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Emma Castelnuovo who passed away on 13th of April 2014 was one of the most influential persons in the domain of mathematics education all over the world. She was a close friend of late Professor Anna Zofia Krygowska, the founder of this journal. Below, it is Emma's biography written by her pupil, collaborator, and follower Daniela Valenti-Gori.

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Emma Castelnuovo: three key ideas

Abstract: This paper focuses on three key ideas that underpinned the pedagogical research and teaching activity of Emma Castelnuovo over a period going from 1944 to 2014:

1. teacher-researcher;
2. dynamic geometry;
3. real-world mathematics education.

The paper will seek to find answers to various questions about previous ideas, among which: Where do these ideas come from? How, when and where are they implemented in secondary school teaching? Can we envisage their future development?

Key words: Emma Castelnuovo; teacher-researcher; dynamic geometry; real-world mathematics education.

Figure 1. CIEAEM 28, Louvain-la-Neuve, 1976: Stefan Turnau, Anna Sofia Krygowska, Emma Castelnuovo, Claude Gaulin, Willy Servais, Guy Brousseau.

1 The teacher-researcher

'L'insegnamento come passione' (Teaching as a passion) is the title of the book published (Zan, 2013) by the UMI (Unione Matematica Italiana) in honour of Emma Castelnuovo's 100th birthday: her passion for all aspects of mathematics education immediately struck anyone who met her. As well as all those who were privileged to work with her. A passion that was contagious, well described by Anna Zofia Krygowska (Nowecki, 1992):

I am, in general, a pessimist. When comparing the two pans of the balance of human life, the black pan – full of our suffering and pain, all the disappointments, all our dream which did not come true – and white pan – filled with moments of good fortune, minutes of satisfaction and contentedness – it seems to me that the black pan is much heavier and overbalances the white one in spite of it all. Nevertheless all this could be surmounted if one has a passion for something, for doing something with all one's heart and soul. Then one may survive even those black moments of life.

Emma Castelnuovo taught mathematics to pupils aged 11 to 14 in a state school from 1944 to 1979, the year she retired, but she carried on conducting educational research and experiences both personally and through her collaborators up to a few months before she died, in April 2014.

Right from the outset she supplemented teaching in the classroom with reflections on the difficulties or mistakes of her pupils and with research on methods and the organisation of subjects in order to improve pupil learning. In these researches she was backed up by an extensive network of mathematicians and pedagogues, who would meet up on a regular basis through national and international organisations that Emma helped to found or develop, first and foremost the Istituto Romano di Cultura Matematica (I.R.C.M), founded in the summer of 1944. Emma herself described the creation of the I.R.C.M. (Castelnuovo, 2007, pp. 19-25):

We [...] *were literally going out of our heads, we wanted to do something, we knew that the teaching of mathematics was selective, but should not be so. So what was to be done? The first thing was to do research.*

So one day, [...], *it was after the summer of 1944, and out of nothing we founded an association, which we called 'Istituto Romano di Cultura Matematica'. Did we ask for an authorisation? A licence? We didn't even think about it! And who would we have asked?*

We just did it. There were three of us involved: Tullio Viola, assistant professor of mathematics at the Istituto Matematico of Rome, Liliana Gigli, who was even younger than me, and that year was teaching at a school that we should never forget, the school for partisans, for former partisans, and myself. So what did we do? We began to scrape together some money, from our own pockets, in order to publish a list of conferences, so as to be able to meet up. But who were we aiming at? The answer was: Everyone: mathematicians, physicists, pedagogues, everyone. We were brave. [...] *We invited colleagues to come on Saturdays, at half past three, to this or that conference.* [...]

Here I should explain how colleagues came to know about these meetings. Well, I would ride on my bike to all the schools, [...] *announcing our meetings. The first time, there might have been about thirty of us. After two or three meetings the hall was full: a hundred people. If we think about it today, a hundred people moving around Rome, without any means of communication, just to go and hear something, just to be able to say: yes, maybe I will get one or two ideas for my lessons* [...].

It's a wonderful thing, something that should not be thrown away or forgotten.

In 1949 the long-established association <u>*Mathesis*</u> resumed its activities, and the new group merged into this organisation, which also published a journal, called *'Il periodico di Matematiche'*. There we can find one of Emma's first articles (Castelnuovo, 1946). Two years later Emma published her first textbook, *'Geometria intuitiva'* (Castelnuovo, TB, 1948), which would set her on the road to international recognition.

Figure 2.

