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Using Student-Friendly Tasks to promote mathematical understanding in a middle school classroom

Abstract: Focusing not only on procedural skills, but also on conceptual understanding, is a very important aspect of teaching mathematics. One possible way to promote students' understanding is to use problem solving in the classroom. The two key elements of designing a problem-solving activity are selection of the task and organization of the instructions. In this article, a new category of tasks, named Student-Friendly Tasks, is characterized, and a way to use these tasks in the classroom is proposed. One task fitting this category has been chosen, and the process of solving it, by a group of students and by individuals, is analysed in detail. The claim is that this proposal can serve as a didactic tool to engage students in mathematical reasoning and help them develop a deep understanding of mathematics.

1 Theoretical background

1.1 Understanding in mathematics

Teaching and learning mathematics are both directly connected to the concept of "understanding". The great mathematician William Thurston stated that understanding is a critical issue in mathematics. He claimed that what mathematicians really want is understanding, and not a collection of answers to some problems (Thurston, 1994). Speaking on behalf of practising mathematicians, Thurston added that "the measure of our success is whether what

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we do enables people to understand and think more clearly and effectively about mathematics" (Thurston, 1994, p. 163). Being such an important part of professional mathematicians' work, understanding should be placed in the centre of interest of mathematics educators.

Although the term understanding seems to be commonly known, it is interpreted differently among both mathematics educators and students. Since the early 70s, a lot of mathematics education literature has been dedicated to the concept of understanding (e.g. Dyrszlag, 1971; Skemp, 1976; Hiebert, 1986; Sierpińska, 1990, 1994; Sfard, 1991; Pirie & Kieren, 1994). Several models of understanding in mathematics have been created as a result of theoretical considerations. Schroeder (1987) presented a review and synthesis of the research from the 70s and 80s regarding this topic, placing the models and conceptualizations of mathematical understanding into three broad categories: 1) epistemological and constructivist models, 2) Piagetian and neo-Piagetian models, and 3) human information processing and cognitive science models. In the next sections, references to constructivist models of understanding, in particular to the one presented by Skemp, can be found.

Skemp (1976) distinguished between instrumental and relational understanding. The former has been characterized as "rules without reasons" or "the ability to apply an appropriate rule to the solution of a problem without knowing why the rule works" (Skemp, 1976, p. 20). Relational understanding, on the contrary, is seen as "knowing both what to do and why" (Skemp, 1976, p. 20). Skemp stated that he had initially not considered instrumental understanding as understanding at all. Later, however, he acknowledged that both terms, instrumental and relational, were used to express different meanings of the same word, "understanding". In his paper (1976), Skemp plays the role of devil's advocate by discussing some supposed advantages of instrumental understanding (e.g. being easier to understand and often quickly providing the correct answer), but then points out a few strong arguments for relational understanding (e.g. being easier to remember and more adaptable to new tasks). The terms "instrumental" and "relational" are used by Skemp (1976) to distinguish between two different kinds of understanding, but also of knowledge, thinking, learning and teaching.

Hiebert (1986) presented a similar distinction by defining two types of mathematical knowledge and understanding: conceptual and procedural. Procedural knowledge refers to computational skills and making use of rules and procedures, while conceptual knowledge gives meaning to those procedures – it is rich with relationships and refers to the structure of mathematics (Hiebert, 1986).

Another model of understanding in mathematics, regarding the mathematical concept, was presented by the Polish educator Dyrszlag (1972). His model of understanding a mathematical concept has been frequently used in Polish mathematics education research (e.g. Sajka & Luty, 2012). Dyrszlag characterized four levels of understanding a mathematical concept: 1) definition level: the student can produce a definition of the concept, recognize/create its designata and non-designata, 2) local complication level: the student can create designata and non-designata meeting some extra conditions, 3) generalization level: the student can point out precedent and subordinate concepts and prove theorems concerning the concept, 4) structural level (the highest level): the student can see the structure of the concept and recognizes analogous structures in different mathematical objects.

Regardless of the terminology, Skemp, Hiebert and Dyrszlag are all strong advocates of shaping a deep – relational, conceptual, and structural – understanding of mathematics. Although creating the appropriate conditions for such teaching is a demanding task for teachers, the authors state that this is the only way of achieving real mathematical skills, which are applicable outside the artificial context of school tasks (Skemp, 1976; Hiebert, 1986).

1.2 Mathematical understanding in the classroom

Currently, most mathematics educators agree that instrumental understanding alone, as per Skemp's terminology, is not enough. They suggest that learning mathematics should balance the students' conceptual understanding and procedural fluency (Krygowska, 1977a; Turnau, 1990; Schoenfeld 1992; Boaler 2002; Bass 2005). Rittle-Johnson, Siegler, and Alibali (2001) suggested that both procedural and conceptual knowledge influence each other in the sense that they develop iteratively: increases in one type of knowledge lead to increases in the other one. The changes in mathematics curricula introduced over the last few decades around the world can be interpreted as proof of this view. There is a common tendency among all nations to shift the centre of attention from proficiency in mathematics algorithms and procedures, which were the cornerstone of past curricula, towards a deeper understanding of mathematics, e.g. Realistic Mathematics Education (Gravemeijer, 1994), Mathe 2000 (Wittman, 2007), the NCTM standards (NCTM, 2000).

In Poland, this tendency can be observed in the new core curriculum (MEN, 2008). The mathematics education section distinguishes educational objectives, defined as general requirements, and educational content, defined as specific requirements. For the third educational stage (middle school), the core curriculum (MEN, 2008, p. 35) mentions the following general requirements:

I Using and creating information.
The student interprets and creates a text of a mathematical nature, uses mathematical language to describe the reasoning and the obtained results.

II Using and interpreting representations.
The student uses simple, well-known mathematical objects, interprets mathematical concepts, and makes use of mathematical objects.

III Mathematical modelling.
The student assigns a mathematical model to a simple situation, creates a mathematical model of the situation.

IV Using and creating strategies.
The student makes use of the strategy clearly defined in the task, creates problem-solving strategies.

