ANNALS OF THE POLISH MATHEMATICAL SOCIETY
5TH SERIES: DIDACTICA MATHEMATICAE 35 (2013)

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Specific skills necessary to work with some ICT tools in mathematics effectively

Abstract: ICT tools have become part of teaching mathematics and focus of much research. Generally, the research has shown that they can be potentially benefitting to pupils' learning of mathematics, however, some limitations have been brought about. It has transpired that learning does not come about without conscious support by teachers and suitable tasks. The aim of the article is to a) summarise some research results which concern two most widely used ICT tools – graphing calculators and dynamic geometry software, mainly from the point of view of what problems pupils faced and what caused the ICT tools not to be used to best advantage, b) present specific skills which pupils must develop in order to overcome these problems, c) suggest some suitable tasks to develop the skills.

1 Introduction

For several decades now, PCs and other ICT tools have been used at schools. These tools are no longer the focus of teaching of informatics but have become an important tool of teaching in many other subjects including mathematics.

The growing use of PCs in mathematics is documented by the report of *European Commission* (Empirica, 2006). It states, among others, that 80% of teachers who use computers for the preparation of lessons are teachers of mathematics, informatics and sciences. 22% of them use PCs in their lessons, too, which is a big percentage when compared to teachers of other subjects. The report also points to the fact that the most widely used programs at schools included in the survey are the ones aimed for mathematics.

Acknowledgment: The paper was supported by the project PRVOUK P15 *The school and teaching profession in the context of growing educational demands.*

The integration of ICT in the teaching and learning process brings about the increase of pupils' motivation and performance in tests of knowledge and skills as has been shown by research. For example, Balanskat et al. (2006) in their meta-analysis looked into 17 qualitative and quantitative research projects on the influence of ICT on the teaching in European schools. The main results consist of the existence of a positive correlation between the length of ICT use and pupils' performance in mathematical tests. Schools which are well equipped with technologies reach better educational results than schools with poor equipment. Teachers, namely more than 86%, agreed that using computers and Internet in teaching increases pupils' motivation.

Positive influence on learning is also evidenced in the study of ICT integration at primary schools in 27 countries of European Union (Balanskat, 2009). The goal of the project was to carry out a comparative analysis of strategies for the use of ICT in teaching at primary schools in the included countries. The benefit of technology including the improvement of pupils' study results, however, manifested itself only if the ICT integration was combined with well thought-out conception and strategy. Research has also pointed to the well-known fact that teachers often feel the lack of methodical support and support from the point of view of the development of competences related to the use of technology.

2 ICT tools in the teaching of mathematics

It is generally presumed that with the use of ICT there will be a change of curriculum and teaching methods. For example, research on the integration of graphing calculators from the 1990s showed that "the paper and pencil skills can be taught in a shorter time, after developing concepts, without a loss of achievement on skills" (p. 43). It also pointed out some necessary changes in curriculum which concern an emphasis on topics such as estimation, rounding and graphic solutions to mathematical problems (Dunham, 2000). Ruthven and Hennessy (2002) say that an important prerequisite for successful integration of ICT is a mutual connection of technologies, curriculum and school practice. Using technologies in mathematics lessons should be seen "as a systemic interaction of a number of dimensions" (Lagrange, 2005, p. 144). The integration obviously represents a complex process in which the availability of technologies (their features and design) permeates curriculum and ways of using them in teaching mathematics. There does not seem to be a "quick way to deal with the complexity of educational use of technology" (Lagrange, 2005, p. 180). On the other hand, a simple change of curricular documents will not guarantee a positive influence of technologies. Tondeur et al. (2007) point out that the integration of ICT in the curriculum does not necessarily result in changes to the teaching process. Moreover, there can be a clash between the curriculum and the actual use of ICT at schools caused by the influence of the climate of a concrete school or teachers' approaches to technologies. Other studies show that technologies, namely computer algebra system and dynamic geometry, are used mostly within the current curriculum (Laborde, 2001, Gawlick, 2002, Ruthven, 2008) while a bigger emphasis is put on experimenting with mathematical objects, discovering and verification of hypotheses (Laborde et al., 2006; Kasten, Sinclair, 2009; Marriotti, 2011). Next, we will concentrate on the types of tasks used with the help of ICT.