Invited to a teachers' course in Sèvre, she received a somewhat controversial reception (Menghini, 2013a, pp. 50-59). French colleagues accused her of *'faire les mathématiques avec les mains sales'*, while Belgian colleagues of the École Decroly congratulated her and invited her to visit their École. This marked the start of an important and lasting collaboration with École Decroly and with Belgian mathematician Paul Libois.

In 1951 she met Jean Piaget. This meeting would be one of Emma's most abiding memories, and she told of Piaget's great interest and acute observations regarding the question of teaching 'angles' to 11-year-olds.

Caleb Gattegno then contacted Emma after having read her book on Geometry, and invited her to sit on an international commission that had just been founded: the CIEAEM (*Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques*) (Felix, 1986).

Témoignage d'Emma CASTELNUOVO.

"Je suis membre de la Commission depuis 1951. C'est GATTEGNO, dont je ne connaissais pas l'existence, qui m'a jointe par une lettre : il me disait qu'il avait eu l'occasion de voir mon manuel de géométrie pour le 1er cycle secondaire, et qu'il se rendait compte que nous avions des idées en commun. A la fin de cette lettre il me parlait de la CIEAEM et il me proposait de devenir membre.

C'était donc en 1951, mais ce n'est qu'à partir de 1954 que j'ai eu la possibilité de participer aux Rencontres.

Figure 3.

The number of publications began to grow, with textbooks and articles, a list of which can be found on the web (Menghini, 2013b).

So right from way back in 1944, Emma appears to embody the figure of teacher - researcher into mathematics education, a subject that would still be a hot topic for discussion some thirty years later (Stenhouse, 1975).

And there was an equally rapid growth in the number of mathematics teachers of all levels that drew inspiration from her ideas and methods. We might therefore view Emma also as a trainer of all those teachers, in terms of both initial training and continuous professional development.

But I know that Emma would not be pleased with such a description: she did not like to train teachers directly through a 'theoretical' illustration of her methods. She preferred to describe the work done in the classroom, or to have it observed directly, or even better to get the young teacher to take part in her experiences. Emma felt it was important for teachers to enjoy the same freedom to acquire experiences that she wanted pupils to have. Her books, articles and workshops had the purpose described below, in her words (Castelnuovo, 1958b, p. 41, original in French):

Dear reader, you will find in this article no advice, no rule for teaching better or for aiding the understanding of pupils, no definite path for a first course in geometry in secondary school. You will find only something you already know: the difficulties encountered when introducing concepts and operations, the most common mistakes made by pupils. These data – and we may now call them data – will be used as a basis to critically appraise your own methods, to analyse your faults, to obtain a serene and objective picture of your teaching.

During the second half of the 20th century the figure of teacher and his training changed significantly (Arzarello, 2013, 1998, 2004 and Hall, 2009). In some countries (such as France) the academic recognition of research into teaching methods as a scientific discipline would lead to a separation between forms of theoretical research, with the definition of subjects, timing and methods of teaching, and teaching practice, for which teachers are trained, with less room for personal intervention. In other countries the figure of teacher-researcher was favoured, in keeping with Stenhouse's recommendations.

Teacher participation in research is a key factor. Our starting point for this position is twofold: the logistic problem of covering the large number of schools and our working within a tradition (in the Centre) which is concerned for teacher participation in research as a basis for the betterment of the teaching. (Stenhouse, 1975)

In Italy there was effective dialogue between motivated teachers and university mathematicians, who came together by means of *'Nuclei di ricerca didattica'* (educational research groups), created in the 1970s at university mathematics Institutes and funded by the Italian National Research Council (CNR). In those years in Rome activities relating to the training of teacher-researchers flourished, driven by Emma Castelnuovo and her growing number of collaborators and supporters (Menghini 2013a, p. 63-69). Initiatives included periods spent at École Decroly, experimental degree theses in mathematics education, grants for research into the teaching of mathematics, the *Pilot Classes* project, training courses on international projects, such as the UK's *School Mathematics Project*, and much more.

But one of the key points for the training of teachers remains teacher-training experience in the classroom, in classes given by more expert teachers (up until 1979 by Emma herself, then by numerous collaborators and teachers trained by her). This was a fundamental experience that led the 'mature' teacher to explain his choices and bring his teaching methods up to date, and allowed the 'young' teacher to begin teaching practice, possibly of an innovative nature, with a sense of tranquillity and self-assurance. This type of training of primary and secondary teachers (TFA) is still the standard nowadays in Italy.

