ANNALS OF THE POLISH MATHEMATICAL SOCIETY
5TH SERIES: DIDACTICA MATHEMATICAE 36 (2014)

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The benefit of the metaphor "contract" for understanding of concept formation in mathematics

Abstract: The paper reports on a design and research project conducted in a secondary school on the Indonesian island Sumba. The aim of this project is to foster students' understanding in learning mathematics and, consequently, to educate them better for their career opportunities. The essential idea is thereby to use the introduction of negative numbers and arithmetic operations on integers as a background motivating the learners to a new cognitive behavior, especially to critical thinking, monitoring and reflection. The new approach to teaching and learning uses the metaphor *contract* as a representation of a specific way of thinking when dealing with systems of concepts. Thereby, the *systemic thinking* in the sense of Sierpinska (Sierpinska et al., 2002; Sierpinska, 2005) and *axiomatic thinking* in the sense of Freudenthal (1963; 1968, p. 7) are meant. To establish this metaphor in students' minds, a learning environment (LE) for the introduction of negative numbers was designed. Since the context used in this learning environment relates to banking operations with assets and debts in a fictitious bank, the learners conceive this context as a kind of game and not as a part of their everyday experience. Their activities are performed in a thought experiment and not as hands-on activities with concrete objects. The problem in designing the LE was how to find access to students' intuitive knowledge related to integers and to reconstruct the intuitively understood way of dealing with these numbers and not how to find a simple way to make arithmetic calculations quickly. The

Key words: metaphor, negative numbers, systemic thinking, axiomatization, local and global organization.

emphasis is placed thereby on the process of reconstruction and precise representation of the intuitive knowledge up to a contractual (axiomatic) definition of addition, subtraction and the inverse of a number in the realm of integers. The access is created through the use of the metaphor "contract". In the LE, the learners have to conclude a contract with the bank in order to achieve that the bank performs banking operations in a way which can be accepted by the students, based on their intuitive knowledge on debts and assets.

Based on students' written works, the paper presents some effects and explains the benefits of using the metaphor "contract". The aim of this qualitative analysis of students' solutions is not to provide the reader with a quantitative explanation of the teaching success, but to document and explain the intended understanding, in particular, the axiomatic view and some aspects of the systemic thinking as they are indicated in the effects of the teaching and learning process documented in this paper. It can be stated that the metaphor "contract" enables the learners to experience mathematics as *human activity* (Freudenthal, 1991, p. 14) – in particular as an "activity of discovering and organizing in an interplay of content and form" (ibid., p. 15) – and to use this experience in order to establish new cognitive habitus in learning mathematics.

1 Introduction

Internationally known problems of teaching and learning mathematics (at school or at university) are the failure of learners in practicing autonomous and critical thinking on the one hand, and mathematical reasoning on the other (cf. Sierpinska et al., 2011; Sierpinska, 2007; Sembiring et al., 2008; OECD, 2014). One explanation for this failure is seen in the fact that learners acquire mathematical knowledge in such a fragmentary form, which hinders independent ways of applying it (cf. Vinner, 1990; Cohors-Fresenborg & Kaune, 2005; Schneider, 2012; Zachariades et al., 2013). And this fragmentation might be seen as a consequence of teaching mathematical content without inner conceptual systematization and without reflecting on such systematization (cf. Zachariades et al., 2013; Sierpinska, 2007).

An explanation of how to prevent the fragmentation in constructing mathematical knowledge has been given by Cohors-Fresenborg (Cohors-Fresenborg, 1996; Cohors-Fresenborg et al., 2002) with the metaphor *cognitive mathematical operating system*. This metaphor means to realize that the organisation of access is an important task when constructing a knowledge network in students' minds. The aim is thereby that in the beginning of any maths lesson, this operating system boots in pupils' minds and organises connections within mathematical knowledge. Empirical research has shown that the construction of such an operating system needs additional time in teaching mathematics, but, consequently, these efforts make it possible to understand the mathematical content in a shorter time than usual (Cohors-Fresenborg et al., 2002; Cohors-Fresenborg & Kaune, 2005).

Given this aim, the question arises of how to enable the learners to establish such a cognitive mathematical operating system in their minds. Furthermore, the question on the cognitive tools navigating this operating system must be discussed. The second question was elaborated by Hans Freudenthal (1973) – already many years ago – in his conception of mathematics as "human activity" (Freudenthal, 1991, p. 14). He placed importance and educational value on cognitive tools which enable the learners to experience mathematics as "activity of discovering and organizing in an interplay of content and form" (ibid., p. 15). Some of these tools, amongst others, are mathematizing, formalizing as well as conceptual (local and global) organizing of mathematical knowledge. Local organization means that mathematical concepts are "analyzed up to a certain rather arbitrary frontier, let us say up to the point where the naked eye as it were, distinguishes what notions mean, and whether propositions are true". This activity is also called *axiomatizing* (Freudenthal, 1968, p. 7). Global organization is explained by Freudenthal as organizing a category of systems (and not organizing a system internally) by looking from the outside and recognizing a common structure in different models. In this process, the ontological bond (like for example marbles, chairs or flowers subjected to arithmetical operations) disappears and different systems are seen form an abstract point of view. In the next step, some statements can be deduced from the abstract description of this structure (Freudenthal, 1973, p. 454).

To date, some aspects of Freudenthal's conception of mathematics are of great importance in teaching and learning theories (for example in *Realistic Mathematics Education* (cf. Sembiring et al, 2008)) and in designing lessons according to these theories. Convincing explanations of how his ideas of conceptual ordering can be successfully implemented in school mathematics are, however, still rare. In particular, the educational value of axiomatizing seems to encounter resistance in teaching practice, presumably due to a lack of faith in the abilities of learners.

The design and research work conducted in the Institute for Cognitive Mathematics (IKM) of Osnabrueck University (Cohors-Fresenborg, 1996; Cohors-Fresenborg et al., 2002) is an exception. This work has shown how to enable learners to establish a cognitive mathematical operating system in their minds. Additionally, it has explained how to allow pupils to understand *axiomatization* and to use it as an essential cognitive tool in their operating systems. Furthermore, the research provided an evidence base showing that pupils' understanding of an axiomatic point of view enables the learners to organize their mathematical knowledge in such a way which hinders fragmentation and makes the access to the constructed knowledge and to the experience in using it easier (Cohors-Fresenborg et al., 2002; Cohors-Fresenborg & Kaune, 2005). This specific point of view facilitates *systemic thinking* in and about mathematics. Systemic thinking means that mathematical concepts are seen as parts of conceptual systems (semantic networks), with conceptual and methodological relations within such systems and not as isolated objects. Consequently, the meaning of a concept is established based on its relations with other concepts. In (Sierpinska et al., 2002; Sierpinska et al., 2011), systemic thinking "in the sense of thinking about systems of concepts" (ibid., p. 278) is described as an essential characteristic of "theoretical thinking", which is essential for gaining autonomy as learners.

The problem concerning the failure of learners in practicing autonomous and critical thinking and mathematical reasoning affect teaching and learning mathematics worldwide, in particular in Indonesia (Sembiring et al., 2008, p. 929). The international comparative study PISA (2003, 2012) has shown that Indonesian students are ranked in one of the last places assessing their *mathematical literacy* (OECD, 2013, p. 25). Even despite huge efforts of the Indonesian government, the ranking has not changed effectively in year 2012 compared to 2003 (OECD, 2004, 2014). More than 75%[1] of the Indonesian population investigated in PISA 2012 was ranked *on level 1* of mathematical literacy *or below it*, and did not even reach level 2, which requires amongst others the ability to employ basic algorithms, formulae, procedures, or conventions (OECD, 2013, p. 41). It seems that despite many efforts to change this situation in the Indonesian reform movement adopting realistic mathematics education (RME) theory, known as *Pendidikan Matematika Realistik Indonesia* (PMRI), there is still much research and developmental work to be done in order to improve students' thinking and achievements in mathematics.

In 2011, scientists at the Institute of Cognitive Mathematics (IKM) of Osnabrueck University were assigned by the German foreign aid organization MISEREOR to support teacher training and the development of the quality of math classes on the east Indonesian island Sumba. According to people responsible for teacher training there, the average level of education of students is very low on Sumba compared to Java or Sumatra. Therefore, it did not seem recommendable for the involvement on Sumba to seek isolated content-based improvements only, but to try out a cognitive reorientation of teaching mathematics. It seemed necessary to design measures enabling the learners

[1] `http://nces.ed.gov/pubs2014/2014024_tables.pdf`

to acquire new cognitive habitus in learning mathematics. The decision was made to start this reorientation at the beginning of a new school form in order to reduce resistance to it and to explain the intended habitus as a socially accepted essential characteristic of learning mathematics in the new school. Grade 7 which constitutes the beginning of junior high school (called SMP in Indonesia) and the beginning of an intensive course of school algebra was chosen as an appropriate starting point. In the next step, measures to foster students' understanding (especially in school algebra) were designed. For this purpose – and in accordance with the curricular guidelines – the content *Integers* was chosen to start with. This content provides a particularly suitable basis for coherent mathematical argumentations, systematization and local ordering of intuitive knowledge and of experiences in dealing with numbers. For this content groundwork (Kaune et al., 2011, 2012) had been prepared by the IKM during teaching projects on Java.

One possible way of engaging the Indonesian learners in the mentioned activities could be the way of providing them first with opportunities to reinvent mathematical concepts and their properties through acting in realistic contexts and, subsequently, with opportunities to master some schemas of reasoning typical for the use of this concept in a specific type of mathematical problems. Undoubtedly, this could bring some short-term results related to the specific mathematical contexts. However, it is unclear how mastering contextual schemas of reasoning could enable the learners to understand how to deal with new abstract mathematical concepts and how to construct and control comprehensive mathematical argumentations.

Therefore, another approach was chosen. Instead of mastering schemata of contextual argumentation, the essential decision was taken to provide the learners with an opportunity to create a system of metaphors in their minds representing the axiomatic view of dealing with a system of mathematical concepts, including the formal representation of these concepts. Given such a metaphorical network, a reasonable way of dealing with a new system of abstract mathematical concepts can be achieved by recourse to these metaphors. For this reason, in the designed approach, the learners are engaged in a mathematical discourse which takes place first in the learning environment (LE) *Banking*[2]. In this LE, they elaborate how to deal with debts and assets (with debit and credit), reflect on their intuitive knowledge and experience, and develop a *contract* that determines how the fictive bank has to calculate

[2]Since the learners living on the island Sumba are not familiar with the context of a real bank, they conceive the LE "Banking" not as a real context but rather as a fictitious scenario or as a play. Therefore, the context is realistic in the sense of Gravemeijer et al. (2003, p. 53), and not real.

the results of banking operations with assets and debts. The contract is motivated by the objective to achieve that the bank performs banking operations in a way which can be accepted by the students, based on their intuitive knowledge on debts and assets. This contract (called *Contract to deal with debts and assets*) is represented in a formal mathematical language and it forms the basis for argumentations concerning the correctness of banking operations. In the process of abstraction from the context of "Banking" to the context of "numbers", the metaphor "contract" explains how to deal with arithmetic operations in the realm of integers. The "contract" used in this mathematical context is called *Contract for dealing with numbers*. In the formal representation of this contract, a mathematician will see a model of the axiomatic system for the commutative additive group together with the definition of "subtraction". Later on, the metaphor "contract" is used to explain the extension of number systems, methods of dealing with term rewriting and equations.