There is a growing body of research which focuses on the types of mathematical tasks set within the ICT environment. One possible typology concerns the role *technology plays in the solutions to the tasks* (Böhm et al., 2004):

- Tasks where the use of PC and software is of little or no help; the solution is faster by-hand.
- Tasks that are solved faster or even trivialised by PC.
- Tasks testing the ability of using PC and software.
- Tasks starting from traditional ones that are extended to PC tasks (e.g., by including formal parameters or using realistic data).
- Tasks difficult, time consuming or impossible to solve without PC.

Zbiek et al. (2007) distinguish *technical* and *conceptual mathematical activity.* The former includes geometric construction and measurement, numerical computation, algebraic manipulation, graphing, translation between notation systems, solving equations, creating diagrams, etc. The latter involves understanding, communicating, and using mathematical connections, structures, and relationships, defining, conjecturing, generalizing, abstracting, etc. The authors stress that neither type of activity is more mathematically meaningful than the other.

Finally, Zbiek et al. (2007) distinguish between the use of technology as *amplifier* and as *reorganiser.* The former accepts the goals of the current curriculum and works to achieve those goals better. The latter changes the goals of the curriculum by replacing some things, adding others, and reordering still others.

The research has shown that in order to utilise the potential of ICT tools, pupils must get a conscious support to develop *certain skills specific to the type of the ICT tools.* Note that by that we do not mean the skills connected to the initial incompetence of pupils to use an unknown ICT tool. Many point out that at the beginning, pupils tend to focus on the technical realisation of, for example, a geometric construction rather than on the mathematical basis of the solved problem (e.g., Hölzl, 1994, 1996, quoted in Gawlick, 2002). This gradually fades as the users gain more experience with the tool, but the specific skills we have in mind can manifest themselves in an experienced user's work as well. In other words, the user is often not aware of his/her deficiency, is sure of his/her competence to use the ICT tool and thus cannot use the tool to best advantage and as a consequence, develops insufficient mathematical knowledge.

In the following text, we will concentrate on two most widely used ICT tools – graphing calculators and dynamic geometry software – for which three questions will be addressed:

a) What does research say about problems pupils face when using ICT which prevent them from using the ICT tools to best advantage?

b) What are the specific skills which pupils must develop in order to overcome these problems?

c) What are suitable tasks to develop the skills?

ad b) The specific skills presented in this paper are based both on research literature from a) and on the author's long-term experience of using ICT in the education of (student) mathematics teachers and on a four-year teaching experience at a secondary school.
ad c) The tasks as well as their solutions come from the data base of the author's teaching (student) mathematics teachers and secondary school pupils. Thus by pupils in the text below, we will mean upper secondary school pupils.

3 Graphing calculators

3.1 Research on graphing calculators

Research on the integration of graphing calculators confirms that they have positive influence on pupils' attitudes to mathematics, success in problem solving and acquisition of new knowledge, namely above all of mathematical concepts. This influence is more important if pupils can use graphing calculators in tests of mathematical achievement (Ellington, 2003; Gersten et al., 2008). At the same time, studies pointed to the necessity of the development of skills which pupils invoke when working with graphing calculators.

There are differences in presenting results in both calculation and graphic modes of the calculator as compared to the mathematical theory. They can function as barriers for pupils to interpret numeric and graphic results correctly, which is further aggravated by the fact that pupils tend to accept the calculator results without scrutiny. Many authors found out that it was a problem mainly for weaker pupils (e.g., Ward, 1998).

Burrill et al. (2002) point out that "the use of handheld graphing technology may further extend students' misconceptions about mathematical concepts, such as increased confusion between rational and real numbers" (p. *vi*). Doerr and Zangor (2000) met the same problem when goniometric functions were investigated, in which values of independent variable were given as rational multiples of number π, however, the calculator worked with rational approximations of these multiples. They also noticed that when solving real-life problems, pupils did not round the numbers and worked with the maximum possible number of decimal places visible on the screen.