Emma's vision for the training of teacher-researchers remains valid, a fact that can be seen by the *International Commission on Mathematical Instruction*'s decision to create *'The ICMI Emma Castelnuovo Award for Excellence in the Practice of Mathematics Education'* (ICMI, 2014).

The Emma Castelnuovo award for outstanding achievements in the practice of mathematics education will honour persons, groups, projects, institutions or organizations engaged in the development and implementation of exceptio-

nal and influential work in the practice of mathematics education, including: classroom teaching, curriculum development, instructional design (of materials or pedagogical models), teacher preparation programs and/or field projects with a shown influence on schools, districts, regions or countries. The Emma Castelnuovo award seeks to recognize and to encourage efforts and ideas and their successful implementation in the field, as well as to showcase models and exemplars of inspirational practices to learn from. A possible list of criteria for selecting among candidates would include:

- *their educational rationale*
- *the problems addressed*
- *the conditions under which the work is taking place (cultural/political context, infrastructure, funding, people involved)*
- *the originality/creativity involved to address problems and bypass obstacles the extent of the influence of the work the quantitative and/or qualitative evidence of its influence the potential to serve as a role model (either for inspiring others addressing).*

2 Dynamic geometry

Anyone with little knowledge of Italy's history might well imagine that the homeland of Maria Montessori and Caleb Gattegno would have had in place well before 1944 an active teaching of mathematics, and of geometry in particular, in all its primary and secondary schools. Yet this was not so!

Emma wrote (Castelnuovo, 1955a) about the teaching of geometry that she had found in schools for children aged from 11 to 14 in her first years of teaching.

The course of intuitive geometry thus begins with some general definitions and concepts. We cannot expect a child of 11 to be able to give definitions straight away. Let us not forget that in order to have a perfect organisation, from a critical point of view, of elementary geometry, as we present it in the intuitive course, we had to wait until the second half of the last century, and, even if we want to stop at Euclidean Elements, centuries of practical and intellectual work were needed to get to that point. And before we could get to Euclid there were, among others, a certain Thales and Pythagoras.

The general definition and concept that we ask the pupil to learn will be for the purpose of exercising his memory, and certainly not his intelligence. In this way the class will be able to say the right things (as a mnemonic exercise), but will not be able to construct, to think, to become interested in or experience *the work of centuries, since what he learns in this way is too perfect, too much beyond his creative possibilities. It will be a class in which the teacher can too easily highlight his own superiority. The class will have a passive participation in the geometry course.*

The proposal contained in the school textbook *Geometria intuitiva* (Castelnuovo, TB, 1948) was thus 'revolutionary', and came at a point in time (post-war era) when there was a strong desire for innovation and reconstruction. This also helps to explain the impact that her proposals also had overseas.

It was also important however for this textbook to be tried out in a school with normal pupils aged from 11 to 14, and for Emma to be able to passionately describe her course and the reactions of pupils, as we can read in her words (Castelnuovo, 1946):

I began the course by recalling the origins of geometry, and getting pupils to understand that it is necessary to calculate areas, restricting the study at the beginning to polygonal figures. The question of areas can be "felt" by children from an early age.

From the area of a rectangle, calculated in the usual way, the youngster easily moves on to calculate the areas of other polygonal figures, and in the end any polygon, by dividing the shape into a certain number of triangles. A thousand examples and applications appear to the pupil living in the city, even more come to the young country dweller, as regards the practical usefulness of these measuring rules.

But in addition to determining the area of a room, a small field, and so on, I believe the child would benefit from working on something smaller, more tangible. Cardboard models, to assemble and disassemble flat figures, give pupils the opportunity to analyse and summarise, developing their powers of observation. From this simple playing around with figures, there is a natural revelation of one of the great truths of geometry: Pythagoras' Theorem. [...]

Returning to the calculation of areas, the pupil realises that the problem is often practically impossible to solve, for example if in the field we want to measure there are obstacles, such as lakes, forests, etc.

This naturally leads to the question of equality; for example, marking out, close to a field, a field that is equal to the one we want to measure. The question of equality then leads to the need to introduce the concept of angle.

So going from the complex (polygon) we gradually arrive at the elements (angle, segment) that go to form the complex.

The same article expressly lists some of the principles on which her course is based:

- find connections to what pupils have already done at primary school;
- replace the descriptive method with a constructive, active and continuous method;
- shift from the concrete to the abstract;
- organise the course in accordance with the historical development of the subject.

