner, 2012).

[22]The aim of this study was to diagnose skills to positively verify hypotheses. It was not about a study on the skills of proving the falsity of mathematical statements. An ability to create counterexamples and denial of mathematical theses requires specially constructed research studies.

(Łobocki, 2000) conducted with some of them. The theoretical basis, which the analysis of student solutions was basing on, was a typology of types of evidential reasoning provided by N. Balacheff (1987, 1990) and its compatibility with anthropo-mathematical direction, assuming a significant impact of natural mental activities of the learners (common in daily life of a human) on the operations of justifying general statements in mathematics. N. Balacheff provides the following, hierarchically structured, typology of student proofs:

- *Naive empiricism (l' empirisme naïf)* – a proof, is in this case, remains at the level of recognizing a specific observation to be sufficient. A property of a particular, specific example is considered by a student as an general property. In the scope of naive empiricism, Balacheff emphasizes the existence of sometimes a little "higher" form of such reasoning involving an alleged conviction deciding about the truth, arising from the structure of the considered situation, not only from the empirical experience. To illustrate "naive empiricism", an example taken from the work of M. Ciosek (1992, pp. 105-106) will be used. The example concerns the following problem:

The table shows a relation between the x number of the sides of a polygon and the number of its diagonals. Complete it:

x	3	4	5	
Number of diagonals		2		14

The table given in this task, one of the students completed as follows:

x	3	4	5	**28**
Number of diagonals		2	**0**	14

The schoolgirl asked by a teacher (immediately after the completion of the table) of the reasons why she solved the task in that way, replied: *I noticed that if x was odd, the number of diagonals was , and if x was even, the number of diagonals was half of it* (p. 106). The schoolgirl observed data in the table with a focus on the discovery of a general relation and, based on the table-illustrated example, she formulated some observation, which she did not treat as a hypothesis requiring verification. She did not make any attempts to verify it (even on other specific examples), and the property of the example she considered the general property. It is this type of recognizing a particular observation as sufficient to justify the general statement is called naive empiricism.

- *Crucial experiment (l' expérience crucial)* – a person solving the problem also reasons on a specific example (or several examples), but realizes the importance of the mathematical context of this example in the direction of its generalization. This generalization is somewhat conscious, but cognitive and language limitations do not allow a student to go beyond the concrete examples[23] . Balacheff illustrates such reasoning with an example derived from the work of two students. The students solving the problem (Martine and Laura) put forward their hypothesis as to how to calculate the number of diagonals of a polygon by analyzing the case of a hexagon; they take into account subsequent vertices of a hexagon and count the diagonals drawn from them (Balacheff, 1990, p 290). The students reason as follows (with reference to Figure 1, Figure 2 and Figure 3):

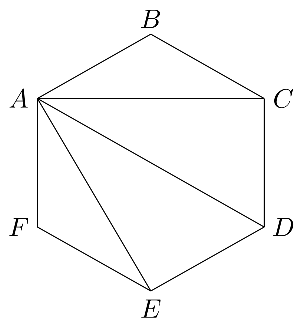

Figure 1.

From the first vertex (A) there are 3 diagonals drawn, and this is the same as the number of vertices of this polygon reduced by 3. From the second vertex (B) there is the same number of diagonals drawn as well:

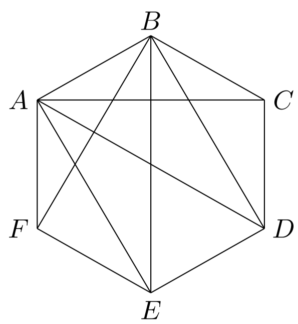

Figure 2.

[23]Balacheff draws attention to the crucial role of such a way of thinking in the transition from empirical pragmatism to logical rationalism (1990, p 289).

From the next vertex (C) there is 1 fewer of new diagonals drawn, from the vertex (D) – 2 fewer, etc.

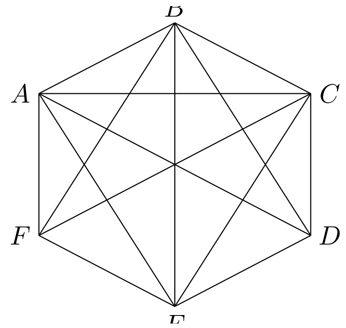

Figure 3.