V Reasoning.
The student provides simple reasoning, justifies its validity.

The commentary of the core curriculum (MEN, 2008) states that the general requirements are a synthesis of the crucial educational objectives with a recommendation for curriculum and textbook authors to take into account not only the educational content, but also the general requirements, supplying teachers with proper tools for their implementation.

However, in practice, it seems that teachers received little support in the implementation of these changes. Despite the teachers' interest in new methods of education geared towards forming an understanding of mathematics, research into education conducted in recent years shows that most Polish mathematics lessons still make use of the traditional model of knowledge transmission (Karpiński, Grudniewska, & Zambrowska, 2013). The teachers also claim that they are not able to achieve both the fulfilment of the core curriculum's specific requirements and the development of higher-order thinking skills in students, as described in the general requirements. As mentioned above, educators agree that teaching constrained to providing knowledge and procedural skills does not nurture the development of a deeper understanding of mathematics. Teaching based on understanding requires engaging the students in creative mathematical processes, such as creating task-solving strategies, justifying one's reasoning and actively participating in mathematical discussions. The aforementioned activities are a significant part of achieving the goal of a deeper understanding of mathematics – the laws, structures and links between different aspects (Schoenfeld, 1992). This is not easy to achieve in a classroom, which makes it even more important to promote approaches and practices among teachers that will engage students in such activities. This was the aim behind the creation of the educational proposal presented hereafter, which, if properly implemented, can be a tool for teaching relational mathematics at the middle school level.

Two other theoretical assumptions, which strongly influence this work, should be mentioned. The first one is the notion that higher-level mathematical skills, such as reasoning, modelling or justifying, are not only the domain of the best students. Teachers often think the opposite and avoid teaching relational mathematics, as they believe they are saving weaker students from failing. The author fully agrees with Piaget (1977), who stated that every normal student is capable of good mathematical reasoning when the emotional inhibitions in lessons are removed. Krygowska (1975) also stated that no student should ever be assumed to fail in terms of their mathematical thinking.

The second assumption concerns students' views related to mathematics. It is known that the students' attitude towards mathematics is an important variable that affects learning and achievement (Schoenfeld, 1992; McLeod, 1994). Unfortunately, this variable often negatively affects the development of mathematical competence, as fear and reluctance towards mathematics can suppress or completely prevent the mathematical development of a student (Lyons and Beilock, 2012; Pieronkiewicz, 2014). Furthermore, research shows that these negative emotions are often the result of the way the students are taught (McLeod, 1994). Fortunately, even though mathematics is a subject which is perceived as frustrating and off-putting, it also has a different aspect. "There is a real joy in doing mathematics, in learning ways of thinking that explain and organize and simplify. One can feel this joy discovering new mathematics, rediscovering old mathematics, learning a way of thinking from a person or text, or finding a new way to explain or to view an old mathematical structure" – said Thurston (1994, p. 171). The personal experience of the author indicates that students of any mathematical proficiency can enjoy mathematical reasoning. Thus, the second important goal of the research reported in this paper was to design an environment in which all students can enjoy mathematics and change their negative attitude towards this subject.

2 Student-Friendly Tasks: theoretical design

The subject of this article are tasks that help promote reasoning and deepen students' understanding of mathematics. The aim of this chapter is to describe a didactic tool which includes this type of task and a proposal on how to use it in class. These tasks, coupled with instructional guidelines regarding their use, could constitute a tool for teachers allowing the to develop a students' understanding of mathematics by engaging them in creative mathematical activities. This proposal is the result of research conducted by the author in the years 2013-2015 at the Pedagogical University of Cracow. The proposal is aimed at middle school level, but can also be used at elementary and high school levels.

2.1 Design principles of Student-Friendly Tasks

One can sometimes hear that this is not about the choice of tasks, but about the way we work with these tasks. Accepting that every task can be extended, opened and enriched, the author believes that there are tasks which have more potential to engage students than others. The focus of the article is on the aforementioned type of tasks called Student-Friendly Tasks. Based on source literature and the teaching and research experience of the author, the tasks are defined by the following features.

(1) *Easy entry.* The task is formulated in an easy, concise way, to make it comprehensible for all students. The first stage of solving the task is easy, which means every student can try working on the task on their own.

(2) *Engagement potential.* The task is interesting to the student. It usually contains an element of surprise, often in the form of an unexpected regularity, which excites the curiosity of the students and makes them want to try to find the answer to the question "Why does this happen?" The value of such tasks in the process of teaching and learning mathematics is emphasised by Ciosek, Ratusiński, and Turnau (in press).

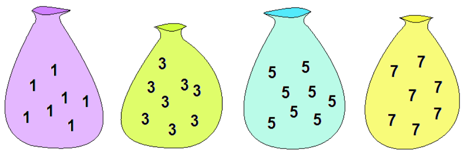
(3) *Mathematical richness.* The task focuses on processes, not concepts. Its content relates to the skills outlined in the core curriculum, but in order for the task to be fully solved, non-standard methods have to be used, which encourages the students to reason and create their own strategies. Such activities help develop relational understanding (Skemp, 1976). Despite an easy start, the task is not trivial: suddenly a problem appears, and overcoming it requires effort, which enables a positive mathematical struggle.

(4) *Potential to generate new problems or new solutions.* After solving the initial task, new questions can be created based on the solution or task itself (Wittmann, 1971; Krygowska, 1977b; Arcavi & Resnick, 2008) in order to create more general and more complex problems.

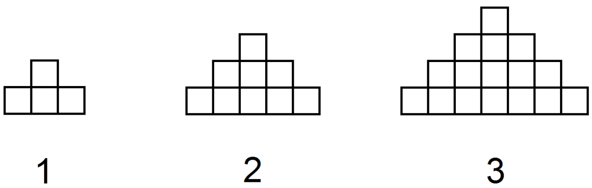
2.2 Examples of Student-Friendly Tasks

Some examples of Student-Friendly Tasks are presented below. Other problems which fit in this category, for different educational levels, can be found e.g. in works of Polya (1945/1973), Mason (1982), Wittmann (2002; 2007), Ciosek and Żeromska (2013), Rożek (2014).