In the above excerpt Emma speaks of *'cardboard models to assemble and disassemble flat figures'.*

But over a decade of teaching, contacts and reflections, both in Italy and overseas, improved Emma's ideas on the use of materials. Here is what Emma wrote 12 years later (Castelnuovo, 1958b, original in French):

We talk about materials, models, devices. Dear reader, please don't expect to see a treasure chest open up before you! It is merely our hope that the one or two ideas we had when coming into contact with pupils can be useful when working with other pupils, and can be extended to give rise to new experiences, and new ideas (p. 41).

But we would like to stress that the material used has to be movable: it is indeed mobility that draws the child's attention and leads him from the concrete to the abstract; it is not the material itself that attracts him, rather the transformation of the material, an operation that, being independent from the material, is abstract. We believe that the material itself provides the spark for forms of operational education (p. 54).

Through the years many examples were added to the research into movable materials: initially small, simple home-made devices to help the teacher realise that he could easily duplicate the construction. Then, in 1964, the production by the publisher of her school textbooks of a box of materials to be used for the construction and study of articulated polygons.

It was in those years that a wide-reaching 'revolution' was taking place internationally, as summed up by Emma (Castelnuovo, 2009c).

In the late 1950s there began to emerge a sense of dissatisfaction with regard to the teaching of mathematics, which no longer appeared to be in keeping with the growing number of children from all social classes entering school. This was compounded by a crisis originating from a sphere outside the school: Russia's launch of the Sputnik, the first artificial satellite. Why was the Sputnik so important?

Because Russia's first space launch provoked an intense debate within the Government, in the United States in particular, and consequently in the spheres of mathematics and the school.

In those years declarations were being made in the USA such as: "Today we need to train a large number of scientists, mathematicians and engineers who, in the near future, will be able to launch more powerful and state-of-the-art satellites than those now being launched in Russia. It is therefore necessary to focus on the scientific training of youngsters from the first years of secondary school; and the study of mathematics should be given priority in schools". The problem was therefore: what sort of mathematics should be taught in schools?

Before drawing up new programmes, the Americans asked international organisations such as the OECD to organise a congress with the participation of specialists from different countries to discuss the problem. This congress was organised and staged in Royaumont, France, in late 1959.

This congress marked an important turning point in time for mathematics syllabuses.

French mathematician Jean Dieudonné took a firm stance, demanding a complete break with tradition. With his "Down with Euclid" slogan, Dieudonné managed to convince the majority of participants to return home and argue for the need to abandon Euclidean teaching in favour of a form of mathematics more in keeping with modern-day research developments.

The study of geometric figures had to give way to important elements of modern algebra, so as to highlight the unity of mathematics, just as, at a high level, the Bourbaki group, to which Dieudonné belonged, was doing.

But at Royaumont only a rough framework was agreed for was to be this great break with tradition. So a year later, in 1960, a panel of mathematicians and experts in mathematics education met in Dubrovnik. During the course of a long workshop a book of recommendations for the teaching of mathematics in secondary schools, divided into two parts, was drafted. There were no detailed syllabuses, only suggestions and considerations, presented in a broad form, so as to be adapted to the traditions of single countries.

The text was published, yet what happened in subsequent years had not been anticipated.

Many countries sought to stress the unitary nature of mathematics, and believed that the best way was to base the teaching of mathematics largely on set theory and structures, closely following the recommendations of the Bourbaki movement.

The Euclidean axioms were therefore replaced, under the compelling name of 'modern mathematics', with stronger, and thus more harmful axioms: harmful, as it obliged pupils to learn general and abstract theories, often with *no connection to reality, no use of geometric figures and no seeking of a spatial vision.*

This is undoubtedly a unifying and general method, since a set can represent many 'mathematical objects', from geometry to numbers. This explains the fascination it may hold for mathematicians and teachers alike; but would adolescents be equally enthralled?

Back in the early 1970s 'modern mathematics' spread like wildfire, with little opposition. But a new crisis was behind the corner. In August 1976, on the occasion of the International Congress on Mathematical Education (ICME3), English mathematician Michael Atiyah, addressing a plenary conference, accused mathematicians of having removed from most countries the teaching of geometry in all secondary schools. Because it is geometry – he said – that sets in motion intuition, and is the first step towards discovery, and that is the link between the physical world and mathematics.

This stance, taken by a mathematician of a certain stature, such as Atiyah, as firmly as that adopted by Dieudonné in 1959 in siding against Euclid, had a great influence on the whole world of mathematics, and also on society.

Yes, on society too, since the gap had been growing between the mathematics taught at school and the mathematics being proposed more and more frequently by the media of the time. And isn't school obliged to educate citizens to follow television programmes, read newspapers, participate knowingly in questions of economic or environmental policy?