In their research, Ward (1998) and Berry and Graham (2005) notice that pupils tend to display the graph of the function immediately after entering its equation without thinking about a suitable range of coordinates. So they display the graph in the previously used range which might be unsuitable for the new graph. Ward (1998) highlights the fact that pupils are often persuaded that to display the graph of function with all its important points (zero points, extremes, etc.), it is enough to use a sufficiently wide range of coordinates. In Ward's research (1998), pupils repeatedly used the button for the automatic enhancement of the range of coordinates without considering properties of the function: "very little critical thinking and much button pushing was occurring".

Much research further highlighted pupils' misconceptions of functional domain and of the range of coordinates on the calculator display (Dunham, 2000). Mitchelmore and Cavanagh (2000) conclude that pupils mostly do not realise the influence of the range of coordinates on the shape of the graph of the function. In their research, the problem of expressing irrational values in the graphic mode appeared as some pupils repeatedly tried to get exact values of zero points by changing the range of coordinates even though the coordinates were irrational.

Other research (Ward, 1998; Dunham, 2000; Berry, Graham, 2005) revealed pupils' insufficient understanding the influence of the choice of scale on the axes (with regard to the number of pixels on the axes), especially when the scales on both axes differed. It can be caused by the fact that the marks on the axes are not labelled on the calculator screen and/or by the pupils' little experience with the shapes of graphs of elementary functions (Mitchelmore, Cavanagh, 2000; Berry, Graham, 2005). This misunderstanding can lead to pupils' misconceptions in terms of the relationship between the properties of the function and their manifestation in the graph.

In order to overcome the above difficulties, Goulding and Kyriacou (2008) suggest using cognitive conflict, i.e., presenting pupils with a "puzzling image" (e.g., displaying a part of the parabola which, due to the choice of the range of coordinates, looks like a line segment) and thus encouraging them to work through their misconceptions. "Students need to be alert to these potential sources of confusion and given good access to the technology so that they can develop familiarity" (p. 16). One of the causes of pupils' misconceptions in terms of the interpretation of the displayed graph can be an insufficient number of problem situations of that kind that they meet. Teachers should use them as much as possible (Ward, 1998; Dunham, 2000). Burill et al. (2002) formulate it succinctly: "students would benefit from confronting limitations of the technology and considering how to make more effective use of the technology. Attempting to explain these limitations can lead to better mathematical understanding." (p. *vii*) More research which deal with the barriers of integration given by the properties of calculators is presented in Ward, 1998) and (Dunham, 2000).

3.2 Specific skills for the use of graphing calculators

From the point of view of numerical computations, a graphing calculator is a computation aid which is used in the same way as a calculator without a graphic mode. For work in a numerical model it is therefore necessary for pupils to acquire skills connected to mathematical calculations; namely, place value estimation of results, rounding results including the judgement whether the acquired value is realistic. When using calculators, it is important that pupils realise that they use a certain representation of real numbers. In the case of infinite and non-periodic decimal expansion of irrational numbers, the calculator works with a finite number of digits; in fact, with a rational approximation of irrational numbers. Calculator usually works with a certain number of significant digits, however, the display only shows 10 significant digits at maximum. Thus the calculator cannot precisely show real numbers which are rational with a periodic decimal expansion and irrational numbers. In a graphic mode, this problem presents itself, for example, in determining coordinates of a given point on the calculator display.

An important prerequisite of more effective work in a graphic mode is the development of skills related not only to numerical calculations but also to geometric imagery and to displaying mathematical objects such as graphs of functions, statistic diagrams, etc. graphing calculators *Casio fx 9860* and *TI 83* are used in illustrations below. Tasks 1 to 9 are suitable for secondary school pupils.

3.2.1 Interpretation of graphs

The interpretation of displayed graphs of functions or curves in view with mathematical theory is sometimes complicated mainly due to the low resolution of a graphic display. Namely, the pupil has to recognise a 'jagged' curve correctly as a, for example, graph of linear function (Fig. 1), $y = 0.2x - 0.5$. However, the experience shows that this does not cause troubles for pupils.