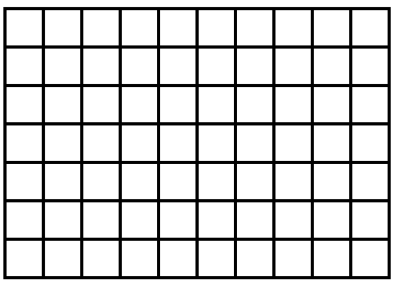
Make 37 (NRICH Project website)
Four bags contain a large number of 1s, 3s, 5s and 7s. Pick any ten numbers from the bags so that their total is 37.



Pyramids (Boaler 2008)
Presented below are three figures built subsequently in accordance with a rule. Draw the 5$^{\text{th}}$ figure. How many blocks would it have? How many blocks would case 10, 100, n have?



The Rose Hotel – based on a task from a textbook by Dobrowolska et al. (2008)
The Rose Hotel has a capacity of 70 in two and three-person rooms. On the following chart, in which every cell represents one person, outline the rooms, connecting the cells in groups of two and three. How many rooms for two and how many rooms for three are there in the hotel?



Adding decimals (Open Middle website)
Use the digits 1 to 9, once each, to fill in the boxes and make three decimals, the sum of which is as close to 1 as possible.

$$\begin{array}{r} 0.\square\square\square \\ 0.\square\square\square \\ +\,0.\square\square\square \\ \hline \end{array}$$

2.3 Student-Friendly Tasks as a didactic tool

The proper implementation of solving Student-Friendly Tasks in the classroom can be a way of applying the relational teaching of mathematics. In order to achieve this goal – deepening the understanding of mathematics by engaging students in creative mathematical activity – the use of such tasks in class should be deliberate and well-planned. A proposal of a working method for Student-Friendly Tasks based on the analysis of literature and conclusions drawn from research conducted by the author is presented below. The research included several repetitions of the cycle of conducting lessons, observation, in-depth analysis, and an assessment of the effectiveness of the implemented pedagogical moves. The conclusions drawn from the analysis were used to improve the lessons in the next cycle.

The lessons including Student-Friendly Tasks consist of four basic phases: 1) solving the easy part of the task – individual or group work, 2) struggling with the problem – individual or group work, 3) whole class discussion on the solution, 4) extending the initial task.

Phase 1: solving the easy part of the task

After the students are presented with the task, the initial phase begins. Due to the structure of the task, the students will easily begin working on the problem, looking for the solution or a part of it. This is a way of warming up and familiarizing the students with the task. The work of the students in this phase is simple, not requiring any advanced knowledge or skills, which makes it possible for every student to participate. At the same time, the work is not typical or formulaic, and require the students to make use of a simple type of reasoning, which gives the students, especially low-achieving ones, significant amounts of satisfaction. This small success in the initial phases of solving a task "befriends" the students with the task and motivates them to persevere and perform harder work, which is required in the next phase of solving the problem.

Phase 2: struggling with the problem

The described tasks are created in such a way that the student eventually stumbles upon an obstacle. The obstacle may manifest itself at different points for different students. During this phase the students can work on their own or in groups, and the teacher should not interfere with the students' work, but observe this work carefully. The teacher should not provide any hints or comments which could help the students find the solution. The aim is to create the opportunity of a productive struggle for the students, which is a vital part of developing an understanding of mathematics (Hiebert & Grouws, 2007). In practice, if a task is interesting and the students have "befriended" it in the first phase, the struggle does not discourage them. During trial lessons with different groups of students, the author often observed the opposite: after one of the students loudly announces his success in solving the task, the other students firmly ask for the answer to not be revealed: "Don't, I want to think about it!". It is worth noting that the ones protesting are often those who usually do not involve themselves in mathematics lessons.

If the students are struggling and failing to find the solution, or, on the contrary, if one of the students manages to solve the task before the others, the teacher should maintain the students' interest and involvement. One possible way is to encourage them to share their ideas and inspire each other. If the teacher notices the students who have already solved the task getting bored, they can be encouraged to refine the solution (proper notation, comments), or be given a modified, extended problem, by being asked an additional question. A different approach is to be used with students who give up when faced with obstacles. They are to be encouraged into further work, e.g. by asking the students who managed to already overcome the issue to share their ideas in front of the class or in a small group. It is important for the teacher to not guide the students to the correct (usually one of many) way of solving the problem, but rather to facilitate communication and inspiration between students.

Phase 3: discussion on the solution

The third phase of the lesson involves discussing the solution with the whole class. It is favourable for as many students as possible to have already solved the task on their own or learn the solution of another student, but it is not necessary. In this part of the lesson, the teacher's role is to orchestrate classroom discussion regarding the solution. Numerous studies during recent decades present the elements necessary to facilitate worthwhile discussion in class (e.g. Lampert, 1990; Cobb, Wood, & Yackel, 1993). Krygowska (1977) claimed that finding the solution usually means the end of thinking about the problem, which is a typical mistake and weakness of our teaching. It should be the beginning of posing new problems, and the first ones can be: *Are we sure this is the correct answer? Are there different ways to get this answer? How are these ways of thinking similar, and how are they different from each other? Which solution seems to be the quickest? Which one seems to be the most general?* It is important to let the students present their solutions and encourage them to comment on and compare each other's work. It is also viable to use various representations of mathematical objects, i.e. arithmetic, algebraic, and graphic.

The teacher should encourage the students to present their own ways of thinking, even if they do not significantly differ from each other and all lead to the same result, and then ask the students to compare them. It is crucial for the teacher to not criticise the students' solutions, as it might discourage them from sharing their thoughts. In order for the students to feel safe and confident, all contributions to discussions made by the students should be recognized, and every mistake should be treated as a natural element of the learning process and an opportunity to grow. Krygowska (1988) stated that if misunderstanding comes to light early enough as the student's error, this is a "blessed error", which can and should be used to make further progress. The teacher should aspire for all ideas and solutions, both correct and wrong, to be analysed and critiqued by the students. This is also a way of establishing the authority of mathematical logics – it should not be the teacher, but the logics of mathematics which decide the truth in a mathematics classroom. It is also worth encouraging students to formulate sentences properly and to use mathematical language correctly, but the main focus should be on making sense of mathematics, not on the correct answer or precision.