Atiyah's position reflected a view that had become widespread. The time was now right. And it was from that very year that many countries, having embraced 'set theory', made significant changes: conferences, meetings, discussions on the subject became more frequent. The questions raised were:

How to re-introduce geometry in secondary schools?

How to connect mathematics to reality at different school levels?

Syllabuses changed radically, textbooks that in the period of set theory had been revered were thrown away [...]

It thus became important to re-introduce geometry into schools: research into active and dynamic teaching of geometry, which Emma had never forsaken, again became popular. And in 1971 Emma, together with her three longtime collaborators, produced dynamic and innovative teaching material, making use of the overhead projector: a series of transparencies placed on top of an overhead projector, in order to show dynamic geometric figures.

Projects and produced materials were presented in lively meetings, at the University or at Emma's home, and everyone was involved in these passionate discussions, including newly graduated teachers like me. The main points for discussion were: "How can transparencies be used as part of an effective learning path? Can transparencies replace the physical handling of materials?" Emma spoke of Clairaut, Gattegno, Décroly... of her experiences with junior high school pupils in order to back up her ideas: "The material shown by the teacher to the whole class is important for capturing the pupils' attention and stimulating their observation, but individual material is of equal importance, with every pupil being able to handle it, construct figures and discover its properties". A host of experiences, including those of other teachers, confirmed the fact that it was fundamental for pupils to link up concepts to tactile or motor sensations: this was the main way of constructing an enduring knowledge.

Now these materials are again usable (Valenti, 2009): the Sapienza University site contains 21 short videos on the use of transparencies for teaching, and every transparency is supported by two worksheets that propose teaching strategies using the two most commonly used dynamic geometry software (Cabri and Geogebra).

It takes no time at all to watch one of the videos[1] illustrating a transparency and to peruse the corresponding dynamic geometry file[2]: it will be natural to consider the dynamic transparency as a 'precursor' of the file produced using dynamic geometry software.

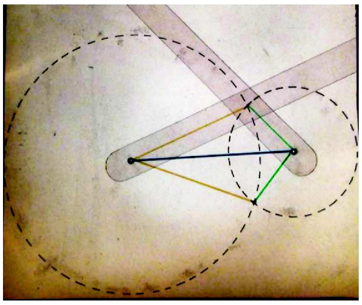

Figure 4.

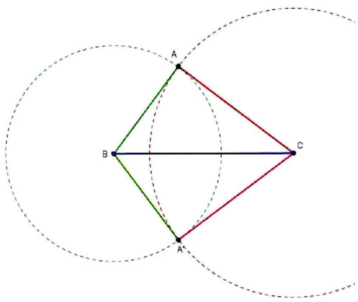

Figure 5.

This causes us to recall that, in France where the teaching of 'Bourbaki mathematics' was pursued with greatest zeal, in 1985 Jean-Marie Laborde, a computer scientist, mathematician and researcher in discrete mathematics, proposed the creation of a "**CA**hier de **BR**ouillon **I**nformatique pour la géométrie", or "Cabri-Geometry software", allowing the exploration of the properties of geometrical objects and their relationships.

The software proved to be a great success, and spread rapidly in schools throughout the world, as it appeared to offer an innovative response to the problem of introducing geometry in schools. Cabri was followed by numerous

[1]http://www1.mat.uniroma1.it/ricerca/gruppi/education/MaterialMat/T2/Tria2_1.mov
[2]http://www.mat.uniroma1.it/ricerca/gruppi/education/MaterialMat/T2/Trigeo2_1.zip

other software applications, in particular Geogebra in the new century, which had the same objective: to work in a "dynamic geometry environment".

Now the term "dynamic geometry" is often associated with a software, while the geometry course proposed by Emma might perhaps be described more accurately as "intuitive geometry course taught using an active method supported by dynamic material".

Emma never became familiar with a computer, towards which she maintained an attitude of 'wary indifference'. When I went to see her at home I often took my laptop with me, so that we could surf the Internet and visit some sites that particularly interested her, such as 'her Association of Spanish teachers' (La Sociedad Madrileña de Profesores de Matemáticas)[3] or the section devoted to the school on the Enciclopedia Italiana site (Treccani/scuola)[4], for which she had painstakingly prepared an article and two video interviews in 2009 (Castelnuovo, 2009a, b, c).