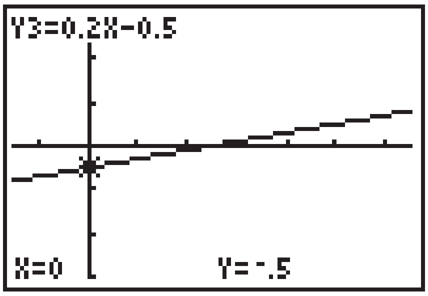

Figure 1. *TI-83*, graph of a linear function.

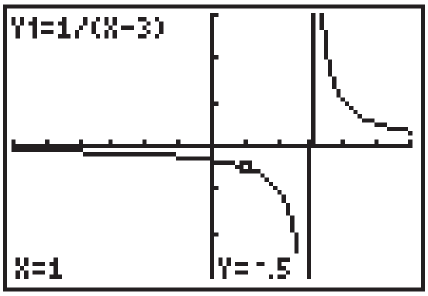

Figure 2. *TI-83*, hyperbola as a graph of a function.

When displaying graphs of functions it is necessary to develop specific skills characteristic for displaying graphs of functions whose domain is made of the union of disjunctive intervals. To illustrate these specific skills, mostly tasks of the second ('tasks solved faster or even trivialised by ICT') and third ('tasks testing the ability of using PC and software') types of Böhm et al.'s classification (2004) will be given. These types are suitable both for developing the skills and for diagnosing them. The last two types of the classification are, in graphing calculators without CAS, connected to higher mathematical demands of the task rather than to specific skills for their use.

Task 1: Graph function $f\colon y = \frac{1}{x-3}$ (Fig. 2).

Especially when they had little experience of using a graphing calculator, pupils tended to think that the displayed vertical line was a graph asymptote. In fact, it is a line connecting the 'last' displayed point with the x-coordinate $x < 3$ and the 'first' displayed point with the coordinate $x > 3$. This line is displayed thanks to the selected adjustment of coordinates on the display in which no pixel on the screen has the coordinate $x = 3$ when at the same time, the option *Connected* (i.e., points of the graph are to be connected) is used for displaying the graph of the function.

As already stated, the calculator setting can be the cause of pupils' interpretation of the graph of a discontinuous function as continuous.

Task 2: Graph function $f\colon\ y = \frac{x^2+x-2}{x-1}$ (Fig. 3).

Some pupils displayed the graph straight away and used the option *Connected.* They got the graph in Fig. 3 (using a simple modification of the formula, we get $y = \frac{(x+2)(x-1)}{x-1}$). At first sight, this graph seems to be the graph of a continuous function, no pixel on the screen with the standard range of coordinates has the coordinate $x = 1$. Only when the pupil identified the point of discontinuity from the formula, he/she could verify this by calculating the functional value for $x = 1$ for which the calculator did not return any value y (Fig. 3, bottom part of the display). In this example, not even removing the option *Connected* and setting the coordinate system in which the pixel with x-coordinate 1 existed helped weaker pupils with the correct interpretation of the graph (Fig. 4). Due to the shape of the graph on the screen which is influenced by its low resolution some pupils did not recognise the point of discontinuity.

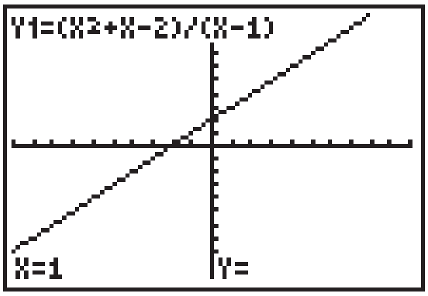

Figure 3. *TI-83*, graph of a discontinuous function (*Connected*).

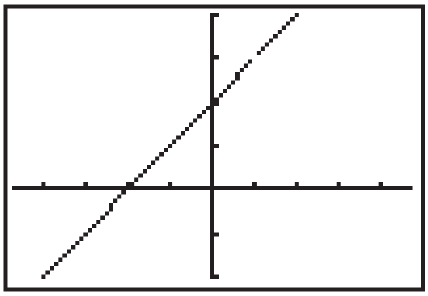

Figure 4. *TI-83*, graph of a discontinuous function (without *Connected*).