Phase 4: extending the initial task

Following class discussion is the last phase, in which the problem is extended or modified. The fourth phase can take on many forms – the students could work in small groups or the entire class could work on the task, or the two forms can be mixed. The aim of this phase is to engage the students into deeper mathematical thinking based on a problem which has already been solved (Wittmann, 1971). The duration and the complexity of this phase can vary from one additional question to a collection of problems related to the initial task.

The entire process of working on the task should take at least 30 minutes. This is necessary because of two reasons: firstly, the initial phase of solving the task should allow all students to have enough time to handle the problem on their own, and the teacher should not accelerate this process; secondly, the more discussion there is regarding the task, the longer it takes, the more modifications there are, and the more facts that are discovered in the last phase, the more opportunities for the students to use reason.

To illustrate the aforementioned four phases, below are elements crucial for every phase based on the three of the tasks presented before.

	Make 37	Pyramids	The Rose Hotel
Phase I: simple but non-standard student's activity	Choosing 10 numbers (more or less thought-out), calculating the sum, and attempts at modifying the set of chosen numbers.	Deducing the shape of the fifth figure and drawing it; calculating (usually counting one by one) how many blocks are in it.	Marking two and three-person rooms on a chart and counting them.
Phase II: struggling with the problem	Why does 37 not appear among the results?	How can you calculate the number of cells of the 100$^{\text{th}}$ pyramid without drawing it? How can you calculate the number of cells if the specific number of the figure is not known (case n)?	How can you find all possible arrangements?
Phase III: possible discussion questions	What was the first thing that made you think the solution does not exist? How can you be sure that there is no solution if you have not checked all possible combinations?	How did you see the growth of the pyramids? How did you find the solution? Can you think of a geometrical representation of the formula for the number of blocks in case n?	How can we be sure that we have found all the solutions? Do you see any pattern in the list of possible solutions? Can you explain that pattern?

Table 1. Examples of the four phases of solving Students-Friendly Tasks

(Continued on next page)

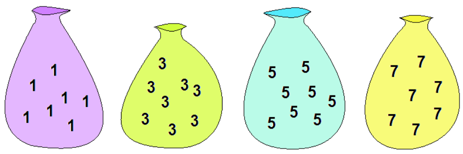
Phase IV: possible extensions of the problem	Find all sums that can be obtained from adding up 10 numbers from the bags. How many ways are there to obtain each possible sum?	How many squares have to be added to the n^{th} figure in order to create the next one? Can a pyramid be created from any number of squares?	What would change in the solution if the hotel had four-person rooms instead of two-person ones? What would change if the hotel had two, three, and four-person rooms?

Table 1. Examples of the four phases of solving Students-Friendly Tasks
(Continued from previous page)

3 Student-Friendly Tasks: empirical study

The aim of this chapter is the presentation and analysis of a task which has been determined to fit the criteria of Student-Friendliness. This task, named Problem 1, is a simplified version of the task "Make 37", available on NRICH project website, designed by University of Cambridge.

> **Problem 1.** Four bags contain a large number of 1s, 3s, 5s and 7s. Pick any six numbers from the bags so that their total is 23.



Presented below is the record of a lesson making use of Problem 1 and its analysis followed by a report on individual observations of the subjects solving this task.

3.1 Example of a lesson making use of Problem 1

The following is an analysis of the record of the lesson during which Problem 1 was solved. It is important to notice that the aim is not to present the perfect lesson to which all other lessons are to be compared, as this lesson can definitely be improved in numerous ways, but rather to give the reader a specific example of making use of a Student-Friendly Task.

The lesson described below was a part of a study conducted by the author in 2014-2015. The aim of this study was to design and test an environment in which low achieving middle school students engage in mathematical thinking and can change their negative attitude towards mathematics.

During the study a group of 24 second and third year middle school students from the same school in Cracow, identified by a school counselor as having poor grades in mathematics, were invited to join an additional mathematics class meeting once a week, in the afternoon, for 10 following weeks. Sixteen out of 24 students volunteered to participate in the study. During the first meeting the students were asked to complete the questionnaire regarding their beliefs about the nature of mathematics and about themselves as mathematical thinkers. Most of them stated that they don't like or even hate mathematics. Answering a question about their mathematical abilities, 5 out of 15 students placed themselves in 10% of the weakest students, 4 other students claimed that they are below average, 6 students regarded themselves as average students, and only one student stated that her mathematical ability is above average.

All 10 classes had similar structure. A meeting usually started with a review of the previous lesson (5 minutes), followed by a math talk (10 minutes) and the main part – solving a Student-Friendly Task (30 minutes). The lesson described in this chapter was carried out by the author on December 15, 2014, with a group of 11 students.

Lesson description

The description of the lesson presented below omits the first several minutes of the lesson as it concerned topics different than the task at hand. The description features commentary made from the point of view of the teacher and it contains an explanation concerning the aims of the teaching techniques used and an interpretation of the students' behaviour. The lesson presented here consists of six specific parts which also appeared when working with other groups of students with the same task.

Presentation of the task and students' spontaneous comments on the properties of the numbers (0-3 min.):

Lesson transcript

The teacher presents Problem 1 on the board. One of the students, indicating the picture, remarks:

S1: *These are all prime numbers!*

The teachers asks whether the rest of the class also thinks that the numbers in the bags are prime and most of the students agree. Then the teacher asks for the definition of a prime number. The students claim that a prime number can be defined as only having two divisors: 1 and the number itself.

After being reminded of the correct definition of a prime number, the students conclude that 1 is not a prime number. After this conversation, another student states:

S2: *But the numbers are definitely odd.*

The teacher does not comment this statement and asks the students to start working on the task.