It was the video dedicated to *'Materials that students can use in geometry'* (Castelnuovo 2009b) that showcased her vision of the role of materials in the teaching of geometry, developing in pupils the capability of active observation, based on connecting multiple sensorial experiences, and stimulating creativity. Recent works (e.g. Swoboda, 2012; Gucler, 2013 or Hegedus, 2014) speak about the importance of these multimodal aspects in teaching-learning processes. In the video Emma deals with the same subject, simply and concisely, using the material she shows. Thus her message arrives loud and clear: it is important to feel the non-deformable triangle in one's hands as well as the articulated quadrilateral, to listen to the light rustle of the turning sticks, to see the shape changing; the movement must be slow, controlled by my hands, perceived by my ears and my eyes. Her final words were to warn that audiovisual materials can induce a passive attitude, when showing images that move too quickly to be observed.

This does not diminish of course the role that is played by dynamic geometry software, which offers an important bridge between reality and geometry and crucial support for mathematical literacy activities (Arzarello, 2002, 2003, 2006 and Laborde, 2004), using a technology (computer, smartphone, etc.) that pupils have had contact with since early childhood. There remains considerable doubt as to the possible and complete replacement of actual materials with virtual ones, inviting teachers to reflect also on the efficacy of the Interactive Whiteboard (IWB) in teaching, a source of many questions and interesting researches (e.g. Moss, 2007).

[3]http://www.smpm.es/index.php/informacion?eprivacy=1

[4]http://www.treccani.it/scuola

3 Real-world mathematics education

Up until now we have focused on intuitive geometry and on dynamic materials, leaving on one side the other main subject of Emma's research into teaching methods (Gario, 2013): interactions between the real world and the teaching of mathematics. This is a question that goes back a long way, and one that Emma had already discovered through her father Guido Castelnuovo and uncle Federigo Enriques. Below are two significant passages written by them.

This is the chief wrong of the doctrinaire spirit invading our schools. We teach you to be suspicious of approximation, which is reality, and to worship the idol of a perfection that is illusory.

We present to you the universe as if it were a building, whose lines have a geometric perfection, yet they appear to us to be deformed and blurred, due to the imperfection of our senses. We should however get it across that the uncertain forms revealed to us through our senses are the only accessible reality, which we substitute, in order to satisfy certain needs of our spirit, with an ideal precision. [...] *There is no better way of reaching the goal than by placing side by side at every step theory and experience, science and its applications*(Castelnuovo, G., 1912).

I mean that, if in a sense every professional school is also – to an extent – formative, on the other hand the most eminently formative school must be able to make use of practical applications, in order to arouse the interest and curiosity of those pupils less sensitive to the beauty of abstract theory, and also to accustom pupils to recognise the abstract in particular concrete examples. (Enriques, 1921).

The above excerpts lead one to understand that rather than interactions, there were often juxtapositions between the real world and the teaching of mathematics. These contrasts became particularly evident in the period of the 'Bourbaki revolution', which we have already seen. In those years the CIEAEM became an international meeting place (and sometimes a place for clashes) between different viewpoints on 'modern mathematics'. So, just attending CIEAEM's meeting, Emma found a wide variety of positions on the subject, and received fresh stimulus to study new contents and compare many alternative educational visions to her own, presented by figures of substantial intelligence and culture. In this way there arose many important collaborations and friendships, based on mutual esteem and trust. The story of these relationships could fill a whole book on their own. Here I would like to recall two people, who were particularly important also for my personal and professional growth: Anna Zofia Krygowska and Hans Freudenthal, in the photograph below in Rome, in 1979, during the International Conference organised at the

Accademia dei Lincei on the occasion of the final year of Emma's teaching career.

Figure 6.

Below are two excerpts that show the perspectives of these two personalities in the debate on the introduction of 'modern mathematics' in secondary schools.

Opinions regarding the pupil's introduction to the axiomatic method and its place in school mathematics as a whole are very conflicting. Some see it as the essential element in mathematical education at every level, and therefore consider initiation into the axiomatic method as an essential condition for modernizing mathematical education. But the teaching aspect of this proposition is not clear and its feasibility has not yet been adequately verified. Others consider that the secondary school pupil is not capable of comprehending the axiomatic concept and does not need it. This negative opinion is often justified by reference to observations made during traditional teaching of deductive geometry. But this cannot be conclusive, since modern concepts of axioms and axiomatization are not the same as traditional concepts. That is also why the question of how to teach the axiomatic method must be treated from a modern standpoint; the results of experiments based on this new concept may differ from those of the traditional school. The question is therefore entirely open and requires objective research (Krygowska, 1971).