Before a broader discussion on empirical data will be presented in Section 4, one statement held by a student in grade 7 indicating some aspects of the intended axiomatic view is presented and explained in the following paragraphs. This might enable the reader to get an idea of how this kind of thinking can be displayed in a small realm of school mathematics.

After two months of participation in the design research project in grade 7 (school year 2014/15), all learners from one project class were asked to complete a questionnaire including some questions and tasks related to contractual calculations and proving (see Examples 5.1 and 5.2 in Section 4). One of these questions was the following (**Question 1**):

Peter says: *I know how to perform calculations with numbers. For this I simply use the arithmetic laws (paragraphs). But proving is more difficult, because I do not know what I can use thereby.*

Question 1: Do you agree with Peter that proving is more difficult than calculating? Give reasons for your answer.

The female student named Josi answered Question 1 as follows: "*No [proving is not more difficult than calculations], because when we calculate, we want to find the final result. And while proving we are looking for a way to justify why the given statement is correct. Thus, this means that numbers in calculating can be seen as substituted variables in proving, or vice versa. When we conduct a proof, we need to know the agreements in the contract (for example: N^+, I^+, K^+, A^+, D^-)*[3]*, the agreements that are not in the contract, like*

[3]Here the student is referring to the following "agreements" called in many school books "rules": N^+ – additive identity element; I^+ – inverse element; K^+ – commutative law; A^+ – associative law; D^- – definition of subtraction.

theorems, and also the kite-rule[4] *as justification for calculations.*" In an interview, the student gave an additional explanation to her statement: "*When we calculate or when we write a proof, we look for justifications for each step of the term rewriting. In both cases, we use the same contract and the same theorems. Thus, for me, there is no big difference whether I have to make some calculations with numbers or with variables.*" This explanation indicates some aspects a systemic thinking: the awareness of some methodological similarities between contractual calculating and proving, in particular of the existence of a system of pieces of knowledge being the basis for calculations and proving; the distinction between "agreements" from a contract (axioms) and between agreements from outside the contract (proved theorems); the view that mathematical statements (theorems) can be justified by the means of a proof.

Furthermore, the student showed that she can use her meta-mathematical knowledge to make contractual calculations and to prove the given theorem (Fig. 1). Each step of her calculations and proof is justified by the appropriate agreements (A^+, K^+, I^+) from the contract describing how to deal with numbers and by the theorem T1[5].

Latihan 1. Hitunglah term!

$$((127 + (-38)) + 38)$$
$$= (127 + ((-38) + 38)) \qquad A^+$$
$$= (127 + (38 + (-38))) \qquad K^+$$
$$= (127 + 0) \qquad I^+$$
$$= 127 \qquad T1$$

Latihan 2. Buktikanlah teorema!

$$((a + (-b)) + b) = a$$
$$= (a + ((-b) + b)) \qquad A^+$$
$$= (a + (b + (-b))) \qquad K^+$$
$$= (a + 0) \qquad I^+$$
$$= a \qquad T1$$

Figure 1. Contractual calculation and proof – solution of Josi.

The solutions give an explanation what the student meant with "substituted variables" in her statement to Question 1. Her justification given to the arithmetic calculation is exactly the same as this one given in the proof. Since she did not use any information on the values of the given numbers, she could substitute the numbers by variables and use the same justification to prove the given theorem. On the other hand, one can see the numbers as the result of substituting some values for the variables given in the theorem.

[4] In contractual calculations, the "kite-rule" is used to justify the result of addition or subtraction of two numbers based on the pre-knowledge from the primary school. This rule is represented by the kite sign.

[5] T1: $(a + 0) = a$

Since these tasks are *not* a part of the realistic context *Banking* and since they do not ask to calculate the result of a booking, these solutions indicate a transfer of experience and knowledge from this realistic context to the *mathematical context*. In order to enable the reader of this paper to understand this transfer and the benefit of the metaphor "contract", the paper puts the focus on the development of the contract with the bank in the LE "Banking" (especially the development of the formal notation with variables), on the use of the metaphor "contract" in mathematics and on students' conceptions related to the use of this metaphor. In Kaune et al. (2011, 2012), Kaune & Nowinska (2011) and Nowinska (2014), some complementary results of the implementation of this LE are discussed.

In the following sections, first the theoretical framework for the design of the microworld "Banking" is presented. There, the didactical concepts and conceptions of school mathematics underlying the design and research project are explained. The next sections explain in more detail the design of the LE "Banking" and, thereafter, the construction of the contract dealing with debit and credit. One transcript from a lesson given on Sumba documents this construction in a real classroom situation. The following section provides an insight into students' solutions to mathematical tasks in order to explain the effects of the teaching and learning process and the benefit of the metaphor "contract". The aim is not to provide the reader with a quantitative explanation of the teaching success, but to document and explain the intended understanding, in particular, the axiomatic view in the sense of Freudenthal. Thereby, the objective is also to meet the requirement "ecological validity, that is to say, (the description of) the results should provide a basis for adaptation to other situations. The premise is that an empirically grounded theory of how the intervention works accommodates this requirement". (Gravemeijer & Cobb, 2013, p. 103) And finally, the designed measures and the achieved effects are discussed against the background of doubts and difficulties experienced at the beginning of the project.

2 Theoretical framework for the design of the microworld "Banking"

2.1 From basic ideas and mental models to microworlds and metaphors

An important question for teaching and learning mathematics is how to establish appropriate conceptions of mathematical objects and how they operate in students' minds (cf. Vinner, 1992, p. 69; Perkins & Unger, 1994, pp. 3f;

vom Hofe, 2006, pp. 142f). A common approach is the development of *basic ideas* (Fischbein, 1989) and *mental models* (called in German "Grundvorstellungen") (vom Hofe, 1998, 2006). This means constituting the meaning of a new mathematical concept through the connection to a known context (vom Hofe, 1998, pp. 97f; 2006 pp. 142f). Mental models play an essential role in the process of concept formation. They also enable the learners to imagine some operations with a new concept and to apply this concept to reality by recognizing the respective structure in real-life contexts, or by modelling a real-life situation with the aid of the mathematical structure (vom Hofe, 2006, 143). In teaching and learning practice, however, basic ideas are often used to create substitutes for formal definitions only and for practical applications of mathematical concepts to reality (cf. Griesel, 1996, p. 18).

The approach established in IKM in projects aiming to improve the quality of Indonesian mathematics lessons differs from the position of generating basic ideas in the following sense: Mental models for mathematical concepts, methods and ways of thinking are used to support the understanding of, communication about and precise formal representation of mathematical objects. This approach starts width elaborating pieces of knowledge that the students are familiar with (from everyday life or from their previous theoretical considerations) and expands these pieces in the context of their experiences into a new system, a *microworld*. The main idea of the resulting microworld is this one described by Schwank (1995): "Microworlds are the external places where the external actions – caused by the cognitive activities – take place; the effects of these actions give feedback to the mental organizations and the development of mental models" (ibid., p. 104). A microworld has to function and to contain the ideas, theories and procedures that are later on utilised in a mathematical theory (Cohors-Fresenborg & Kaune, 2005). "To function" means that in such a microworld an intellectual approach can be tested and experience is gained before the mathematization is carried out. Subsequently, the behaviour which became evident in the microworld is clarified and formulated, which is to say formed, and then written down, i.e. formalized. Whether the intended meaning is being resembled by what has been said or noted is reflected again and again during this process. Thereby one reflects upon the usage of the formulation and the understanding by the reader. Since this microworld has been developed and learners have gained some form of experience of acting in it, a *metaphor system* (Lakoff & Johnson, 1980) can be created, which can be utilised anytime by the student as an evidence base explaining how parts of mathematical knowledge relate to each other. One can say that this experience provides a metaphorical meta-description of a semantic network in the sense of Davis & MCKnight (1979).

Thus, the aim is not only to create sense-constituting mental models for isolated concepts and to motivate the learning of mathematics by applications, but rather to generate and facilitate intellectual behaviour which plays an important role in an understanding *dealing with systems of concepts* and in the process of *concept formation.* In the designed approach, the development of sustainable mental models (e.g. when dealing with numbers and algebraic transformations) is considered as more important than the mediation of factual knowledge and the practice of calculation techniques (cf. Cohors-Fresenborg, 2008).

2.2 Differentiation from the concept of Realistic Mathematics Education

Given the fact that the introduction of negative numbers and calculations with integers constitute the starting point for the formation of the intended cognitive mathematical operating system in students' minds, and in particular for the formation of an axiomatic view, an appropriate approach to teaching and learning this content had to be designed. The problem was not how to find a simple way to make arithmetic calculations quickly, but rather how to find access to students' intuitive knowledge related to integers and to reconstruct the intuitively understood way of dealing with these numbers.

The new approach of introducing negative numbers follows to some extent the subject-specific teaching learning theory *Realistic Mathematics Education* (RME), going back to (Freudenthal, 1973), (Gravemeijer, 1994). However, it uses the idea of a *realistic* context (i.e. *experimentally real* and not necessary "real" in the common sense) in an unusual way as it aims at reconstructing some *ways of thinking* typical for dealing with mathematical concepts and not only of some *properties* of mathematical objects. Consequently, it differentiates from many approaches to teach and learn mathematics that are based on RME.

> Of course, by saying the starting points are experientially real, we are not suggesting that the students had to experience these starting points first hand. Instead, we are saying simply that the students should be able to imagine acting in the scenario, as king for example (Gravemeijer et al., 2003, p. 53).

The teaching concept starts with an analysis of balancing account transactions. Students acting in this realistic context describe banking operations and their features using the symbolic (mathematical) language agreed with the fictitious bank. This experience is subjected to learners' reflection. In this way, mathematics is reinvented. Gravemeijer and Cobb (2006, p. 63) describe the process in their theoretical framework as follows: "This requires the instructional starting points to be experientially real for the students, which means that one has to present the students problem situations in which they can reason and act in a personally meaningful manner". The objective of the guided reinvention principle is that the mathematical knowledge that the students develop will also be experientially real for them.

The RME approach offers a basis to reconstruct properties of mathematical objects; in this case, negative numbers and the related operations. It does not, however, explain explicitly how to reconstruct the cognitive behaviour typical for systemic thinking within a locally ordered mathematical field. This way of thinking relates to the axiomatic foundation of a number system expansion as it would be characteristic of 20^{th} century mathematics. In line with the RME approach, it seems impossible for the students to develop an understanding of how these parts of the integer theory, which do not appear in one joint real scenario, work together. Furthermore, the fact that some of these parts do not appear in a natural way in the every-day reality (like multiplication of two negative numbers) may cause the feeling that these parts appear unexplained out of nowhere. This may lead some learners to the conclusion that mathematics cannot be understood.

In order to prevent this, the microworld "Banking" guides students to develop a contract with a bank, such that all future bank transactions have to be justified on the basis of this contract. The contractual reasoning used to justify calculations conducted by the bank must be given in the form of a complete chain of reasons so that no doubts arise whether the controlled calculation is correct or not. This experience in contractual reasoning is followed by the abstraction from the idealised bank to a "Math Bank" conducting calculations with numbers representing not only amounts of assets and debts. In this case, the Math Bank's contract is an agreement on how to operate with integers. The formal representation of this contract and the methods of dealing with it are exactly the same as in the case of the previous contract for banking operations. This helps to understand how to conduct contractual calculations and argumentations in mathematics. With the recourse to the metaphor "contract", algebraic proofs can be understood as contractual "calculations" with the reference to the accepted "contract" and statements justified on the basis of this contract. We see such an metaphorical projection of the mathematical content onto the previous experience as an essential cognitive mechanism that facilitates systemic thinking and enables the learners to experience some "acts of understanding" (Sierpinska, 1994). Thus "reality", in the sense of Freudenthal (1991, p. 17), in the microworld "Banking" also refers to the experience of making contracts and holding to them.