Teacher's commentary

I did not intend to discuss the properties of the numbers in the bags immediately after presenting the task. I was afraid that focusing at the very beginning on the fact that all the numbers are odd would help the students solve the problem quickly, which, in turn, would not induce the "positive struggle". This is why I refrained from commenting on the remark made by student S2.

I used the previous remark regarding prime numbers to discuss the common misconception that one is a prime number. What the students did not remember regarding definition of the prime number was that the numbers have to be larger than one, or that the two divisors have to be different numbers.

Calculations (4-9 min.):

Lesson transcript

The students start solving the task. Five groups of two are formed and one student chooses to work alone. After a minute one of the students exclaims:

S3: *We've got it! Three 7s and two 1s.*

S4: *No, it's five numbers...*

A few minutes pass with the students intensely searching for an answer. Then some questions arise:

S5: *Can we subtract numbers?*

S2: *Can the sum be 24?*

S6: *Can I multiply the numbers?*

The teacher reminds the task to clarify the doubts of the students, and they continue searching for the solution. In a short period of time, almost all groups find a way of choosing five or seven numbers which add up to 23, and a way of choosing six numbers which add up to 22 or 24. Suddenly, one of the students states:

S3: *We've got* 24 *with six numbers!*

T: *You do? Can you present your solution?*

Then a student from a different group protests:

S5: *No, don't! We're still thinking.*

The teacher approaches students S3 and S4 to verify their solution, which turns out to be incorrect. The students keep searching for the correct solution.

Teacher's commentary

For this part of the lesson, I tried to be almost invisible to the students. I did not give any hints and restricted my remarks to a minimum to elongate the time of group (or solo) work as much as possible, in order to facilitate the students' "positive struggle" with the problem.

When the student suggested that her group found an answer consisting of six numbers, I knew it was incorrect, which is why I asked them to present it. If I had suspected the answer to be correct, I would have made sure that they did not reveal it – it was too early and I did not want the rest of the class to stop working. The students' protest to get to know the right answer shows that their work was not only a valuable, but also an enjoyable experience.

Students expressing doubts that the conditions of the task can be fulfilled (10-12 min.):

Lesson transcript

Uncertain students' voices can be heard: "I don't think this can be solved", "This seems impossible". After some time, the teacher states:

T: *I noticed some of you suspecting this task cannot be solved. Which of you think this is the case?*

Several hands are raised.

T: *Think about why that would be the case. What is the matter with these numbers that no six numbers can be added up to* 23*?*

Teacher's commentary

At first, I intentionally did not comment on the statements regarding the task being impossible to solve, but when some of the students became bored with constant failure in finding the right numbers, I decided to draw their attention to the doubts they expressed.

If I were to change anything in this part of the lesson, I would limit my last remark to just "Think about why that would be the case", leaving the students enough room for them to ask "what is the matter with these numbers?" themselves.

Justifying that fulfilling the conditions of the task is impossible (13-17 min.):

Lesson transcript

After some time passes, one of the students tells the teacher:

S7: *This can't be done, because odd numbers cannot result in an even number.*

The teachers asks S7 to clarify her idea to the three students listening to her and to ask them what they think about that. While the student S7 is trying to explain her idea, the other groups keep working. After a while, a student from another group exclaims loudly:

S2: *I know why it's impossible! Six is an even number, and* 23 *isn't! I get it!*

Then she starts explaining to the students in her vicinity:

S2: *If we add two odd numbers, the sum will be even...*

After that the teacher says to the whole class:

T: *Listen, some of you already figured out why a sum of* 23 *cannot be made using six numbers. It is indeed impossible. Now our goal is for everyone here to understand why this cannot be done. Who would like to explain that? Who's going to try?*

Several students raise their hands. The teacher asks student S7, who was the first to discover that the task cannot be solved, to pick a student who would explain why this is true. She chooses S1, who steps in front of the class and explains:

S1: *The point is to create an odd number, but the bags only have odd numbers in them. But adding for example two* 1*s results in an even number. Two* 3*s results in the same thing, same with two* 5*s and two* 7*s. Odd numbers can't sum to an odd number.*

T: *Who understands this explanation?*

Everyone confirms. The teacher asks another student to rephrase the explanation.

Teacher's commentary

When some of the students reported finding an explanation, I did not comment, but asked them to discuss it in small groups (at this time some of the pairs merged into groups of four). When two of the groups found the solution, the remaining students stopped concentrating on their own work, and started to listen to those explaining their solutions, which is why I decided to discuss the solution with the whole class then.

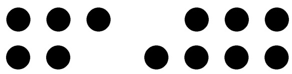
The explanation provided by the students had its faults, but everyone could understand it. Keeping in mind that the group I was working with consisted solely of students with poor grades and a dislike for mathematics, I refrained from formulating the sentence in a more formal way (e.g. "the sum of any two odd numbers is an even number"), as it may have been too complicated for them. The fact that the students understood the concept was more important to me.

Graphic interpretation of the solution (18-19 min.):

Lesson transcript

T: *Let's try creating a graphic interpretation of what you said. S1 said that two odd numbers cannot result in an odd number. Why does this happen? Think about how this could be drawn.*

After a short discussion, the students and the teacher conclude with a following drawing.



Teacher's commentary

The task was impossible to solve, because the sum of six odd numbers cannot be odd. Noticing the fact that the sum of two odd numbers is always even was key to solving the problem. To make a connection between algebraic and geometric representations, I asked the students to create a graphic interpretation of this fact.

Formulating questions related to the task and attempting to find answers to some of them (20-30 min.):

Lesson transcript (extracts)

The teacher facilitates a discussion regarding Problem 1 which lasts several minutes. Questions related to the task are asked.

(...)

T: *Is it possible for the sum of five numbers to be* 23*?*

All students confirm being able to add five numbers up to 23 for several times while before. When asked why this was possible, one of the students replies:

S8: *Because there are only odd numbers there, the result is supposed to be odd, and the amount of numbers we're using is also odd.*

Other remarks appear:

S9: *We got it from seven numbers! It's possible!*

S2: *Of course it is, because seven is odd too.*

(...)