The shape of the function and interpretation of its properties from the graph is also connected to adjusting the range of coordinates on the calculator screen.

3.2.2 Adjusting the range of coordinates on the display, scaling

When displaying graphs of functions, pupils work with a certain model of Cartesian coordinate system as the graphic mode is realised through displayable points, so called pixels. The coordinates of points-pixels depend on the number of pixels in horizontal and vertical directions and on the chosen range of coordinates.

Task 3: Graph function $f :: y = e^x - 6$.

It often happened that a pupil chose such a range in which the x-coordinate of pixels changed by a constant step, e.g., 0.5, and thanks to the maximal precision of determining results, the functional value of the exponential function $y = e^x - 6$ in 'neighbouring' points of the graph was the same (Fig. 5 and 6). It led to an incorrect image that the function was constant on a certain part of its domain.

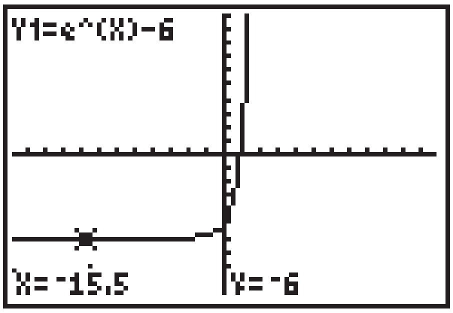

Figure 5. *TI-83*, $f(x)$ for $x = -15.5$.

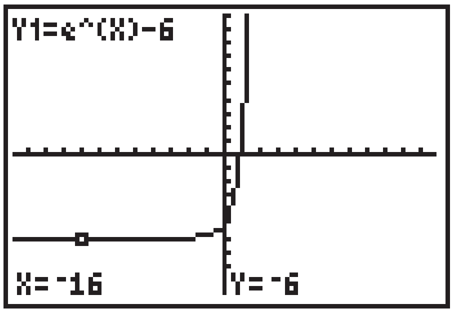

Figure 6. *TI-83*, $f(x)$ for $x = -16$.

Thus, work in a graphic mode requires skills related to adjusting the range of coordinates on the calculator display in such a way that the displayed graph corresponds to mathematical theory. Based on the knowledge of properties of the function and processed data, the pupil should choose such a range of coordinates on the x and y axes for which the graph with all important points such as zero points, global extremes, etc., is displayed (so called *complete graph*). This skill should be developed in connection with the knowledge of the properties of analysed functions (Demana et al., 1994).

Task 4: Graph function $f\colon y = \sqrt{20 - x}$.

Again, we observed that pupils often used a standard range of coordinates in which $x \in \langle -10, 10\rangle$, $y \in \langle -10, 10\rangle$, as in this range, many elementary 'school' functions are displayed with all important properties. For the function above, however, they acquired a graph which seemed to be a straight line (Fig. 7). Only after they realised that the domain of the function was right-closed interval $(-\infty, 20\rangle$ and the image of the function was left-closed interval $\langle 0, \infty)$, they chose a more suitable range, for example, the one with $x \in \langle -6, 21\rangle$ and $y \in \langle -1, 7\rangle$ (Fig. 8).

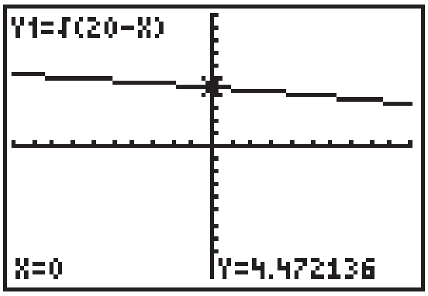

Figure 7. *TI-83*, unsuitable choice of coordinates.

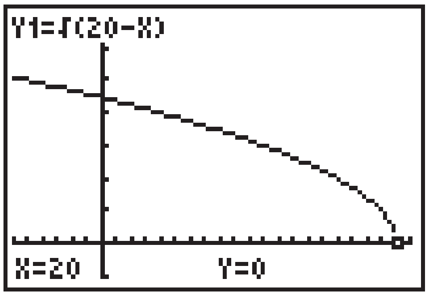

Figure 8. *TI-83*, suitable choice of coordinates.