Worldwide in teaching practice, the introduction of negative numbers is often motivated by the use of multiple realistic contexts. Then, these contexts are used to illustrate addition and subtraction on integers. In general, each of these contexts is used to explain only a few (and often only one) of the arithmetic laws ("rules") describing how to deal with addition and subtraction. Additionally, some of these "rules" are introduced on the basis of observations of certain regularities of arithmetical operations using the method described by Freudenthal (1986, p. 435) as "induction extrapolatory method". As a result of a deep analysis of the most popular schoolbooks for teaching mathematics in Poland, Ceglarek (2013) worked out that the way how these books use realistic contexts to introduce negative numbers and the arithmetic laws for addition and subtraction with integers hinders a mathematical discourse and an explanation of how these different "rules" work together. It seems that this approach is typical not only for Polish books, but also for many of these used in Indonesia and presumably in many other countries, too. This may hinder mathematical argumentation and systemic thinking.

In encyclopaedic books, one may let the presented network of concepts and methods grow by simple addition of new ones, but understanding does not grow that way (cf. Sierpinska et al., 2002; Wittmann, 1981). Single objects are interesting for conceptual understanding. This point is elaborated in the framework of Anthropological Theory of the Didactic (ATD) (Bosch & Gascón, 2006; Chevallard, 2007, 2006) and in the description of mechanisms in mathematical human thought provided in Cognitive Science Oriented Mathematics Education by Davis & MCKnight (1979).

ATD provides a model of mathematical activity in terms of *praxeologies* consisting of two parts – the practical part (*praxis* – a type of tasks and a technique – a way of solving these tasks) and the theoretical part (*logos* – a discourse intended to provide explanation and justification for the technique). This theory classifies mathematical praxeologies into a sequence of increasing complexity. "The "simpler" ones – *point*-mathematical organisations –, built up around a single kind of problem, can be linked according to their theoretical background to give rise to *local*, *regional* or *global* praxeologies that respectively cover a whole mathematical theme, a sector or a domain" (Bosch und Gascón, 2006, p. 59). The absence of a joint background linking the point-praxeologies into more complex structures may cause difficulties in understanding.

Davis and MCKnight (1979) give some implications concerning the construction and the use of mathematical knowledge as a semantic network of concepts, operations and experiences related to them. The authors point out that *frames* (a specific information-representation structure that a person can build up in his or her memory and can subsequently retrieve it from memory when needed), *procedures* (sequential processes being parts of frames) and "meta-language" (describing frames and the experience of using it) play important roles in generating, structuring and using mathematical knowledge. The construction of mathematical knowledge in the form of frames (understood as semantic networks of concepts, operations, procedures and experiences in using them) can only be successful if the experience of using mathematical concepts and operations is subjected to learners' reflection (metacognitive thoughts) (ibid., p. 98). These meta-thoughts are aimed to provide explanations of how pieces of mathematical knowledge relate to each other and to situations in which they can be used. They result in descriptors (ibid., p. 96) of frames, or in other words in meta-knowledge, and can have a metaphorical character. Thus, guiding the learners to look ahead on their experience in learning mathematics is of great importance in avoiding knowledge compartmentalization.

2.3 Axiomatic method

In mathematics, the axiomatic form of a system of concepts can be seen as the intended final result of reflection within this system. In the cognitive behaviour of a mathematician, a system of axioms does not only function as an isolated element of his knowledge (e.g. a description of mathematical constructs). The axiomatic approach became a universal way of thinking, guiding the thinking processes when dealing with systems of mathematical objects, the previous knowledge and intuitions. This approach activates specific questioning techniques and strategies which enable a mathematician to (re)organize and systematize his or her knowledge. Thus, a system of axioms functions as a metaphor for dealing with concepts and for systemic thinking.

In terms of school mathematics, such a metaphor can be constructed when dealing with legal *contracts* in the sense that only the written agreements count; not additional knowledge that is constructed in the minds of laymen when reading the written texts. Still, it has to be checked whether not explicitly mentioned statements in the contract remain valid as well as additional agreements. What is meant by this in particular is that it has to be tested whether these "new" agreements can be reasoned on the basis of the contract or whether they contradict already validated agreements. Consequently, even in situations in which an explanation by the access to realistic contexts reaches its limits, an established contract motivates and creates the possibilities to promote explanations and justifications. Mathematical relations, so-called formulas and theorems, are to be derived from paragraphs of the given contract and from the already proven theorems.

The difference between a *contract* for dealing with numbers – how it is explained to the students – and *a system of axioms* of Abelian groups – how mathematicians understand it – is the following.

On the one hand, a contract determines anew how calculations are supposed to be conducted; therefore, a contract is concerned with implicit definitions of "addition" and "formation of inverse". However, it lacks the transition to the fact that the contract deals with variables for functions without specifying the meaning of these functions, but specifying only a framework which must be met when interpreting the variables. In students' eyes, the *contract for addition* and the *contract for multiplication* are two different contracts; in the eyes of mathematicians, these are two different interpretations of one system of axioms. Consequently, contractual calculation can be imagined as an application of an axiomatic theory in the sense of Euclid, but not yet as an application of an axiomatic theory as it has been established by Hilbert (1899).

Both approaches have in common that they determine how these contracts or rather axioms have to be handled when applying in calculations, in explicit definitions and even in proving. Even from the point of view of a contract, the plausible calculation rules (arithmetic laws) have to be proven gradually because it is only understandable in this way that the contractual agreements really cover everything that needs to be controlled. The formal presentation is the same in both approaches. The only missing detail in the contractual approach is the last abstraction that e.g. both symbols of the two-digit functions "addition" and "multiplication" can be replaced by one new symbol in both contracts, and that in the next step this new symbol is interpreted as the name of a two-digit variable for functions. The same is done for the formation of the additive or multiplicative inverses. In the design research project on Sumba, this idea of abstraction and interpretation was made more accessible to the Indonesian teacher trainees who teach in grade 7. Doing so, the metaphor "find and replace" used in text processing proved as a useful tool.

3 The design of the teaching and learning sequence

The following sections outline how the above mentioned theory-based considerations are operationalized in the design of the LE "Banking".

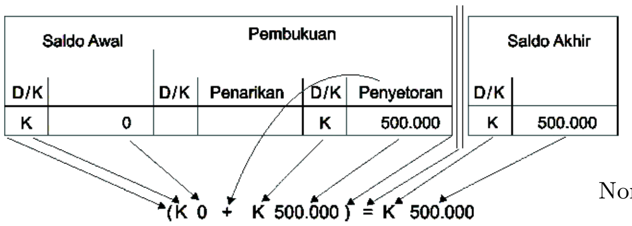
3.1 Construction of mental models in a realistic context

The introduction of negative numbers is based on the intuitive knowledge of the learners about how to deal with debts and assets in a bank.

In the designed (fictional) scenario, the learners have the opportunity to discover characteristics of and relationships between paying asset and debts in and paying asset and debts out, and to describe them. The description of these properties and relations is realised first in the symbolic language of a bank (see Fig. 2 from the student book for grade 7^{th} in Kaune & Cohors-Fresenborg (2014, p. 16)). Here the letters D (for debit; indon. – *d*ebet) and K (for credit; indon. – *k*redit) are used to identify the corresponding debit and credit amounts (debt and assets amounts)[6].

Using this language, the transaction recorded in the account card in Fig. 2 has to be written in a shorthand notation as follows[7]:

$$(\mathrm{K}\ 0 + \mathrm{K}\ 500.000) = \mathrm{K}\ 500.000.$$



Saldo Awal – opening balance
Pembukuan – transactions
Penarikan – withdrawal
Penyetoran – deposit
Saldo Akhir – closing balance
Nomor Rekening – account number

Figure 2. Symbolic representation of a booking operation.

In a next step, this notation is simplified even further. Instead of D and K, only one sign is used to differentiate debit and credit amounts. The debit amounts are characterized with a minus sign:

$$\text{old notation: D } 50.000 - \text{K } 25.000) = \text{D } 75.000$$
$$\text{new notation: } ((-50.000) - 25.000) = (-75.000)$$

In this algebraic mathematical notation, the brackets and the equal sign have their counterparts in the account card, as indicated by the arrows in Fig. 2.

[6] In the microworld "Banking", the term "debt(s)" is used to describe a sum of money that a bank holder owes to a bank. The term "assets" describes a sum of money that a bank holder owns. Instead of "debt(s)" and "assets" the terms "credit" and "debit" are used by the learners as well and in such a case they have the exact same meaning as "debt(s)" and "assets" respectively. That makes the meaning of the terms "debit" and "credit" different from the meaning postulated by a definition in the *real* banking situation. Our experience has showed that this has never caused problems by the learners.

[7] In Indonesian notation of numbers, thousands are separated by a dot as shown in figure 1 and 2 taken from an Indonesian schoolbook. In order to keep the number notation coherent in the whole paper and thereby facilitate the reading of the texts, English notations (thousands are separated by a comma) are neglected.

The more difficult case in the context of the bank world, which is described in the language of mathematics as subtraction of a negative number, corresponds to the operation of paying out a debt amount. This interpretation seems limited at first and for many outsiders also absurd as no *real* bank pays out a debt amount to a customer. However, transactions which are not performed by any *real* bank are conceivable in a realistic scenario in the above mentioned sense of Gravemeijer et al. (2003, p. 35). The case of paying out debts arises in several situations as a solution to a given problem. One of these situations is the merging of two accounts, and another is the undoing of an incorrectly booked (paid in) debt amount. The second situation is represented in Fig. 3. Here, the debt amount of 38.000 Indonesian rupiah was accidentally paid in twice.

Nomor Rekening								3115986771		
No.	Tanggal	Saldo Awal		Pembukuan				Saldo Akhir	
	2011	D/K		D/K	Penarikan	D/K	Penyetoran	D/K	
18.	27-06	K	138.000			D	38.000	K	100.000
19.	27-06	K	100.000			D	38.000	K	62.000
20.	29-06	K	62.000	D	38.000				

Saldo Awal – opening balance
Pembukuan – transactions
Penarikan – withdrawal
Penyetoran – deposit
Saldo Akhir – closing balance
Nomor Rekening – account number

Figure 3. An account card with a falsely booked debt amount.

The second deposit has to be cancelled. One possibility of doing so is to pay out the deposited debt amount which can be expressed by the following term:

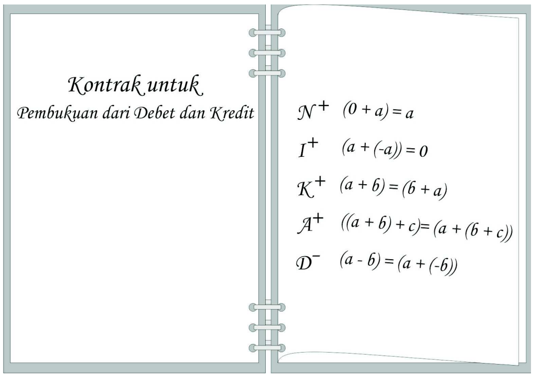
$$(62.000 - (-38.000)) = 100.000$$

The closing balance (cf. Fig. 3) accounts for 100.000 Indonesian rupiah.