Systematization is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematizing, not its result. Its result is a system, a beautiful closed system, closed, with no *entrance and no exit. In its highest perfection it can even be handled by a machine. But for what can be performed by machines, we need no humans. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics.* [...]

It is a pity that most of the criticism against modern mathematics is made with no knowledge about what modern mathematics really is. It is a pity, because there is ample reason for such criticism as long as mathematicians care so little about how people can use mathematics .... I am convinced that, if we do not succeed in teaching mathematics so as to be useful, users of mathematics will decide that mathematics is too important a teaching matter to be taught by the mathematics teacher. Of course this would be the end of all mathematical education (Freudenthal, 1968).

In the first passage Krygowska reflects on pedagogical aspects, and is cautious in making judgements before seeing the results of careful experimentation. In the second excerpt Freudenthal is a convinced and passionate supporter of mathematics as a human activity and of its teaching being closely related to real-world applications. These passionate beliefs were behind Freudenthal's many initiatives in those years: in 1968 he founded the journal *Educational Studies on Mathematics.* The article cited here appeared in the first issue of the publication; in 1971 he founded in Utrecht (Netherlands) the IOWO, now called Freudenthal Institute, fundamental in developing and disseminating internationally a particular approach to mathematics education: 'realistic math education' (Van den Heuvel, 2001).

In those years Emma too was very active, teaching in school, conducting teaching research and doing other important things:

- writing the enduring book on mathematics education (Castelnuovo, 1963), translated into Spanish, French, German and Russian; in 1964 it received an important national award from the Accademia dei Lincei;
- participation in Committees which, in 1963 and 1979, reformed Italian schools for children aged 11 to 14 and its syllabuses;
- a textbook dealing with all aspects of mathematics for pupils aged 11 to 14 (Castelnuovo, TB, 1952, 1966, 1970);
- organising in 1971 and 1974 two 'Mathematics exhibitions', with pupils showing a large audience the 'mathematics they built at school', with the aid of posters and many self-built devices. Materials and the results of these exhibitions are described in two books (Castelnuovo, 1972 and 1976). The title of the second book recaps one of the main *leitmotifs* of Emma's teaching: *Matematica nella realtà* (*Mathematics in reality).*

Emma became president of the CIEAEM in 1979. In 1980, at the Mexico Conference, she summed up CIEAEM's position (Castelnuovo, 1980) in the 'heated' debate on interactions between the real world and the teaching of mathematics:

Now, in these last ten years, perhaps because of the influence of the young men's movement of '68, we have been led to serously reconsider the didactical side. And we have understood that it's precisely with the teaching of mathematics that we can act in opposite directions: we can use mathematics as a selective arm by teaching too abstract theories and, this way, driving off most of the students; but, on the contrary, we can use mathematics as a means of collaboration, by making the students aware of the importance and utility of this science, even in a social perpsective. Now – and this is the opinion of several among us – we can realize this direction only by stimulating our students with real problems. This idea imposes a deep study about the meaning of the interaction of mathematics with other disciplines, in a frame that always respects the autonomy of each of them.

It is in this direction that I see the development of our works in the coming years.

Allusions to a work for future years anticipated the decade-long work leading to the publication of a series of mathematics textbooks for students aged from 14 to 19 (Castelnuovo et al., TB, 1984a, b, 1986a, b, 1988, 1992). These texts continue the lessons given in texts written for pupils aged from 11 to 14 (Castelnuovo, TB, 1988, 1998), and retain the same spirit: each topic starts from the connection to the real world, and gradually moves towards abstraction and formalisation, before finally returning to reality, which will be seen 'with new mathematical eyes'. Special care was always taken with the choice of problems raised by scientific, economic or social reality that was as close as possible to the daily life and interests of pupils. This meant that all authors, and Emma above all, were required to conduct in-depth, passionate research and studies, broken down into a number of often overlapping stages:

1. *Study of the problem,* with the involvement of friends and collaborators having more expertise about the question, in order to fully appreciate the mathematics involved, historical roots, all applications in relation to the real world and foreseeable pedagogical difficulties.
2. *The teaching material project* to support pupils' activity and sharpen their physical and mental actions.
3. *Experimentation in the classroom*, often with the involvement of colleagues and trainee teachers.

4. *Reflections on experimentation*, to improve the method, when unexpected difficulties were encountered or wise questions were raised by students; this often made it necessary to return to point 1.
5. *Drafting the text*, taking special care with language, which must be clear but not over-simplified, the organisation of the subjects, which must be easy but not too schematic, and illustrations, which must be able to effectively visualise concepts.
6. *Editing the text*, and consequent discussion with friends and collaborators, which often meant having to go back to points 1 and 5.