Task 5: Graph function $f\colon y = 4x^3$.

Fig. 9 shows the graph of the given function in an unsuitable range of coordinates because the quick growth of this power function cannot be seen from the graph. Only when pupils (and student teachers alike) used their theoretical knowledge of the graph of power functions, they used the command for *adjusting the scales* on the axes (i.e., the tick marks are an equal distance apart on each axe) and thus got the graph which better corresponded to the theoretical mathematical knowledge (Fig. 10).

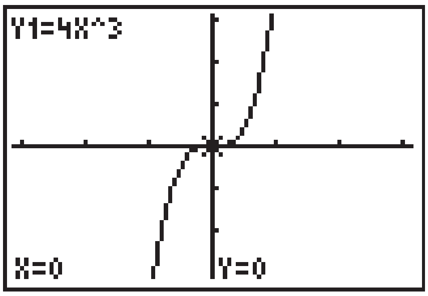

Figure 9. *TI-83*, unsuitable choice of scale.

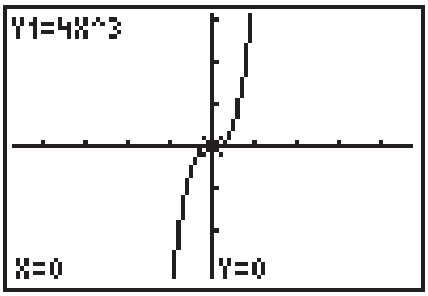

Figure 10. *TI-83*, suitable choice of scale.

Similarly, an unsuitable setting of coordinates can influence the inclination of the line (i.e., the perception of its direction). Thus, the angle included between perpendicular lines seems to be acute, etc. Another typical example is displaying a curve in a parametric form – a circle which is displayed as an ellipse.

For graphing periodic functions such as sine and cosine, it is suitable to develop pupils' ability to choose the coordination setting in view with the period of the function. Task 6 illustrates problems which secondary school pupils met when graphing these functions.

Task 6: Graph function $f\colon y = \sin(100x)$.

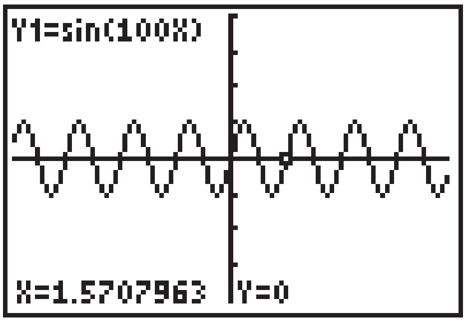

Figure 11. *TI-83*, scale *Trig.*

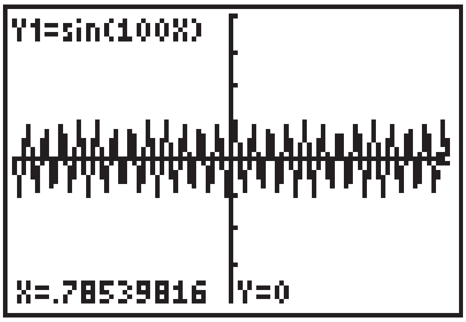

Figure 12. *TI-83*, suitable choice of scale.

Pupils often used the calculator command *Trig* or *ZTrig* which automatically displayed the graph in the range of $x \in (-2\pi, 2\pi)$. Thus they got the graph in Fig. 11 which looks the same as the graph of $y = \sin(4x)$. The displayed graph, however, does not correspond to the formula $y = \sin(100x)$ as this function has a period of $\frac{2\pi}{100}$, i.e., 0.02π. In the selected scale on the $x-$axis in Fig. 11, the difference of x-coordinates of the neighbouring pixels is bigger than $0.02 \cdot \pi$. When the pupil just decreased the scale on the $x-$axis without thinking about the period of the function, he/she got the graph in Fig. 12 which is unclear. Instead of $x \in (-2\pi, 2\pi)$, it is more suitable to take the period into account and use $x \in (-0.02\pi, 0.02\pi)$.