The students easily come to the conclusion that the operation of paying out a debt amount results in the same outcome as depositing the corresponding credit amount. Also, the operation of paying out a credit amount results in the same final balance as the deposit of the corresponding debt amount. This knowledge and experience are used to explain the operation of paying out by referring to deposit (paying in). In the later step of the mathematization, this serves as a basis to reconstruct the definition of subtraction.

3.2 Systematization of conceptions

An important step in constructing the metaphor "contract" is the systematization and further development of mental models, in particular the local ordering of intuitive knowledge and experiences of the learners on how to deal with debts and assets in the fictitious bank. For this purpose, a situation in the classroom is created in which a bank customer has to negotiate a contract with the bank. The contract should specify normatively how the bank handles bookings and how it records them. In the future, all bookings in the bank should be based on this contract. The result (*Contract for dealing with debit and credit* (CDeb) looks like this:



$\mathcal{N}^+$: opening an account
$\mathcal{I}^+$: closing an account
$\mathcal{K}^+$: bookings order
$\mathcal{A}^+$: bookings order
$\mathcal{D}^-$: calculating the result of paying out

Figure 4. Contract for dealing with debit and credit.

The shortcuts for the names of the paragraphs in Fig. 4 prepare the future use of the contract for calculations with numbers: $\mathcal{N}^+$ – additive identity element; $\mathcal{I}^+$ - inverse element; $\mathcal{K}^+$ – commutative law; $\mathcal{A}^+$ – associative law; $\mathcal{D}^-$ – definition of subtraction. The next step is the abstraction from the idealized bank to the "Math Bank". In this step, the contract for dealing with debit and credit is interpreted as a contract describing calculations with numbers. As a result of this insight, the contract gets the new name *Contract for dealing with numbers* (CNum).

Hereby, the numbers with the sign "–" are interpreted as negative numbers, and the unsigned as positive numbers. The plus sign for the two-digit operation of deposit is interpreted as the two-digit addition sign, and the two-digit minus sign for paying out as the subtraction sign.

In the step of the reconstruction of the mathematical knowledge, the learners use their experience in dealing with contracts in order to control step-by-step mathematical calculations and their correctness in accordance with the new contract. From the existing paragraphs of the contract further conclusions are derived, i.e. – from the mathematical point of view – new theorems are to be proven. Thus, abstraction and formalization are of essential importance in this approach.

3.3 The role playing game

The following paragraphs are concerned with the construction of the previously described contract CDeb. They aim to provide the reader with the possibility to follow the process of abstraction from the given banking operations with concrete amounts of debts and assets to the formal representation of these operations. Therefore, it is recommendable to read the presented transcript with the focus on the symbolic and algebraic representations used to emphasize the essential ideas of the discussed bookings and to abstract from the concrete examples.

The method used in the classroom to contract CDeb is role playing game. The play is being led by two teachers. One of them plays the part of the bank manager, the other the part of a customer who negotiates terms for dealing with accounting transactions. The students' task is to imagine themselves as employees of the bank helping and advising the manager. They are allowed to answer questions of either the customer or the manager, issue proposals, and contribute their own comments. Their visible active participation is very limited, but it is important that they are given the chance to experience the discourse (led by these two teachers) concerning the interplay between formal banking operations and their abstract representation, the content and the form of each agreement in the contract, and also the use of these agreements. They observe how examples of banking operations are investigated with the aim of emphasizing these properties which are independent from the given values of debts and asset. In the next step these essential properties are represented in a symbolic way. This representation is the result of the abstraction performed ("played") in the discussion between the bank manager and bank customer.

Because of the importance of the interplay between the content and the form the contract, the discourse has to be conducted in the described way so that the interaction between the conceptions of how to deal with debts and credit on the one hand and the formal representations of these conceptions on the other hand is highlighted and depicted clearly.

At the time of the role play, the students have already learned how transactions are written down on an account card and how they are formally represented. Still, for the first time the students use variables for debts and assets amounts in order to determine dealings with transactions normatively.

Course of the Role Play

In the following paragraphs, general information about the course of the Role Play with the introduction of CDeb are given. Thereafter, selected fragments of the transcript of one single lesson, in which CDeb was developed, are presented and analysed with the focus on the interplay between the concrete Banking operation and the formal notation.

The role play proceeds according to the following schedule:

1. The bank customer and the bank manager introduce themselves. The customer explains the purpose of his visit. He wants to open an account with the bank and negotiate terms for dealing with bookings, which should be listed in a contract. The manager welcomes the customer, introduces him to his colleagues (students), and describes his bank as very customer-friendly.
2. The customer enquires about conditions for opening an account at this bank from the manager who explains these to him. When opening a bank account, the opening balance is zero Indonesian rupiah, and the final balance is exactly the amount of the first deposit. The bank allows the customers a sum of money as well as a debt amount when effecting the first transaction. The deposit of debts means to "transfer" a debt amount from one account to another.
3. The bank manager explains transactions when opening an account with selected examples to the customer. Finally, the manager and the customer agree on completing a contract in which it is approved that an account with any money and debt amounts can be opened. The first agreement is listed in the following form: $(0 + a) = a$.
4. The bank client enquires about conditions for closing an account at the bank from the manager who explains these to him. The final balance has to be zero Indonesian rupiah. The bank has decided to close each account with the operation of paying in (although other banks use the possibility of paying out the final balance in order to close an account). The opposite of the final balance has to be deposited to the opening balance. The opposite of a debt amount is the appropriate amount of credit (and vice versa).
   The bank manager explains transactions when closing an account with selected examples to the customer. Finally, the manager and the customer agree on a new paragraph of the contract. This paragraph clarifies that an account with any opening balance can be closed by a corresponding deposit. The second agreement is listed in the following form: $(a + (-a)) = 0$.
5. The bank customer tells the bank manager his wish to merge two accounts. He asks whether the order of transactions when joining two accounts (depending on which of the two accounts should be closed) has an effect on the closing balance in the resulting account. The bank manager provides specific examples that this is not the case. The merging of two accounts is allowed. In this case, the order of the transactions that are made for this purpose does not affect the closing balance. The customer wants these conditions apply to any opening balance which the manager accepts. He provides a general example of two accounts with opening balances a and b, and shows how the closing balances of the first or rather second account are calculated. The customer and the manager agree on a second paragraph, which determines that the order of the transactions when merging two accounts does not have any effect on the closing balance.

This is listed in the following form: $(a+b)=(b+a)$.

6. The bank client enquires about conditions for merging three accounts. Analogue to 5, the manager and the customer agree on a further paragraph, which states that the order of transactions for merging three accounts has no effect on the final balance. This is listed in the following form: $((a+b)+c)=(a+(b+c))$.
7. The bank customer is surprised that the contract says nothing about payouts. He would also like to have the option for withdrawals.
   The bank manager guarantees that this is possible. Nevertheless, his employees use a special method for calculating the final balance after a withdrawal. On the basis of their experience, the withdrawal of a certain amount of money leads to the same balance as the payment of the corresponding opposite amount (and vice versa). To calculate the final balance after a withdrawal, the bank employees list the appropriate deposit and, by that, determine the outcome of this operation. In other word, the bank attributes the withdrawals to the corresponding deposit. The customer accepts this approach which is listed in the contract in the following form: $(a-b)=(a+(-b))$.
   Finally, both the manager and the customer agree that no more paragraphs for payouts are needed. Dealing with payouts can be learned by understanding that each operation of paying out can be replaced by the suitable operation of paying in. In the next step, properties that apply to payments can be used to learn more about payouts.
8. The manager and the customer sign the negotiated contract with the effect that they bind themselves to the paragraphs. If one of them breaches a paragraph, a lawsuit can be filed. Finally, all employees (i.e. all students) sign this contract (Fig. 6). In the course of conversation, every paragraph is given a name and an abbreviation for that name. The abbreviation is listed in the contract (Fig. 5).

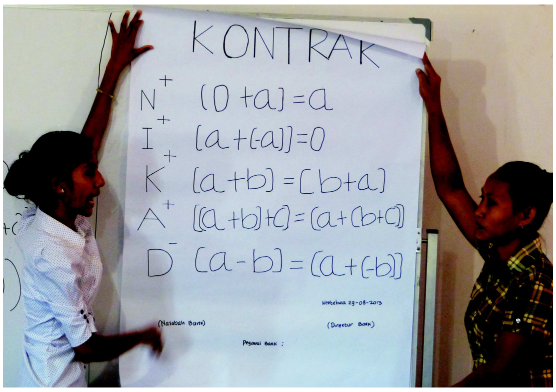


Figure 5. Two teachers (manager and customer) present the negotiated contract.

Figure 6. Students sign the contract.

Transcript of the classroom situation with the introduction of CDeb

In the following paragraphs, selected fragments of the transcript of one single lesson with the introduction of CDeb are presented. Since the beginning of grade 7, this class has been incorporated as a *project class* into a design research project conducted by the IKM. The role play was performed in lesson 12. Each of the previous lessons lasted 60 to 80 minutes, with one of the lessons having an organisational character. The role play itself lasted 54 minutes. It was conducted by two students (prospective mathematics teachers) who had been further trained for their work in the design research project.

Due to reasons of space, the transcript is restricted to the introduction of the first paragraph, and in some places the teaching activities are summarized. This is highlighted by the sign "[...]". The summary of the teaching activities is written in italics, followed by the sign "*".

The teachers are: Ibu[8] Misery (bank customer) and Ibu Olif (bank manager). The students play the part of the employees who are involved by Ibu Olif.

It is noticeable that sentences in the transcript are not always formulated completely and accurately. Still, this corresponds to what was said. In addition, when specifying debts and credits, the currency is not always mentioned (here: Indonesian rupiah). This is a sign of a certain lack of precision of the mentioned facts and a lack of control (here: mostly in the cases of the teachers) whether what is said corresponds to what is meant. Square brackets indicate additional comments that relate to activities in class or to the meaning of the used vocabulary. They are supposed to assist the reader in understanding the text.

Introduction of the first paragraph

A bank customer comes to the bank and asks about the terms for opening an account.

Ibu Misery	Oh, thank you. Ibu, I would like to ask you a question. Suppose I have an amount of 100.000. Can I open an account with an amount of 100.000 or not?
Ibu Olif	[*addressing a member of her staff*] Er, what do you think, Wati.
Wati	Yes, this is okay.
Ibu Olif	Yes, this is okay, Ibu.
Ibu Misery	Oh, it is okay, so, then, how is it recorded, Ibu? [...] **Ibu Olif explains that every new customer gets a new card account.*

[8] *Ibu* (abbr.: *Bu*) (Indon.) can be literally translated as "mother" and is used in speech much like the English words "ma'am" and "lady".