I believe that the texts prepared in this way, after extensive testing, have proven to be effective in helping pupils to achieve an important goal: that of reaching levels of abstraction and formalisation consistent with a pre-university mathematical education for all pupils, and, what is more, reaching such a level gradually, aware of the motives and the value of the mathematical path followed, with the added possibility of further study for those intending to undertake university studies in the sciences.

Most of the texts for students aged from 14 to 19 have recently been turned into multimedia lessons that can be accessed by everybody, free of charge, at the Enciclopedia Italiana website (Valenti, 2014). An example is the lesson on the exponential function[5].

Over the past 20 years the teaching of mathematics in relation to the real world has spread worldwide, due in part to *Programme for International Student Assessment* (PISA) since the year 2000. Below are some quotes that give an idea of the spread of such teaching.

It is extremely important to develop an adequate vision of mathematics, not reduced to a set of rules to be memorised and applied, but recognised and appreciated as a framework within which to raise and tackle significant problems and to explore and perceive relations and structures that occur and recur in nature and in human creations (Ministry of Education, Italy, 2012, p. 49).

Mathematics provides tools to be able to act, choose and decide in everyday life [...]. *The main elements of mathematics are acquired and mastered mainly through problem solving, especially taking as a starting point real-life situations* (Direction générale de l'enseignement scolaire, France, 2009, p. 4, original in French).

As the basis for an international assessment of 15-year-old students, it is reasonable to ask: 'What is important for citizens to know and be able to do in situations that involve mathematics?' More specifically, what does competency

[5] http://www.treccani.it/scuola/lezioni/matematica/Funzione_esponenziale

in mathematics mean for a 15-year-old, who may be emerging from school or preparing to pursue more specialised training for a career or university admission? It is important that the construct of mathematical literacy, which is used in this document to denote the capacity of individuals to formulate, employ, and interpret mathematics in a variety of contexts, not be perceived as synonymous with minimal, or low-level, knowledge and skills. Rather, it is intended to describe the capacities of individuals to reason mathematically and use mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena (OECD, 2013, p. 24-25).

If the principal objective of teaching the subject is to make students able to apply their mathematical knowledge and skills to everyday problems, they must acquire the ability of modeling the real problem in the abstract mathematical world, and demodeling the found solution of the abstract problem, or interpreting it back in the reality of its origin (Turnau, 2008).

In the same years there have been numerous examples of *real-world mathematics education* that have uncovered a number of weaknesses. In some cases the teaching of mathematics in secondary schools centring only on *mathematical knowledge and skills in relation to everyday problems* has developed good skills in this field, but has left pupils, even 18-year-olds, at an overly elementary level of abstraction and formalisation, with no access to the *abstract mathematical world*. In other cases teaching has remained focused on formalisation and calculus techniques, adding one or two 'pseudo-real' problems or 'PISA-type' tests, which has left students with a contradictory and misleading vision of mathematics. This state of affairs is damaging for students' possible access to university education in the sciences, and may be one of the reasons behind the *declining interest in science studies among young people* (Global Science Forum, 2008).

I believe that Emma Castelnuovo's texts for students aged from 11 to 19 could be a rich source of ideas for more effective *real-world mathematics education* initiatives.

I also believe that a good way to end here is by quoting the particularly appropriate words aimed by Emma at all teachers at the end of her term of office as CIEAEM president (Castelnuovo, 1981):

Please do not present mathematics as something done and dusted, something that you know about and your pupils are ignorant of. Stimulate their interest in subjects that they can feel, experience. Let the theory emerge from the concrete, from reality, even if you are teaching older pupils. To do this you need to study, read, think, reconstruct.

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Emma Castelnuovo: trzy główne idee

S t r e s z c z e n i e

Niniejszy artykuł koncentruje się na trzech kluczowych ideach, które stały się podstawą badań pedagogicznych i działalności nauczycielskiej Emmy Castelnuovo w okresie od 1944 do 2014: Są to

1. Nauczyciel-badacz;
2. Dynamiczna geometria;
3. Edukacja matematyczna odnosząca się do rzeczywistego świata.

W artykule autorka stara się znaleźć odpowiedzi na różne pytania związane z tymi kierunkami. Są to między innymi pytania: W jaki sposób te idee się zrodziły, skąd pochodzą? Jak, kiedy i gdzie znajdowały one zastosowanie w nauczaniu matematyki na poziomie szkoły średniej? Czy możemy przewidzieć ich dalszy rozwój?