3.2.3 Rounding results, estimations, checking the numerical and graphic outputs

Based on my experience from teaching at the secondary school and on results of some research (see section 3.1), especially weak pupils have a non-critical approach to both numerical and graphic results acquired from the calculator. This can be eliminated by developing the skill of correct rounding numerical results and of calculation estimations with regard to checking the results (see also Doerr, Zangor, 2000). Similarly, in the graphic mode it is necessary to develop and practise the skill to recognise the correctness of the output on the display, that is, whether the graphic output, graph, table, diagram, etc. corresponds to the formula of a function, given values or task conditions. Before displaying the graph of a function on a display, we often asked pupils to realise what they know about the properties of the function and what the graph should look like. They were asked to draw the graph of the function in their exercise books before displaying the graph on the screen and comparing the two graphs.

As already stated, when the calculator without CAS technology works with irrational numbers, it uses their rational approximation. This way of displaying irrational numbers in calculation results manifests itself in the graphic mode, too.

Task 7: Find the roots of equation $x^2 - 3 = 0$.

When looking for the roots of the equation in a graphic way, the pupil determined the roots as the $x-$coordinates of zero points of the quadratic function. One root is irrational number $\sqrt{3}$, however, the display only shows an approximate solution 1.732 050 808 (Fig. 13).

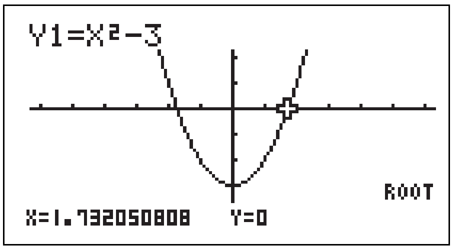

Figure 13. *Casio*, an approximate solution of the equation.

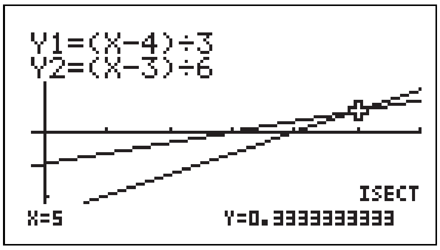

Figure 14. *Casio*, determining the intercept.

When working with rational numbers with an infinite periodic decimal expansion in a graphic mode, we get approximate solutions, too.

Task 8: Solve the set of equations $x - 3y = 4$, $x - 6y = 3$ with two real unknowns.

By graphing two lines and finding their intercept, the pupils got the values of $x = 5$ and $y = 0.333\ 333\ 333\ 3$ (Fig. 14). The solution to the set is, however, an ordered pair [5, 1/3], thus, it is suitable to develop pupils' ability to recognise that the result $y = 0.333\ 333\ 333\ 3$ is approximate and corresponds to $y = \frac{1}{3}$. Recognising the calculation result as a rational number with an infinite periodic decimal expansion is related to the number of digits in the period and to the precision of the calculator. If the number of decimal places of the displayed result does not correspond to at least double the number of digits in the period (eventually increased by the number of digits in the preperiod), a pupil cannot notice the period. The pupils convinced themselves of the correctness of the estimation of the value $y = \frac{1}{3}$ by substituting it into the equations.

An important skill for using a graphing calculator is a critical judgement of results acquired in computations and in a graphic way and justification of the solution. Especially when they still had little experience, the pupils tended to argument that “the result is given by the calculator and thus it must be correct”. They have to realise that it is not the proof of correctness. They should be able to decide if the result is precise, approximate or only a hypothesis which should be justified in some other way (see the illustration).

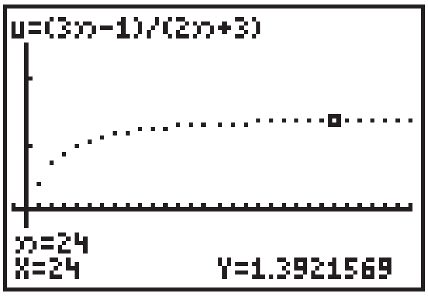

Figure 15. *TI-83*, part of the graph of a sequence.