	Each booking is listed on it. However, the bank also uses abbreviated notations in order to communicate easier with bank customers about individual bookings.
Ibu Olif	[*addressing the members of her staff*] Okay. When Ibu Misery has opened an account card here, how much is the initial balance of Ibu Misery's account, yes? Er, Nadin?
Ibu Olif	Zero rupees.
Ibu Misery	I see.
Ibu Olif	As Ibu has just opened an account with us, the initial balance of her account is then zero rupees. And here, because Ibu has an amount of 100.000 rupees and has paid it in when opening the account, our bank will then write it that way. [*The whiteboard is divided into two parts. Ibu Olif writes the following equation on the right side of the dividing line:*]

$$(0 + 100.000) = 100.000$$

	(4sec) So Ibu's final balance will also be 100.000 rupees, Bu.
Ibu Misery	Oh, 100.000 rupees, yes.
Ibu Olif	Ibu's final balance, which amounts to 100.000 rupees, corresponds to the amount that Ibu paid in when she opened the account. [*Ibu Olif draws red boxes around the numbers*:]

$$(0 + \boxed{100.000}) = \boxed{100.000}$$

Ibu Misery	I see. So, if Ibu, if I pay in 100.000, it means that the final balance is 100.000. Can I replace it with another amount, Ibu?
Ibu Olif	Oh, yes, of course, Bu. [*Ibu Olif wipes out the amounts.*]

$$(0 + \boxed{}) = \boxed{}$$

Ibu Misery	Suppose Ibu, I pay in 200.000. Does the notation remain the same as this?
Ibu Olif	[*addressing the members of her staff*] What do you think, my bank employees, is it the same or not? Jean.
Jean	It is the same.
Ibu Olif	It is the same, Bu.
Ibu Misery	Oh, it is the same.
Ibu Olif	Yes, how much does Ibu deposit here, Bu?
Ibu Misery	Let's assume 200.000.

Ibu Olif Yes, 200.000 rupees. Then Ibu's final balance automatically will be 200.000 rupees, Ibu. [*Ibu Olif fills in the boxes, while she speaks*:]

$$(0 + \boxed{200.000}) = \boxed{200.000}$$

(11sec) Yeah, so it is like that, yes: Here, Ibu's final balance of 200.000 rupees corresponds to the amount that Ibu paid in when she opened the account. [*While Ibu Olif says this, she points to the corresponding numbers.*]

Ibu Misery Oh, now I see, Ibu. I want to ask something else, Ibu.

Ibu Olif Yes, please.

Ibu Misery Suppose I have an account later, an account at Ibu's bank. After that, I have debts at my company. I want to pay off the debts and open a new account, Ibu. Will this work with my debts?

Ibu Olif [*addressing the members of her staff*] What do you think my bank employees? Will it work to open an account with debts paid out from another account, or won't it? Yes, er Des?

Des It will work.

Ibu Olif It will work, Ibu.

Ibu Misery Oh, it will work, er, that is good, what a generous bank.

Ibu Olif Well, it is also called generous bank, Ibu.

Ibu Misery If you don't mind, I would like to see what the notation looks like, Ibu. I'm new here.

Ibu Olif It is exactly the same as before, when Ibu put the money onto it, er paid it in, yes Bu. Here the initial balance is still zero rupees, because Ibu has just opened an account. And because Ibu deposited the money here, how much it is, Bu? [*Ibu Olif writes the equation piecewise.*]

Ibu Misery Assume a debt amount of 20.000.

Ibu Olif 20.000. (7sec) 20.000 rupees, yes, then Ibu's final account will be a debt amount of 20.000 rupees. [*Ibu Olif writes the following on the whiteboard*:]

$$(0 + \boxed{200.000}) = \boxed{200.000}$$
$$(0 + (-20.000)) = (-20.000)$$

[...] * *The same way as in the case of the examples with credit amounts, Ibu Olif explains the structure of the equation that describes opening an account with a debt amount of 20.000 rupees. She stresses that* *the final balance (20.000 rupees debts in this case) corresponds exactly to the deposit amount (20.000 rupees debts in this case). While explaining this, she draws red boxes to illustrate the explanation*:

$$(0+\boxed{200.000})=\boxed{200.000}$$
$$(0+\boxed{(-20.000)})=\boxed{(-20.000)}$$

Ibu Olif	So, er Ibu's final balance corresponds to the debt amount paid in while opening the account. (4sec) Would you like to ask more questions, Ibu?
Ibu Misery	Oh yeah, another example, Ibu.
Ibu Olif	Yes.
Ibu Misery	As I said before, Bu, assume I would like to open an account with another debt amount.
Ibu Olif	Ah yes. If I open an account with another debt amount, okay. [*addressing the members of her staff*] So, will it work when Ibu opens an account with another debt amount? Will it work or not? Er, Erlin.
Erlin	It will work.
Ibu Olif	It will work, Bu.

$$(0+\boxed{200.000})=\boxed{200.000}$$
$$(0+\boxed{})=\boxed{}$$

Ibu Misery	Oh, it'll work. This bank is very good, er, thus we can open an account with a debt amount, now I see.
Ibu Olif	That's why we named the bank 'generous bank', Bu.
Ibu Misery	Oh yeah, that's right.
Ibu Olif	So, how much money would you like to deposit here, Bu?
Ibu Misery	Let's assume a debt amount of 50.000 Ibu.
Ibu Olif	A debt amount of 50.000. Then Ibu, Ibu's final balance is also a debt amount of 50.000 which corresponds to the debt amount paid in when opening the account. [*Ibu Olif writes in the empty boxes*:]

$$(0+\boxed{200.000})=\boxed{200.000}$$
$$(0+\boxed{(-50.000)})=\boxed{(-50.000)}$$

Ibu Misery	Oh, this means, if I would like to open an account, I can pay in a credit amount as well as a debt amount when opening it.
Ibu Olif	Yes, that's right, Bu.
Ibu Misery	Hmm, this bank is very good. If you don't mind Ibu, I would like to conclude a contract for the opening of an account with debts as well as credits. Can we integrate this in a contract?
Ibu Olif	Eh, it's so, Ibu yes, if we see these two bookings, [*Ibu Olif points to the two bookings on the whiteboard*] er, I think, I and my staff here agree that it is not enough to write it in a contract, for example, if we want to sign a contract. Because there may be other ways, Ibu can open an account with other credit or debit amounts and, therefore, our bank writes it in this form. [*Ibu Olif writes the equation (0 + a) = a on the whiteboard and she explains at the same time.*]

$$\begin{array}{l} (0 + \boxed{200.000}) = \boxed{200.000} \\ (0 + \boxed{(-50.000)}) = \boxed{(-50.000)} \\ \hline (0 + a) = a \end{array}$$

If Ibu opens an account with the initial balance of zero rupees and pays in an amount equal to a, then Ibu's final balance will also be a.

Ibu Olif	And a replaces here, Ibu, er yes, it represents the credit or debt amounts paid in by Ibu while opening an account.
Ibu Misery	I see.
Ibu Olif	This is how it is, Ibu. Is there any explanation that you don't understand yet, Ibu?
Ibu Misery	Yes, I am confused by this sign, Ibu. [*Ibu Misery points to the box around the number (-50.000) on the right side of the equal sign*]
Ibu Olif	Okay, actually we do not have signs like this one in our bank, Ibu [*Ibu Olif points to the boxes*]. First, we think this way, first we think zero plus box, (5 sec) zero plus box (8 sec) is the same box. [*While explaining this Ibu Olif writes the third line on the whiteboard*:]

$$\begin{array}{l} (0 + \boxed{200.000}) = \boxed{200.000} \\ (0 + \boxed{(-50.000)}) = \boxed{(-50.000)} \\ \hline (0 + \boxed{}) = \boxed{} \\ (0 + a) = a \end{array}$$

But Ibu, in our bank, the computer which we use here cannot programme them [meaning the boxes], therefore we agree to use this notation [*Ibu Olif points to the equation* $(0 + a) = a$], and all my employees have already learned this here, a notation like this one.

Ibu Misery Yes, Ibu, do you mind, Ibu? I would like to write it in the contract on the whiteboard, Ibu.

[...] **Ibu Olif writes the heading 'Contract' and the equation for the first paragraph on the left side of the whiteboard*:

KONTRAK

$$(0+a)=a$$

Analysis of the Transcript

In the above described teaching sequence, the first paragraph ("Paragraph about Opening an Account") was listed in the contract. Based on examples with money and debt amounts, specific cases were considered first. The usage of the red box plays an important role in the introduction to the formal notation as they illustrate a relationship between the amount of the initial and final balance. Furthermore, communication is facilitated through this connection. After wiping the numbers from the box, only the equation "zero plus box equals box" is on the board. The boxes fulfil the role of the placeholder. Therefore, they must be filled with content, so that the equation can describe an actual transaction to open an account. At the same time, they express the core idea of these transactions for any case. The equation is the first step of formalisation. In the next step, the box is replaced by the character a. The aim of this is to interpret the variable a as a placeholder for any value. Now, wherever the variable occurs, it has to be replaced with the same value. This needs to be done in the same way as it was visualized for the red boxes.

3.3.1 *Comments to the introduction of the second paragraph*

In the further course of the role play game, the bank manager and bank customer agree on the second paragraph ("Paragraph about Closing an Account") as the result of the discussion concerning the problem of how to close an account. Again, they use specific examples from which they develop the formal notation. These examples are the following:

$$((-55.000)+55.000)=0$$
$$(30.000+(-30.000))=0$$

In contrast to the first paragraph, the red boxes cannot be drawn around the numbers representing the values of the debts and assets amounts. They also cannot be drawn around the variable a in the second paragraph: $(a+(-a)) = 0$. The reason for that is that the notation $(-a)$ is understood as the opposite of the opening balance a. If the opening balance is a debt of 200.000 Indonesian rupiah, $(-\mathit{200.000})$ is written down instead of a. The opposite of this amount is the credit of 200.000 Indonesian rupiah. The variable $(-a)$ is replaced by *200.000* and *not* $(-(200.000))$. It is not until later during the transition to CNum, when the difference between the notation (syntax) and the meaning (semantics) is reflected with the students. If the paragraph I^+ is used during the proof of a mathematical theorem, the form of the paragraph must be considered, not its interpretation in the micro-world "Banking".

The symbolic representation $(a + (-a)) = 0$ appears to be a description of already known and intuitively understood relation. In conclusion, the bank customer and the bank manager agree to add the new paragraph to the contract. In order to be able to communicate quickly in the future when talking about the paragraphs, the customer suggests to label these and add abbreviations to the contract. The first paragraph takes the name "Paragraph about Opening an Account" and the abbreviation N^+. This reminds us of the amount of the opening balance (zero Indonesian rupiah-indon. "nol Rupiah") and of the operation of the deposit (represented by the plus sign). The second paragraph is named "Paragraph about Closing an Account" and the abbreviation I^+. This reminds us of the procedure that the opposite amount (Inverse) to the actual balance must be paid in whenever you want to close an account. Again, the plus sign represents the deposit.

Remarks to the process of abstraction

With the introduction of CDeb an important goal has been achieved. When the idea of a realistic bank is blocked out and when only the paragraphs are taken into account, the contract determines how to deal with numbers (CNum). A mathematician looking at CNum will recognize in this contract the axiom system for a known mathematical structure. Probably, he would say that the contract is concerned with content appropriate for a university lecture and not for students in school. To assess the performances of the learners and to appreciate them properly, however, one has to try to reconstruct the processes of abstraction that led to the contract. The creation of the contract CDeb is not simply a goal by itself. The following paragraphs explain the intended metaphorical character of it.