T: *You said before that by choosing six numbers from the bags, you can sum them up to* 24*. Is this really possible? What about 22?*

The students confirm and give examples of six numbers which add up to 24 and 22. It is revealed that there are several unique sets of numbers having this property.

T: *What about different sums? Can six of these numbers be summed up to* 4*,* 6*,* 11*?*

The students answer the questions and try to justify their answers.

(...)

T: *My next question is: What else can we get by summing up six numbers from the bags? What other sum can we have? Try writing down all the numbers that you can get by doing this. Identify the lowest and highest number you can get.*

The task does not seem obvious to the students. After several clarifying questions and teacher's explanations, the students work in groups for several minutes. At first, the six numbers are picked at random and added up to even sums, but as the time passes, some of the students begin to work more methodically, starting with the lowest possible number, 6, and trying to obtain increasingly bigger even numbers. Finally one of the students exclaims:

S4: *You can get every even number from 6 to 42!*

The teacher asks the students to think about what S4 said at home, announcing that they will be continuing this discussion during the next lesson.

Teacher's commentary

Further discussion aimed at encouraging the students to reason with the newly discovered fact regarding the addition of odd numbers. Some of the remarks indicated a deep understanding of the issue. I was especially pleased by how the remarks of the students showed their ability to draw conclusions (e.g. "It's impossible to add it up to 11, because it's an odd number").

Analysis of students' behaviour

It is worth noting that the lesson described above featured the act of understanding. The fact that despite many empirical attempts at solving the task, none were successful (and, importantly, for nobody in the class), causing the students to suspect: "What if the requirements cannot be fulfilled?" This was the turning point in their work. From this moment on, the students tried to find the answer to a question which was different than the initial one. They wanted to know why this happens and how to explain the fact that it is impossible to find a solution. The students concentrated on the details of the task once again. They analysed them in a different way: not as numerical values – the numbers 1, 3, 5 and 7, but by what common property they have. This was when they noticed, for the second time, that all of the numbers are odd.

They also perceived the number of elements, six, as an even number and the expected sum, 23, as an odd number. Then they realised (some of them faster than others) the relations between the properties of the elements of the sum and their quantity, as well as the properties of the number 23 as the sum of the elements in question: The sum of an even number of odd elements is an even number. This is why the requirements of the task cannot be fulfilled. This minor act of understanding exhibited by the students can be defined as relational understanding, as described by Skemp (1976). The task from the described lesson proved to be interesting for the students. This manifested in the fact that they did not stop searching for answers despite their initial failures. On the contrary, continuous failure led to greater interest in the task, which was an important factor in their subsequent success.

Another evidence of students' engagement and enjoyment were their own comments posted in a survey conducted at the end of the study. Here are several symptomatic examples of the answers given by students asked to comment on the class:

- *Thinking on these classes was easier than usually.*
- *We learned to think deeply and look at the problems from different perspectives.*
- *I learned that I am able to think.*
- *I was surprised that it is so enjoyable when everybody thinks about the problem and then we solve it together.*

It is important to note that providing entertainment to the students was not the main aim of the lesson. Providing students with knowledge was also not the main aim. Telling the students that the sum of an even number of odd elements cannot be odd would approximately take a minute. What is therefore the benefit of devoting almost an entire lesson to work on a basic mathematical fact? It is important that the aim of Student-Friendly Tasks is not the transfer of knowledge or equipping the student with a specific set of mathematical skills. The aim is the development of deep mathematical understanding by engaging in "doing mathematics" through a process of inquiry and sense-making (Schoenfeld, 1992). Such a process was apparent in the lesson described above, with the students' different mathematical activities being its symptoms. The mathematical content in this case was not (and should not be) too advanced in order not to hinder mathematical reasoning.

3.2 Individual observations

In order to get to know the process of solving Problem 1 by an individual student, individual observations of 12 other people solving Problem 1 were carried out by two observers. The subjects were: mathematics students with a specialisation in teaching (5 people), middle school students (three second-grade students and one third-grade student), and three mathematicians. All of the participants, apart from one of the mathematics university students, expressed an interest in the task. However, the ways of solving the problem differed and the differences regarded two issues. One of them was the method of choosing the elements of the created sums. The second difference regarded whether the subjects questioned the possibility of fulfilling the requirements of the task on their own or after being prompted by the observer. As an example, two solutions are presented below: Jakub's (third grade) and Adam's (second grade).

Jakub's and Adam's solutions

Jakub's work

(J1) After three initial attempts (as seen in the upper part of the paper) the student asks the observer: *Does this task have a solution?* The observer responds: *Try finding that out on your own.*

(J2) After another attempt (as seen to the right of the vertical line) Jakub says: *If the number two was there, it would be possible.*

(J3) Jakub tries five more times and states: *The number I get is either too large or too small.* After a moment of pause, he adds: *I don't even know any more.*

(J4) The observer proposes: *Look at your calculations and notice the results you obtained when you added six elements together.* Jakub looks at his last two attempts and says: Here, I managed to get either 22 or 24. After a moment of looking at his calculations, he exclaims proudly: *Oh, if we add six of these numbers, the result has to be even, because these numbers are odd! So there's no way of adding 6 odd elements together to get an odd sum. It would be possible if the number of elements was odd.*

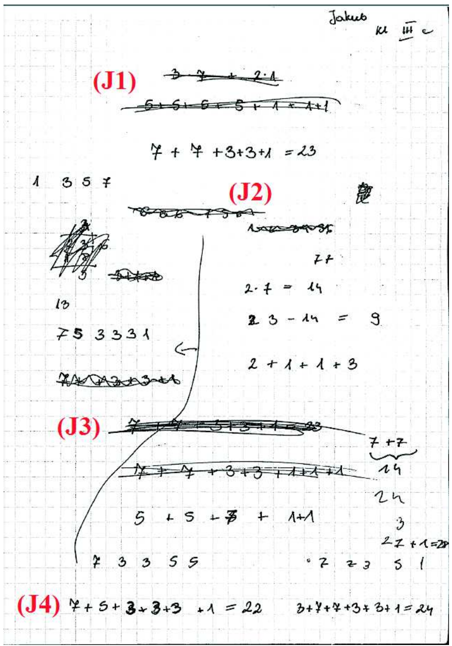


Figure 1. Jakub's notes

Adam's work

(A1) At first, Adam calculates the sums of six identical numbers.