Now it is enough (say students) *to add all the numbers of new diagonals drawn from successive vertices and in that way we will always get a result.*

Then, the observed persons state: *it still must be checked on some very large number of vertices.* Such a verification (for a decagon) students treat as a sufficient criterion to justify the correctness of the hypothesis formulated by them. It's still a reasoning based on empiricism, but with an attempt to perceive generality of the formulated property of a mathematical object.

- *Generating example (l' exemple générique)* – a person solving the problem shall, in this case, make general observations, analyzing a concrete example. It's about a specific way of reasoning of a person who knows some example of an object that meets the verified hypothesis and consciously asks themselves a question, what special property that considered example has. A person asks about the characteristics of that example in connection with the analyzed hypothesis.
- *Mental experiment (l' expérience mentale)* – it is a situation in which a person solving the problem refers to a particular mathematical object (therefore being at the level of acting on a concrete), but the process of reasoning at no time is referenced to the particularity. Such reasoning requires from a person who is carrying out the reasoning procedure a total decontextualization and elimination of concreteness of that considered example (to generate a general solution). A result of this thinking process is not yet a valid deductive inference, although its conduct requires that a person solving the problem has advanced cognitive structures and it occurs only at the level of a language. The mental experiment is based on the intermediate stages of proving, it is its lower level, and it can relatively easily be transformed into a proof. Balacheff states that

such reasoning can rarely be observed (1990, p. 291), it requires from a person to have large language skills and that knowledge was organized at quite a high level.

The list of possible justifications for the correctness of reasoning shall be complemented with:

- *Deductively oriented reasoning* – already discussed in previous chapters, inference based on the rules of logic from the previously accepted resources of knowledge. It is the most desirable method of reasoning and the closest to a valid mathematical proof, or itself being such a proof already. It requires a relatively high level of mathematical and logical knowledge, and much freedom in the use of mathematical language. Conducting this type of reasoning (even with a small degree of formalization) is still complicated and frequently requires complementation from the knowledge directly unrelated to the topic, but necessary to conduct reasoning.

I shall present the results of the research on school students. The diagnostic tests distinguished four groups of such students, representing different levels of mathematical education. And so they were as follows:

- The students of the fourth grade of elementary school (SP IV),
- The students of the sixth grade of elementary school (SP VI),
- The students of the third grade of middle school (G),
- The students of the second grade of high school (L).

The number of persons participating in the research study have been presented in the following table 4:

<table>
<tr><th>Group name</th><th>Number of persons in the group</th><th>Total number of respondents in the study</th><th>Total number of analyzed solutions to the problems</th></tr>
<tr><td>SPIV</td><td>71</td><td rowspan="4">241</td><td rowspan="4">1012</td></tr>
<tr><td>SPVI</td><td>51</td></tr>
<tr><td>G</td><td>69</td></tr>
<tr><td>L</td><td>50</td></tr>
</table>

Table 4.

Each group of respondents was given to solve a set of mathematical tasks (see: Żeromska, 2013); they had unlimited time to solve them. They were asked to write the most extensive explanations, comments and answers to their solutions. Then, the collected documents were subjected to a several-aspect analysis. An important issue in the construction of the sets of research tasks was to maintain a similar nature of the subjects of certain tasks for all research groups to be able to compare the qualitative results obtained after the analysis of the solutions of subsequent age groups (for the purpose of suitable description of the study results).

For the purpose of the described research study, both of types of reasoning (called by Balacheff *naive empiricism* and the application of *crucial experiment* prevail) I called the *simple use of a concrete*. I am thinking, therefore, of a situation in which the basis for formulating a conclusion on validation of the test hypothesis is a concrete example (or several such examples). We should bear in mind that even if this is not only naive empiricism, the verified example is, according to a student "discretionary"[24], i.e. random or e.g. regarding objects characterized by large numbers, then the basis for inference is the student's belief that the occurrence of a general property in a particular case is a sufficient reason to recognize the truth of the general property[25]. That is how I understand *the simple use of a concrete*.

884 student solutions were analyzed for identifying reasoning involving the consideration of specific examples. Each of these solutions has been studied with particular attention to the role of a concrete example in this reasoning[26]. The quantitative data (tab. 5) clearly show that, for the considered stages of teaching mathematics, students justify general statements given to them mostly by examining them on a specific example. The study revealed that only at the higher stages of education, and only some students, combine analysis of an example with asking a question how to determine the sought value, regardless of which specific example we talk about.

We can clearly see two methods of justification used by students while they were solving problems by this example.