3.4 The metaphorical character of the contract dealing with debit and credit

The contract dealing with debit and credit (CDeb) functions as a bridge between intuitive knowledge and the experiences learners gained in the micro-world "Banking" on the one hand and in mathematics on the other. At first glance, the contract fulfils the role that is assigned to a model in the concept of RME. "Models are attributed the role of bridging the gap between the informal understanding connected to the "real" and imagined reality on the one side, and the understanding of formal systems on the other". (Van Den Heuvel-Panhuizen, 2003, p. 13). In RME, "models are seen as representations of problem situations, which necessarily reflect essential aspects of mathematical concepts and structures that are relevant for the problem situation, but that can have different manifestations. This means that the term "model" is not taken in a very literal way. Materials, visual sketches, paradigmatic situations, schemes, diagrams and even symbols can serve as models [...]" (ibid., p. 13).

But unlike in this description of a model, the contract CDeb does not appear to deal with debits and credits as a solution to a problem situation. The situation leading to this contract had a clear goal – the customer wants to conclude a contract with the bank. The only problematic situation at the beginning is the content of the contract. Since the development of the contract is led by teachers, this does not state a problem for the learners which they have to solve themselves.

Unlike a "model" in RME, the contract will not be further generalized to deal with debit and credit, or applied to analogous situations in a realistic context. The aim is to experience the application of the contract and to expand it in order to facilitate the understanding of the application of an axiomatic system in mathematics. The next step for the learners in the textbook (Kaune & Cohors-Fresenborg, 2014) is to understand that an analogous contract for multiplying and dividing appropriate numbers can be developed. By that, a "cognitive shift" should be achieved in the contemplation of the contract: The contract for dealing with debit and credit and it its application explain paradigmatically the cognitive mathematical behaviour in broad areas of mathematics (e.g. the expansion of the number range). Since the form of a contract can differ, it is accurate to label the contract as a metaphor for explaining mathematical activities and not as a model responsible for solving contextual problems. It is not the final form of the developed contract that has to be used by the learner later but the way of thinking related to the role and use of such a contract when dealing with new concepts.

4 Analysis of Students' Solutions

This section presents students' written works and explains the use of the metaphor "contract". The intended "understanding" related to the use of a contract can be seen as a process consisting of some "acts of understanding" (Sierpinska, 1994, pp. 28ff). These acts are mental experiences of a learner by which the learner relates what is to be understood to something else – to the "basis" for this understanding. Since the formal representation of CDeb is exactly the same as the formal representation of CNum and, furthermore, the technique of calculating, controlling and reasoning remains in both cases the same, the shift from dealing with banking transactions to dealing with numbers and variables does not make any essential difference in the cognitive behavior of the learners; such a difference can be seen between calculating and proving within the same context, because proving requires more strategic reflection and planning than calculating. The new situation with numbers and variables can be understood through the reference to the previous context.

Example 1

An example of contractual calculations is documented in the following. This is a transcript of a lesson in grade 7. After the transition of CDeb to CNum, the learners were confronted with a challenging task of stepwise controlled calculations for the first time. The female student Tanja has solved part b of this task correctly. The teacher (Ibu Misseri) marked her solution with the comment "Bagus!" (correct!):

Fill in the gaps!

d) $(((-74) + 15.243) + 74)$

$= ((15.243 + (-74)) + 74)$ K^+

Bagus!

$= (15.243 + ((-74) + 74))$ A^+

$= (15.243 + (74 + (-74)))$ K^+

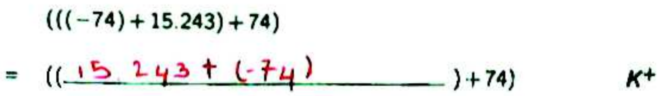
$= (15.243 + 0)$ I^+

$= (0 + 15.243)$ K^+

$= 15.243$ N^+

The teacher asked Tanja to present and explain her solution in front of the class. While explaining the solution, Tanja showed her ability to analyze the structure of the mathematical expressions and to provide the right justifications for each step of the calculation:

Tanja [*Tanja explains the first and the second line.*] Here K^+ is written. Here is minus 74 first, they write here minus 74. [*Tanja points to the number (-74) in the first line.*] I imagine it being a, and I imagine 15.243 being b. Thus, if, if it is K^+, then we turn it, we change it, we turn it. Thus, here I write minus 74 behind it and 15.243 here. [*She uses her pencil to point out the numbers.*] How do you find it, my friends, is my answer correct or false?



Tanja's explanation of the first transformation indicates an adequate understanding of the first term rewriting. She refers to the commutative law $(a+b) = (b+a)$ and conceives of the two numbers (74) and 15.243 as numbers that were put in the variables a and b. Originally, this formal notation was used to express that the order of bookings to be executed – in order to combine two accounts – does not affect the final balance. Tanja shows that she can properly apply this formal notation of the commutative law in a context-free example.

Example 2

Another example of using the stepwise controlled calculation is presented in Fig. 7. Here, one task from a test taken by the students in grade 7 is shown together with the solution provided by the student Martinus. The test was taken by the students in 2013, three months after starting to conduct the design research project in this class. The kite-symbol written by Martinus is used to justify the replacement of addition of 27 and (-14) with the result of this operation. The theorem T used as a justification for the last step of the calculation has to be reconstructed by the students. Here, the letter T does not indicate a theorem which is already known by the students. The learners have to "discover" it. There are several different ways to record this theorem. One of them is presented in the correct solution of Martinus:

Soal 2

a) Berilah alasan untuk setiap pengubahan bentuk! Di dalam satu pengubahan bentuk digunakan satu teorema T yang baru. Teorema ini telah dituliskan.

$$(((((-2) + (-(-2))) + 13) - 17) - (27 + (-14)))$$
$$= (((0 + 13) - 17) - (27 + (-14))) \qquad I^+$$
$$= ((13 - 17) - (27 + (-14))) \qquad N^+$$
$$= ((13 - 17) - 13) \qquad \Diamond$$
$$= ((13 + (-17)) - 13) \qquad D^-$$
$$= (-17) \qquad T$$

Task 2
a) Justify each term rewriting. For one term rewriting a new theorem T has been applied. This theorem has already been written down as an argument.

b) Tulislah teorema T.

$((a + (-b)) - a) = (-b)$

$(((((-2) + (-(-2))) + 13) - 17) - (27 + (-14))) = (-17))$

b) Write down the theorem T.

c) Bilangan mana yang telah ditempatkan di variabel?

(2, 13, 17, 27, dan 14) $a = 13$, $b = 17$

c) What numbers have been substituted for the variables?

Figure 7. Solution of Martinus.

The solution of Martinus indicates a correct understanding of reasoning in contractual calculations. If the last step is to be justified by a theorem, then this theorem must be applicable for other numbers as well. To find this theorem, the structure of the term $((13 + (-17)) - 13)$ and its relation to the term (-17) must be analyzed. Martinus recognized correctly that the number 13 occurs twice in the first term. He represented this fact with the variable a. He also recognized that the number in the second term is exactly the same as one number in the first term. He represented this relation with the variable b. Finally, he was able to indicate the numbers which have to be used for the substitution of these two variables in this theorem in order to get the right statement $((13 + (-17)) - 13) = (-17)$ as result.

The metaphor “contract” helps to understand the sense of the question on the theorem T. It also motivates the hypothetical and analytical thoughts concerning the structure of the given terms and the relations between them. The question was not to prove the theorem T. This would be the next task to assure that the last step in the calculation is correct in the sense of contractual agreements. Since a thought experiment with an interpretation of the term $((13 + (-17)) - 13)$ (for example in the microworld “Banking”) and an intuitive understanding of addition and subtraction show that the result must be

(−17), the theorem T appears to be plausible. Therefore, a proof of this theorem – based on the agreements included in CNum – would be a way to make sure that CNum does not contradict the expectations concerning addition and subtraction with integers.

The essential benefit of the metaphor "contract" in dealing with systems of concepts is that this metaphor helps to understand the sense of proofs for some facts that seem to be evident. One could think that proving such facts does not make sense since they are evidence-based and therefore they must be "right". But, in fact, when dealing with a contract, proving such facts has the aim to assure oneself that the given contract is a right result of the process of systematization and abstraction (cf. Cohors-Fresenborg, 2008).

Example 3

In the following, another task from the test taken by the learners in grade 7 is presented. The task is:
Task 5: Write down a proof for the theorem T13: $(b + (b - b)) = b$!

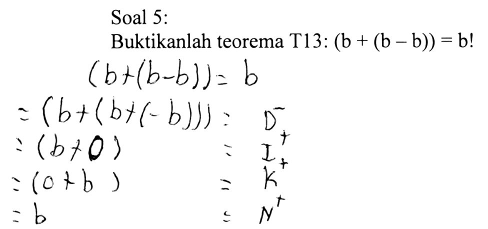


Figure 8. Solution of Karolinus.

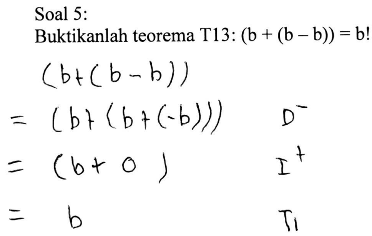


Figure 9. Solution of Ensa.

The solution of the male student Karolinus is syntactically incorrect. Since Karolinus rewrites only the term $(b + (b - b))$, he should have written the first line without the signs "$= b$". In each of the next lines all the signs "=" standing on the right side cause syntactical errors. Despite the errors, the solution indicates a correct thinking process of the student. Each term transformation is justified with the correct agreement.

Ensa's (female student) solution is correct. Unlike Karolinus, Ensa uses the theorem T1 proved before and therefore her reasoning is shorter than this of Karolinus. The use of T1 hints at the fact that Ensa sees a difference between the agreement N^+ $(0 + a) = a$ and the theorem T1. This is an important precondition for understanding how an axiomatic system works.

An interpretation of the theorem T13 and an intuitive understanding of addition and subtraction let assume the plausibility and the "correctness" of this theorem, but the intuition and reference to numeric examples cannot serve as justifications in a proof. Neither Karolinus nor Ensa used their intuition and pre-knowledge as a simple justification that the term $(b - b)$ has the value 0. Instead, they justified it by recourse to the definition of subtraction (D^-) and to the agreement I^+. The theorem $(a - a) = 0$ had not been discussed in the lessons before the test was written. The metaphor "contract" hinders the use of intuitively understood and experienced facts (but not revised by the means of a proof) as reasons in an argumentation. Intuition tells one what should be accepted as "correct" or "true" and it motivates the search for a contractual justification but it cannot be accepted as an argument when contractual dealing with proofs.