(A2) Then, he calculates all possible sums of two different numbers. The student contemplates his notes (A2) for a while, after which he exclaims: *This can't be done, because the sum of six elements will always be even. The sum of two elements will always be even, because we're adding two odd numbers together. Six elements means a sum of three even numbers, so the result is even.*

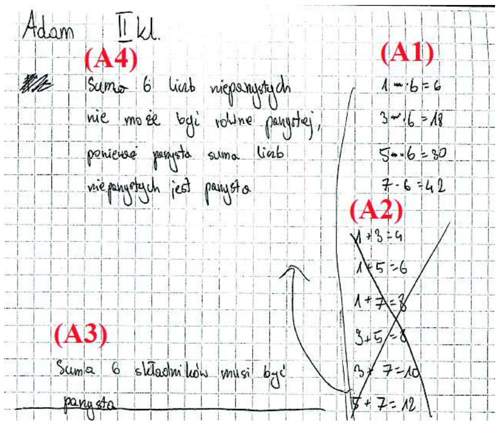


Figure 2. Adam's notes

(A3) After this remark, he writes "The sum of any 6 odd numbers cannot be an even[1]number, because the sum of odd numbers is always even".

Comparison of Jakub's and Adam's strategies

The final results are very similar in both cases, but the task solving process differed significantly between the students. Jakub, like the majority of subjects, starts with choosing six numbers, trying to obtain the number 23. He tries three times (J1), choosing different numbers every time, which results in sums different than expected. This causes him to doubt whether the task can be solved, but since the observer does not clear up this issue, Jakub disregards his doubts and continues trying to choose the numbers (J2). He notices that if the bag contained the number 2, the task could be solved, but he does not reflect on the properties of the number 2 and how it differs from 1s, 3s, 5s, and 7s. He does two more approaches (J3), obtaining the numbers 22 and 24. Then he gives up. When the student stops working on the task, the observer decides to give him a hint: it is suggested that he investigates the sums he obtained when adding six numbers together. In light of this, Jakub considers the numbers and, after a while, discovers the correct answer (J4).

Adam uses a completely different approach to solve the problem. What is unusual in his solution is that in his first attempt, he does not choose six

[1]This sentence is erroneous. The word should be "odd", not "even" in the place indicated by an asterisk. Adam did not correct his mistake, as he did not later read what he wrote down. But what he said before shows that he can correctly explain the fact that the requirements of the task cannot be fulfilled.

random numbers from the bags, acting methodically from the very beginning. He starts with calculating sums of 6 equal addends (A1). When he realizes that the approach of adding the same numbers does not work, he decides to calculate the sums of every pair of numbers from the given set (A2). These sums are all even numbers. The fact that adding up six numbers from the bags is equivalent to adding up three of the "pairs" quickly leads him to the conclusion that it is impossible to get a sum of 23. He writes down his answer with a partial explanation (A4). In writing that "a sum of any 6 odd numbers cannot be an even", Adam makes a mistake, but it is obviously one of carelessness, as his verbal explanation is correct.

It is interesting that, although both Jakub and Adam had enough knowledge and conceptual understanding to solve the problem, only Adam did it without any help. This can be seen as a result of the metacognitive skills (Schoenfeld, 1985) he used while solving the problem. He kept track of his actions and made decisions based on the observations made in his previous attempts. When a particular approach (A1) did not work for him, he shifted to a different method (A2). On the contrary, Jakub mostly made guesses. They were not completely blind guesses, but also not very well-thought-out ones. Although this method did not work for him, he repeated it many times, and finally gave up. It is possible that this student's mathematical identity was not very strong and he did not regard himself as a powerful mathematical thinker. If he had more mathematical confidence, he probably would have followed his intuition and observations: doubts whether the solution exists, noticing that the number two in a bag would help. Those were very good clues which could have led him to the solution, but he gave up on them very quickly. However, after a minor intervention of the observer, Jakub was able to infer the solution from his previous work on his own.

During the observations, eight people out of twelve solved Problem 1 without any help, like Adam, while the remaining four needed some guidance, like Jakub. One should be aware that in a group of students solving such a task in a classroom, there will be Adams, who have already developed some problem-solving skills, as well as Jakubs, who need support to cope with the task successfully. The teacher's difficult role is to identify the students' needs and help them if they are stuck. But the amount of support should be as small as possible, because thinking and struggling with the task are necessary for the students to actually learn from this activity (Hiebert & Grouws, 2007).

Extension of Problem 1: an example of Adam's work

One possible extension of Problem 1 is listing all the possible sums which can be obtained by adding six numbers from the bags. To give an example of a student's reasoning conducted when solving this task, an example of Adam's performance is presented below.

Adam's work on the extended problem

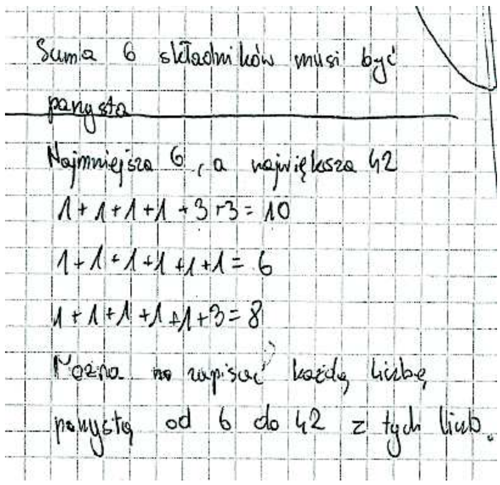


Figure 3. Adam's work on the extended problem

(1) The observer asks: *What sums can be obtained from adding up six numbers from the bags?* After a moment Adam writes "The sum of six elements must be even. The smallest is 6 and the largest is 42".