[24]Students, in a variety of situations requiring the use of an implicit universal quantifier, often reasoned that: the property is to occur "for all" the elements of a given set, so I take some "discretionary" element and verify. In view of the arbitrary choice, I state that the property actually holds for all the elements of this set (Klakla, Klakla, Nawrocki, Nowecki, 1992; Ciosek, 2005).

[25]We should bear in mind that Balacheff distinguishes two other situations in which the basis of inference is a specific example, calling it *the generating example* and *the mental experiment*. However, both of these types of reasoning I treat as requiring a certain awareness that reasoning based on the concrete is an insufficient measure to show generality. Both of these types are characterized by a pursuit of the generality, which is a manifestation of a specific divergence from the concrete.

[26]Of course, I am aware of the fact that the tasks used for the research study, probably give only a partial picture of the entire spectrum of strategies used by students to justify mathematical statements.

Problem

Justify that if the price increases by p%, and then by q%, we obtain the same result as if we first increased the price by q%, and then by p%.

One method included an analysis of a specific example (*the simple use of a concrete*), as in the following solution:

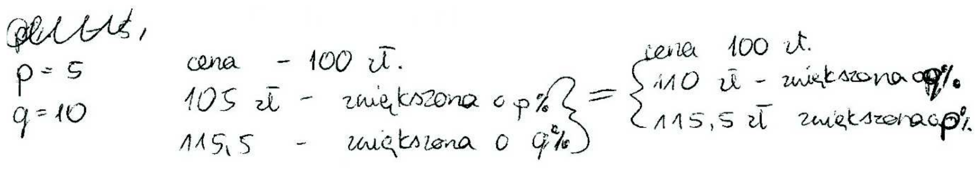

Figure 4.

The second method showed the student's focus on the symbolic calculations and consisted in the demonstration of equality in the case of any percentage values, as in the following notation:

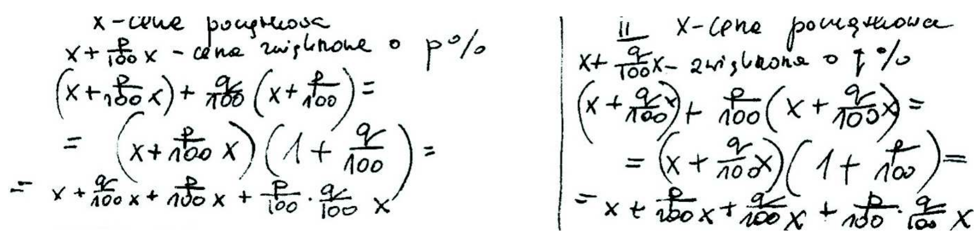

Figure 5.

Unfortunately, the number of students who solved this problem in first way was much more then the others. Let us now take a look at the numerical distribution of the students' solutions (tab. 5).

Study group	SPIV	SPVI	G	L
Number of solutions based on the analysis of concrete examples (*the simple use of a concrete*)/number of all solutions	176/194	159/179	252/280	189/231
Per cent of this type of justifications (of approximately)	90,7%	88,8%	90%	81,8%

Table 5.

These data shall be illustrated on a diagram showing the direction of quantitative changes:

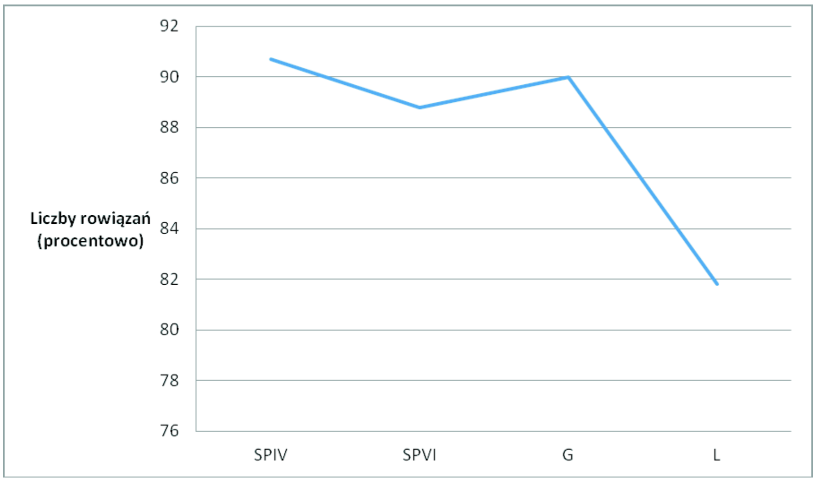

Diagram 1. Justification by *simple use of a concrete.*

This data and the course of the line on this diagram entitle to formulate some conclusions:

- Among the types of reasoning used by learners of mathematics at the school levels of this learning, the methods called by Balacheff *naive empiricism* and the application of c*rucial experiment* prevail.
- Methodological awareness of students at successive levels of learning develops to a small extent. The vast majority of students at all stages of learning apply the method of justifying general mathematical statements based on illustrating the truth of these theses with concrete examples.
- Students learning mathematics in the majority (approximately 80%) do not reach a higher than the first level of understanding of the role of the reasoning of proving type in mathematical theory. Many students use justification of general mathematical statements on specific examples, but with a noticeable tendency to go beyond the concrete. Even a long-term mathematical education does not ensure sufficient understanding of the role of a proof in a mathematical study, and in particular, a mutual relationship between a theorem and a proof. *Simple use of the concrete*, at the level of mathematics studies, is a justifying reasoning, which is often enough for a student to determine the correctness of a general mathematical statement (Żeromska, 2013).

5 Recapitulation

The considerations presented in this paper show some extract from the didactic research (see Żeromska, 2013) issues regarding a rich and complex social process, which is the process of school teaching and learning of mathematics. Against the background of these considerations, some conclusions, hypotheses and reflections arise, which can become an inspiration to many thoughts and, perhaps, specific actions, which could have a positive influence on the outlined situation. The results of the described theoretical and research analyses emphasize the richness and depth of the phenomena with which we are dealing with. And although the goals, the progress and the results of mathematical education are still a major concern to various individuals and institutions associated with education of mathematics, there are a lot of questions that we are still looking for the answers to. Together with attempts to answer some of them, other research questions arise, some difficult and even unsolvable, but on the other hand, important, requiring specific actions.

One of the broad research areas is the issue of identifying the causes of so many problems associated with students' understanding of mathematical methods, in particular with justifying general statements. This issue has been of interest to theoreticians and practitioners of mathematical education for a long time. However, although new attempts to answer the questions about the cause of students' excessive difficulties and failures are continuously being made, it is still impossible to get unambiguous and constructive answers. Frequently, the cause of the occurrence of students' difficulties is believed to be the deficiency in their knowledge and skills. Sometimes, those causes are sought on the teachers' side, criticizing their methods of teaching. Sometimes, the excessive overload of thematic content in the curricula of mathematics is blamed. Probably, the truth lies somewhere in the middle, and an attempt to make a comprehensive diagnosis of this status quo requires a very broad and in-depth analysis of all the factors that may have some influence on this status quo. The analyses must be carried out here from different points of view and with a focus on deep understanding of the complexity of the process which we are dealing with.

One of these points of view is the anthropomathematical approach outlined in this work. It suggests:

- A significant **expansion of the analysis area of the process of teaching-learning mathematics.** The point is that the results achieved by students in the subject of mathematics should be looked at not only from the point of view of mathematical correctness of students' activities. The cognitive process which we are dealing with here, runs in a particular social system, the influence of which we must take into account.

- Consideration of **the influence of non-mathematical ways of thinking of man** on their reasoning carried out within the mathematical activity.
- Abandoning the image of mathematics as presented in teaching, as a field of infallible, absolute truths, and **conferring on mathematics a character of humanistic activity of man.** At the same time, due to the specific nature of mathematics and its complex epistemology, it should be assumed in advance that:
  - the course and results of this activity may be subject to imperfection and prone to failure;
  - the final results of this activity carried out by different people (within the same subject) can be varied as to their mathematical advancement.

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Dowodzenie zdań ogólnych w matematyce i uzasadnianie stwierdzeń przez uczących się matematyki. Perspektywa antropomatematyczna

S t r e s z c z e n i e

Artykuł zawiera rozważania teoretyczne oraz opis pewnego badania dydaktycznego. Refleksje teoretyczne dotyczą metodologii matematyki, roli dowodu matematycznego oraz zagadnienia uzasadniania zdań ogólnych w rozumowaniach matematycznych. Rozważania teoretyczne obejmują także założenia badawczo-teoretycznego podejścia do zagadnień dydaktyki matematyki, nazwanego antropomatematyką. W artykule opisano również badanie dydaktyczne będące diagnozą sposobów uzasadniania zadań ogólnych stosowanych przez uczniów na wszystkich poziomach edukacji matematycznej. Jest to fragment szerszych badań opisanych w mojej monografii pt. *Metodologia matematyki jako przedmiot badań antropomatematycznych* (Żeromska, 2013).