Example 4

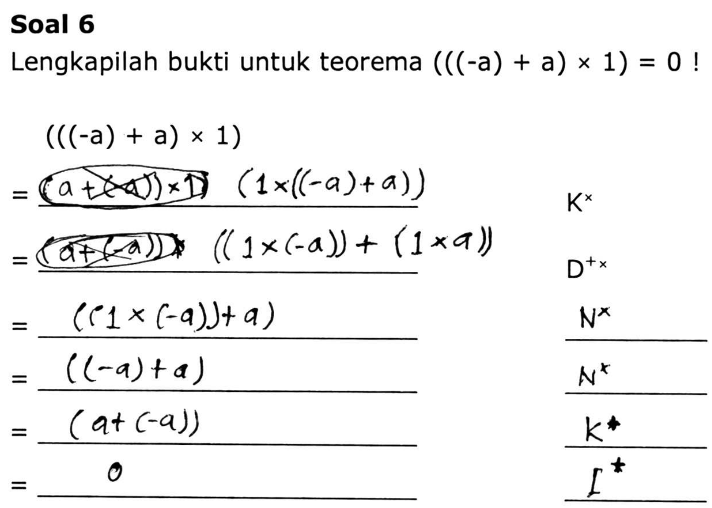
An example of the first extension of the contract for dealing with numbers is presented in Fig. 10. Here, new agreements for multiplication and the distributive law are included. The contractual basis for the three arithmetic operations includes more agreements (axioms) then the one for addition and multiplication only. However, the cognitive behaviour related to the use of this new contract, in particular to contractual calculations and reasoning, is in both cases the same. Thus, one benefit resulting from the use of the metaphor "contract" is that the learners know what needs to be done in general. The experience in dealing with multiplication, division and with the multiplicative inverse has to be reflected, systematized and represented in a precise form. As a result of this process some agreements will be recorder in the contract and other will be written down as theorems. In this process, the students should be guided to reinvent the definition of division and the agreement $I^\times$ for the multiplicative inverse.

$\mathcal{N}^+$	$(0 + a) = a$	$\mathcal{N}^\times$	$(1 \times a) = a$
I^+	$(a + (-a)) = 0$		
$\mathcal{K}^+$	$(a + b) = (b + a)$	$\mathcal{K}^\times$	$(a \times b) = (b \times a)$
$\mathcal{A}^+$	$((a + b) + c) = (a + (b + c))$	$\mathcal{A}^\times$	$((a \times b) \times c) = (a \times (b \times c))$
$\mathcal{D}^-$	$(a - b) = (a + (-b))$		

$$\mathcal{D}^{+\times} \quad (a \times (b + c)) = ((a \times b) + (a \times c))$$

Figure 10. An extended version of the contract for dealing with numbers.

The use of this contract is presented in Fig. 11. There, one student's solution for a task requiring proof activities is presented. The task was given to the students in the test mentioned before, in Example 2 and Example 3. Here, the task is: *Task 6. Complete the proof of the theorem* $(((-a)+a)\times 1) = 0$!



Soal 6

Lengkapilah bukti untuk teorema (((-a) + a) × 1) = 0 !

$(((-a) + a) \times 1)$	
$= \cancel{(a + (-a)) \times 1)}\ (1 \times ((-a) + a))$	$K^{\times}$
$= \cancel{(a + (-a))}\ ((1 \times (-a)) + (1 \times a))$	$D^{+\times}$
$= ((1 \times (-a)) + a)$	$N^{\times}$
$= ((-a) + a)$	$N^{\times}$
$= (a + (-a))$	K^{+}
$= 0$	I^{+}

Figure 11. Marlen' solution.

The proof written by the female student Marlen is correct. Her solution indicates some mistakes in rewriting the term according to the given paragraphs and her correction of these incorrect terms. Each of the next steps is justified with the right paragraph from the extended version of the contract.

Example 5.1

The solutions to the tasks presented in the above examples and in the Introduction (Task 1 and Task 2 in Fig. 1) present correct contractual calculations and proofs. It is worth mentioning that for this kind of tasks there is no fixed plan telling the students how to find the way from particular starting point (an arithmetic term or a theorem) to the required finishing point (the value of the term or the proof for the given theorem). There is no fixed plan telling them what to do at each step and therefore their solutions indicate more than simple instrumental understanding (Skemp, 1977). Solving these tasks, particularly writing a proof, requires awareness of what to do in general and what to use as the basis knowledge.

In the Introduction part of this paper, the **Question 1** is presented there to give the reader of this paper an idea of how systemic thinking can be displayed in school mathematics, in the realm of integers. In the following paragraphs the answers of all 22 students from one project class (grade 7) will be discussed. The class consisted of 23 students, but one of them was absent on this day when the question was posed. It seems that 77% (17 out of 22) of the pupils are aware of what contractual proving consists of. Their meta-mathematical knowledge expressed in their statements to Question 1 can be interpreted as an indicator of their systemic thinking in a small part of school mathematics.

Question 1 ("Do you agree with Peter that proving is more difficult than calculating? Give reasons for your answer.") presented in Introduction was a part of the questionnaire that the students had to fulfill two month after the start of the project. This question asks whether proving is more difficult than calculating. An answer to it requires that the learners reflect on their own experience in contractual calculating and proving, and that they give reasons for their opinion. Since the aim is to find out how the students think, the analysis of their written answers has an interpretative character. An answer "Yes, ..." can be justified with various reasons, indicating that the student does not know what (contractual) proving consists of or that he or she is conscious about some cognitive challenges differentiating proving from calculating (despite their methodological similarities). Similarly, an answer "Yes, ..." can be justified with different reasons indicating that the learner knows exactly what to do in contractual calculating and proving, or that he or she understands the idea of neither of these activities, and therefore assesses both of them as understandable and very difficult.

As the result of an interpretative analysis, the answers given to Question 1 can be classified into 3 categories.

Category 1: Answer "No ..." with reasons indicating that the student is aware of some methodological similarities between contractual calculating and proving, or he or she emphasizes the cognitive endeavor needed in both cases to justify each step.

One of these students reasoned that calculating is more difficult than proving and not vice versa. The reason is that when you calculate with numbers you have to make some additional cognitive efforts to think about the values of these numbers and to calculate with these values.

Altogether, 9 answers can be classified into this category. The reasons mentioned by the learners are the following:

– In both cases (calculating and proving), it is clear what can be used. In both cases the same contract (the same paragraphs) and the same agreements from outside the contract (theorems) can be used to justify each step.

– Both (calculating and proving) require justifications, and calculations can be justified too. In both cases, the same technique can be used.

– If one concentrates and works carefully, one can solve both correctly.

One of the answers classified into Category 1 is presented in the Introduction. Another example of a male student is:

> *No [proving is not more difficult than calculating], because if we cannot prove we cannot know the result of what we calculate.*

In an interview conducted by the teacher, the student explained his written answer as follows:

> *Theorems can be used as reasons for calculation steps. If I have not proven one theorem, I cannot use it as a reason for my calculation step. For example, if I have not proven the theorem* $(a - 0) = a$*, then I cannot simply justify in one step that* $(2 - 0) = 2$*. And therefore proving is not more difficult than calculating.*

The answer indicates the awareness of the student that contractual calculations can be justified by the means of theorems, it means of proved "rules" allowing the students to make his calculations shorter. It also shows that the student conceives proofs as a means to justify mathematical statements and not only as a special kind of mathematical exercises. The reason why proving is not more difficult for this student than contractual calculating is still, however, not elaborated in this answer. Since, in the further part of the questionnaire, he also wrote correctly what can be used to write a proof ("agreements from the contract like for example N^+ $(0 + a) = a$; agreements from outside the contract, it means theorems; the kite-rule"), one can assume that the learner is aware of the technique used for contractual calculating and proving.

Category 2: Answer "Yes ..." with reasons indicating that the pupils are aware of an additional cognitive endeavor differentiating proving from calculating. These answers suggest that the students are aware of methodological similarities between contractual calculating and proving (because they mention in the questionnaire what can be used in calculating and in proving), but that they also see some cognitive challenges (concentration, planning and the need to develop a strategy) in proving and therefore agree that proving is more difficult than calculating.

8 answers can be classified into this category. The reasons mentioned by the students are the following:

- When we calculate, we make some operations with the given numbers and go forward to the final result. If we write a proof, we know what should be there at the end and we try to find a way and reasons to justify the final result.
- When we write a proof, we have to look ahead and make an effort to figure out how we can get to the final "result". Furthermore, we have to justify each step in our plan.

One of the answers classified into Category 2 is the following one:

> *Yes [proving is more difficult than calculating], because if we calculate, we use the given numbers. If we make a proof, we have to use the appropriate contract to justify the already given final "result" as it is given there in the theorem.*

The answer indicates that the student knows what to use in proving. The difficulty in proving is to justify the given final "results"; it means the algebraic expression standing usually on the right side of the equality sign in the given theorem. The algebraic expression standing on the other side is used as the starting point in the proof.

Category 3: Answer "Yes ...." without reasons or with the reason that it is not clear what can be used in proving.

5 students agreed that proving is more difficult than calculating but they did not write reasons for their answers (4 of them) or they wrote that they simply do not know how to write a proof (1 of them).

In this case one cannot say that the students are aware of some methodological similarities in contractual calculating and proving. Additional interviews are needed to describe the cognitive profile of these students.

Example 5.1

The questionnaire given to the learners included also the two following tasks (presented also in Fig. 1 in Introduction) and **Question 2** related to them:

Task 1. Calculate the value of the term:	**Task 2**. Write a proof for the theorem:
$((127 + (-38)) + 38)$	$((a + (-b)) + b) = a$

Question 2. Was for you one of these tasks more difficult than the other? If yes, which one was more difficult? Give reasons for your statement.

It is interesting to mention that only these students whose statements to Question 1 concerning the difficulty of proving were classified into the category 3 solved at least one of the above tasks incorrectly or did not solve it at all. This was the case in 4 out of 5 answers in this category. All of these 5 students stated that Task 2 is more difficult for them than Task 1. Their statements are not elaborated.

All other students (whose statements to Question 1 concerning the difficulty of proving were classified into the category 1 or 2) solved both tasks correctly or their solution included at most a few syntactical errors or one incorrect justification for one step in the proof or calculation. All learners whose statements to Question 1 were classified into the category 1 assessed Task 1 and Task 2 as easy and therefore having for them the same degree of difficulty. They did not give other reasons than those already mentioned in their answers to Question 1.

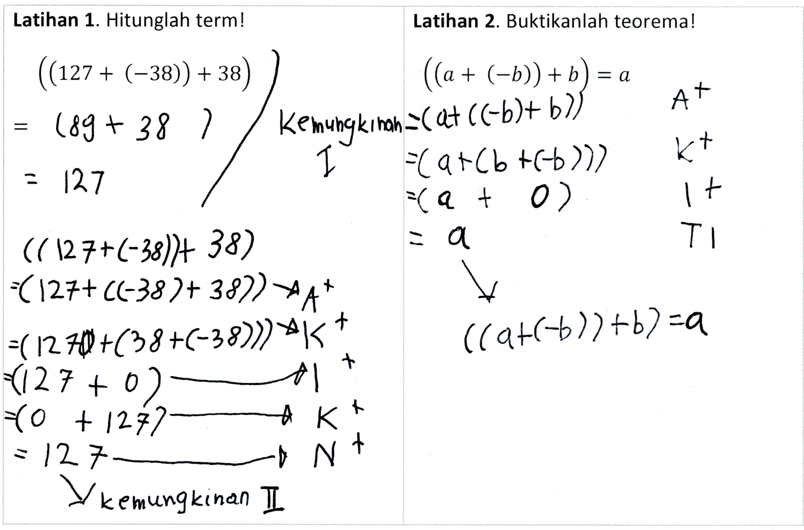
3 out of 8 students whose statements to Question 1 were classified into the category 2 assessed Task 1 and Task 2 as easy and therefore having for them personally the same degree of difficulty. The other 5 stated that Task 2 is more difficult than Task 1. They did not give other reasons than those already mentioned in their answers to Question 1.

An example of such a statement is the following one given be a male student:

> *I think that Task 2 is more difficult than Task 1 because I have to concentrate in order to get to our goal, it means to a.*

This answer indicates that the student is aware of some additional cognitive achievements needed to create the proof. Furthermore, he is also able to perform this achievement as it can be seen in his solutions to Task 1 and Task 2 (Fig. 12).