(2) The observer asks: *Is it possible for the sum to be* 10*?* Adam confirms and writes down the sum $1 + 1 + 1 + 1 + 3 + 3 = 10$.

(3) The observer asks: *And what about other even numbers?* After a moment Adam writes two next lines of calculations and says: *It is possible to obtain all these even numbers. Six is a sum of six* 1*s. When we exchange one* 1 with 3, we'll get 8. The sum of 10 I've already wrote – it was created by exchanging another 1 with 3. You can continue exchanging 1s with 3s, getting 6, 8, 10, 12, 14, 16 and 18. The last number, 18, is a sum of six 3s. In the next step we can exchange 3s with 5s, getting 20, 22, 24, 26, 28, 30, and then 5s with 7s, obtaining 32, 34, 36, 38, 40, 42. So you can get any even number between 6 and 42.

The reasoning presented above can be regarded as a mathematical proof of a particular property of the addition of natural numbers. The property itself is not of great importance, but the fact that the student proved it on his own is crucial. In his reasoning, Adam made use of the relationship of the numbers from the bags he had noticed, which can be regarded as an great example of relational reasoning.

4 Conclusions

The proposal of using Student-Friendly Tasks in the classroom, as described in the article, seems to be a potential way of teaching relational mathematics at the elementary or middle school level. Therefore, it addresses the need for didactic tools which teachers can use to promote mathematical reasoning and understanding among their students. Using Student-Friendly Tasks provides students with a chance to think, to struggle, and to make their own mathematical discoveries. The analysis of the "bags with numbers" task solving processes indicates that, during their work on the problem, the students indeed engaged with creative mathematical activities and deepened their understanding of mathematics. The author asked three other teachers (one of elementary and two of middle school level) to use one of the Student-Friendly Tasks in their classrooms, in the way described in this article, and they all also reported positive outcomes. All three of them claimed that most of their students and the teachers themselves had enjoyed the lesson, and two of them added that they were really surprised by the level of student engagement and persistence in solving the problem.

There are two possible directions for further research on the Student-Friendly Task topic. One of them is to identify or modify existing tasks, or design new ones which meet the principles of Student-Friendliness outlined in this paper. The second is to run a test study with a larger group of teachers in order to test the potential of using Student-Friendly Tasks to improve classroom practice.

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Wykorzystanie „zadań przyjaznych uczniowi" do rozwijania rozumienia na lekcjach matematyki w gimnazjum

S t r e s z c z e n i e

Laureat medalu Fieldsa w dziedzinie topologii, Wiliam Thurston, stwierdził w jednym ze swoich artykułów (1994), że tym, czego matematycy naprawdę potrzebują w swojej pracy nie jest kolekcja odpowiedzi na stawiane pytanie, ale *rozumienie*. We wspomnianym artykule Thurston wyraził opinię, że miarą sukcesu zawodowego matematyka jest to, czy jego praca zachęca innych do pogłębienia rozumienia i myślenia o matematyce w jasny i efektywny sposób. Ten wybitny matematyk interesował się także zagadnieniem kształcenia matematycznego. Uznawał, że *rozumienie* powinno zajmować centralną pozycję wśród celów nauczania matematyki.

Niniejszy artykuł, uwzgledniający przytoczoną powyżej opinię Thurstona, dotyczy zagadnienia „nauczania matematyki dla rozumienia". Jako podstawę teoretyczną dla pojęcia rozumienia autorka przyjmuje znany już wcześniej konstruktywistyczny model rozumienia w matematyce opracowany przez Skempa (1976). W modelu tym wyróżniono dwa rodzaje rozumienia: instrumentalne oraz relacyjne. Pierwsze z nich scharakteryzowane zostało jako umiejętność zastosowania odpowiedniej reguły do rozwiązania danego zadania, ale bez znajomości powodu, dla którego ta reguła działa. W przeciwieństwie do rozumienia

instrumentalnego, rozumienie relacyjne polega na tym, by wiedzieć zarówno co robić, jak i dlaczego.

W światowej literaturze dydaktycznej z kilku ostatnich dekad coraz wyraźniej wskazuje się na konieczność zmiany stylu nauczaniu matematyki z algorytmicznego (instrumentalnego) na taki, który promowałby rozumienie relacyjne i angażował wszystkich uczniów w twórcze aktywności matematyczne (np. Krygowska,1984; Turnau,1990; Schoenfeld,1992; Boaler, 2002). Potrzebę takich zmian uwzględniają programy nauczania matematyki w różnych krajach na świecie. W Polsce uwidacznia tę tendencję dokument oświatowy, jakim jest Podstawa programowa (MEN, 2008). Jednakże badania przeprowadzone przez IBE (Karpiński, Grudniewska, Zambrowska, 2013) wskazują na to, że w powszechnym nauczaniu matematyki w Polsce dominuje transmisyjny model przekazu wiedzy, kładący nacisk na umiejętności algorytmiczne, natomiast niewiele czasu poświęca się na realizację celów ogólnych kształcenia matematyki, w tym na prowadzenie rozumowań oraz uzasadnianie. Jedną z przyczyn takiego stanu wydaje się być brak dostatecznego wsparcia nauczycieli do prowadzania lekcji w sposób inny niż tradycyjny, pozwalający na matematyczne uaktywnianie uczniów.

Niniejszy artykuł zawiera propozycję pewnego środka dydaktycznego, którego stosowanie przez nauczycieli na poziomie gimnazjum potencjalnie sprzyja rozwijaniu relacyjnego rozumienia matematyki. Przedmiotem tej pracy są zadania nazwane „zadaniami przyjaznymi uczniowi". Ich cechą zasadniczą jest prosta treść, wyzwalająca zainteresowanie, którego jednym z przejawów jest naturalne pytanie „dlaczego?", stawiane przez samych uczniów. W artykule zawarta została dokładniejsza charakterystyka tej klasy zadań, a także ich przykłady. Przedstawiona została również relacja z lekcji poświęconej rozwiązywaniu wybranego zadania wspomnianego typu, a także analiza procesu rozwiązywania tego samego zadania przez innych uczniów obserwowanych indywidualnie.