It is noticeable that the student solved Task 1 using two different strategies. First, he made the calculation without changes in the structure of the term. It is unclear why he did not justify each step of this calculation with the kite-agreement. In the second approach, he changed the term in order to use the fact that there is a pair of two opposite numbers (38 and (-38)). Consequently, he saved the effort to think about the values of all numbers given in this term. Both strategies are correct and both can be used in agreement with the contract. The students can decide by themselves which strategy gives them more confidence. The chain of reasons used by this student to prove the given theorem is shorter than this one used in the second way of calculating the term with numbers. In the proof, the student used Theorem 1 (T1) instead of K^+ and N^+.


Latihan 1. Hitunglah term!

$((127 + (-38)) + 38)$

$= (89 + 38)$

$= 127$

Kemungkinan I

$((127+(-38))+38)$

$=(127+((-38)+38)) \to A^+$

$=(127+(38+(-38))) \to K^+$

$=(127+0) \to I^+$

$=(0+127) \to K^+$

$=127 \to N^+$

Kemungkinan II

Latihan 2. Buktikanlah teorema!

$((a + (-b)) + b) = a$

$=(a+((-b)+b))$ A^+

$=(a+(b+(-b)))$ K^+

$=(a+0)$ I^+

$=a$ TI

$((a+(-b))+b)=a$

Figure 12. Contractual calculation and proof – solution of Willy.

5 Discussion

The aim of the reorientation of teaching mathematics on the east Indonesian island Sumba is to enable the learners to acquire new cognitive habitus in learning mathematics. To make this possible, the learners should be facilitated to construct a kind of a cognitive mathematical operating system (CMOP) in their minds. The metaphor CMOP represents the intended structure and organization of access to the constructed mathematical knowledge. The decision was made to start the construction of CMOP with axiomatizing. It means that the learners should understand the process of axiomatization and, consequently, be able to use the axiomatic view as a cognitive tool guiding their thinking when dealing with semantic networks of concepts. This specific view should be gained by the learners with the metaphor "contract" which represents how to deal with mathematical knowledge and abstract concept formation in an understanding way.

The first content in grade 7 (introduction of negative numbers) is designed for this aim so that students are provided with opportunities to learn how intuitive knowledge is ultimately abstracted to an axiomatic theory in mathematics and represented formally. Students' experiences with this content also include mathematical activities of implicit and explicit defining and proving. This experience is gained first when dealing with transactions of debts and assets in a fictitious bank and, subsequently, it is abstracted into a contract dealing with integers. In this contract (system of axioms for Abelian groups), further operations on numbers are defined explicitly and common conventions describing how to deal with algebraic signs are formulated as theorems which are proven in parts.

One central aspect in this framework is a role play game in which the contract for dealing with debit and credit is being developed and represented in a formal language. At first glance, the written result of the role play game (i.e. the contract dealing with debit and credit) appears to be very "abstract", "formal" and "extraneous to school". In order to understand this contract, the reader must be able to reconstruct (in a thought experiment) the abstraction processes from the realistic context of banking (*realistic* in terms of *imaginable* for the learner and not necessarily *real* as part of the learner's everyday experience of reality; cf. Gravemeijer et al. 2003, p. 53) to the formal representation of properties of banking operations. The success of this thought experiment depends on the reader's precise thinking about his own thinking (meta-thoughts).

The intended understanding and cognitive habitus of the learners, in particular the axiomatic view in the sense of Freudenthal (1963; 1968, p. 7) and some aspects of the systemic thinking in the sense of Sierpinska (Sierpinska et al., 2002; Sierpinska, 2005), are documented in students' written works. The aim of the qualitative analysis of students' solutions is not to provide the reader with a quantitative explanation of the teaching success.

The analysed solutions indicate competences of the individual students in comprehensible dealing with formal representations, gradually controllable (contractual) calculations, proofs and argumentations. Of course, in order to facilitate students' systemic thinking, further opportunities for projecting the cognitive behaviour related to the use of the metaphor "contract" into new mathematical context are necessary.

The starting situation of the design research project on Sumba was that authorities responsible for teacher training on Sumba doubted prior to the commencement of the project that Sumbanese learners could have the ability to autonomous reasoning, critical controlling and to an understanding of formally noted mathematics. Therefore, it was important to provide a "proof of existence" for the feasibility first which was achieved by the first diagnosis. The achieved effects encourage proceeding with the research and development work. Consequently, authorities responsible for teacher training on Sumba and other Indonesian islands were astonished at the extent to which even the female students were able to engage discourse and argumentations, as well as to initiate such discourse and to reflect on its consistency. On Sumba, such cognitive dominance of women does not match the traditional role allocation of women in society.

A long-term efficacy in increasing mathematical competences of students on Sumba cannot only result from practicing contractual calculations and proofs as a metaphor of gradually controllable argumentations in mathematics. Rather, a metacognitive-discursive classroom culture, which is oriented towards mental processes of the learners, plays a central role. Our observations so far (Nowinska, 2014) have shown that consequences of these methods are represented in student productions on Sumba in a similar manner as published in the previous feasibility study (Kaune et al., 2012; Kaune & Nowinska, 2011).

Thus, it can be stated that the repeated implementation and evaluation of the teaching concept presented in this paper has shown that the intended goals are achievable as effects of teaching and learning mathematics on Sumba. However, the question of feasibility has to be separated from the question of sustainability of knowledge and of the competences of the learners. Formulating statements regarding the sustainability requires – amongst other things – an investigation to be conducted with a certain time lag after the implementation of the designed means and methods in the project lessons. Moreover, given the long-term aim of the intended reorientation, a broader content has to be considered. This is necessary to elaborate students' "acts of understanding" (Sierpinska, 1994) in other fields of mathematics, where the metaphor "contract" will be used to facilitate systemic thinking. Therefore, more attention is put on the evaluation of the sustainability of the effects in the continuation of the development and research work. These shall include further mathematical contents taught in grade 7 and in grade 8 starting in school term 2014/15. With the long-dated work (2017) it is aimed to gain more empirical findings and to meet the requirement "ecological validity, that is to say, (the description of) the results should provide a basis for adaptation to other situations. The premise is that an empirically grounded theory of how the intervention works accommodates this requirement" (Gravemeijer & Cobb, 2013, p. 103). This is also a way to give the reader an opportunity to gain insight in how the theoretical considerations and the designed learning environment work in a real situation. "By describing details of the participating students, of the teaching learning process, and so forth, together with an analysis of how these elements may have influenced the whole process, outsiders will have a basis for deliberating adjustments to other situations" (ibid., p. 103). However, whether the teaching concept presented in this paper will encounter acceptance or rather resistance in another situation depends on conception of teaching mathematics underling the new situation and on a certain philosophy and view of mathematics related to this conception.

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Pozytywny wpływ metafory „umowa" na rozumienie procesu tworzenia pojęć w matematyce

Streszczenie

Artykuł przedstawia koncepcję i wstępne efekty projektu rozwojowo-badawczego prowadzonego w klasie siódmej w szkole indonezyjskiej na wyspie Sumba. Celem projektu jest wspieranie uczniów w rozwoju rozumienia w procesie uczenia się matematyki i, w konsekwencji, w przygotowaniu ich do życia zawodowego.

Istotną cechą charakterystyczną tego projektu jest wykorzystanie problematyki wprowadzenia liczb ujemnych jako kontekstu motywującego uczniów do rozwoju u nich pożądanych postaw poznawczych, w szczególności do krytycznego myślenia, kontroli i refleksji. Opracowana w ramach tego projektu

koncepcja uczenia się i nauczania matematyki wykorzystuje pojęcie „umowa" jako metaforę reprezentującą specyficzny sposób myślenia w odniesieniu do posługiwania się systemem pojęć i ma na celu rozwój tego sposobu myślenia u uczniów. Chodzi tu o myślenie systemowe (systemic thinking) w sensie opisanym przez Sierpinską (Sierpinska et al., 2002, Sierpinska, 2005) oraz myślenie aksjomatyczne w sensie opisanym przez Freudenthala (1963, 1968, s. 7). W celu skontruowania tej metafory w umyśle uczniów opracowane zostało nowe „środowisko uczenia się" (learning environment) wprowadzające uczniów w problematykę liczb ujemych i działań na liczbach całkowitych. Środowisko to wykorzystuje kontekst transakcji bankowych, długów i pieniędzy lokowanych na koncie w fikcyjnym banku. Z uwagi na ten kontest, uczniowie pojmują to środowisko jako pewnego rodzaju grę w kontekście realistycznym (w sensie opisanym w pracy (Gravemeijer et al., 2003, s. 53)), a nie jako kontekst realny nawiązujący do otaczającej ich rzeczywistości. Aktywności uczniów w tym środowisku nie są aktywnościami konkretnymi, lecz raczej wyobrażonymi. Są one pewnego rodzaju experymentem umysłowym.

Problem w stworzeniu tego środowiska polegał na tym, aby znaleźć dostęp do intuicyjnej wiedzy uczniów na temat liczb całkowitych i zrekonstuować intuicyjnie pojęty przez nich sposób postępowania z liczbami całkowitymi (a nie na tylko na tym, aby umożliwić uczniom szybkie nabycie sprawności w wykonywaniu działań). Akcent położony jest tutaj na rekonstrukcję wiedzy i jej precyzyjną reprezentację rozwiniętą do „umownej" (aksjomatycznej) definicji dodawania, odejmowania i addytywnej odwrotności liczby w zakresie liczb calkowitych. Metafora „umowa" dostęp ten umożliwia w następujący sposób: w środowisku utworzonym w celu wprowadzenia liczb ujemnych, uczniowie konstruują umowę z fikcyjnym bankiem po to, aby w oparciu o nią zmusić bank do wykonywania i zapisywania transakcji bankowych w sposób odpowiadający oczekiwaniom uczniów, zgodnie z ich intuicyjną wiedzą dotyczącą pieniędzy i długów.

W oparciu o prace pisemne uczniów w artykule omówione są efekty wynikające z posługiwania się metaforą „umowa". Z uwagi na cel, jakim jest wyeksponowanie wpływu tej metafory na rozumienie przez uczniów procesu konstruowania pojęć w matematyce oraz sposobu posługiwania się powiązanymi ze sobą pojęciami, analiza prac uczniów ma charakter jakościowy. Na wybranych przykładach wyjaśniony został sposób rozumowania uczniów w zakresie wykonywania działań na liczbach całkowitych, sposób argumentowania, dowodzenia i konstruowania twierdzeń. Prace pisemne wskazują na to, że uczniowie nauczani według omówionej w tym artykule koncepcji pojmują dowody jako sposób weryfikacji przypuszczanych lub zaobserwowanych zależności (a nie tylko jako pewnego rodzaju ćwiczenia matematyczne) oraz wykazują świadomość metodologiczną w zakresie wykonywania działań, argumentowania i dowodzenia. Analiza tych prac prowadzi do stwierdzenia, że odpowiednio użyta w procesie nauczania metafora „umowa" stała się dla uczniów narzędziem poznawczym umożliwiającym uczniom doświadczanie matematyki jako „human activity" (Freudenthal, 1991, s. 14) – w szczególności jako „activity of discovering and organizing in an interplay of content and form" (ibid., s. 15).