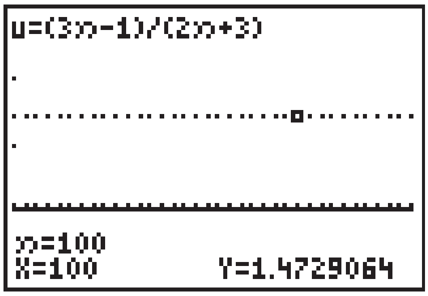

Figure 16. *TI-83*, another part of the graph.

Task 9: Investigate the convergence of a sequence given by the n-th element, e.g., $a_n = \frac{3n-1}{2n+3}$.

The pupils investigated convergence in a graphic mode. After entering the formula of the sequence into the calculator and displaying its graph for the first 30 elements (Fig. 15, in the mode of *Dot* and *Seq*), they could see the values of the elements of the sequence. By changing the range of values n, they realised that the values of the elements were approaching number 1.5 (Fig. 16, $n = 100$) and they concluded that this was indeed the correct result. It is, however, only a hypothesis as we can only see a certain part of the graph on the screen. The hypothesis must be justified in a calculation:

$$\lim_{n\to\infty} \frac{3n - 1}{2n + 3} = \lim_{n\to\infty} \frac{n\left(3 - \frac{1}{n}\right)}{n\left(2 + \frac{3}{n}\right)} = \frac{\lim\limits_{n\to\infty}\left(3 - \frac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(2 + \frac{3}{n}\right)} = \frac{3}{2} = 1.5.$$

4 Dynamic plane geometry software

4.1 Research on dynamic plane geometry software

Dynamic geometry programs enable us not only to draw quickly and precisely but they also include tools which markedly enhance their didactic value for teaching geometric topics. However, studies have pointed to the fact that mastering these programs and their effective use is connected to certain difficulties (Gillis, 2005; Laborde et al., 2006; Vaníček, 2009). Misunderstanding of the conception or philosophy of the dynamic geometry software can lead to mistakes which influence pupils' learning process and his/her success in problem solving (see, e.g., Vaníček, 2009).

Diagrams in plane geometry play a double role (Laborde et al., 2006). They refer to theoretical objects and at the same time to graphic-spatial properties which can give rise to a perceptual activity (Laborde, 1998). One of the goals of teaching geometry is to teach pupils to see this distinction. Research shows that it is often not the case, that pupils work with diagrams as if "it were possible to read the properties of the theoretical object, which is represented by the diagram, by only looking at the diagram" which results in the assumption that "it is possible to construct a geometrical diagram using only visual cues, or to deduce a property empirically by checking on the diagram" (p. 277). This has consequences in the use of dynamic geometry software, too.

Research has amply shown that pupils, especially with little experience with the software, often construct figures *by eye* (which can work well in traditional static geometry). They modify the figure in such a way that it looks correctly without using a correct geometric procedure (Gillis, 2005; Vaníček, 2009; Laborde et al., 2006). For example, a pupil constructs an object seemingly of required properties (e.g., isosceles triangle), however, when the object is moved by its determining element (such as a vertex of the triangle), it loses the properties as the construction was not properly made. Another typical example is the construction of the circumscribed circle of the triangle when the pupil constructs a circle and modifies the figure by dragging so that the circle goes through all the vertices instead of using the intercept of axes of sides. Another example is a construction of two divergent straight lines on which the pupil can see their intercept but for the computer, this intercept does not exist. This type of constructions is called *soft* as opposed to *robust* ones (Healy, 2000), in which the object does not lose its properties when dragged. According to Laborde et al. (2006), this phenomenon relates to the pupil's view of the problem. The pupil sees two separate worlds, "the mechanical world of the computer diagram (in our terms, the spatial) and the theoretical (or geometrical)" (p. 285).

From the point of view of *dependence of objects*, it is important to distinguish, for example, the case in which circle k is constructed through point B or other way round, in which point B is placed on the constructed circle k. In the former case, point B is a free, independent point and can be moved freely which results in the change of the circle k radius. In the latter case, point B is on circle k and can only be moved within this circle; point B is a point on the object and thus it is partially dependent. In the static geometry, these two relationships are expressed by the same record $B \in k^2$. Talmon and Yerushalmy (2004) speak about the *child-parent relationships* and consider their understanding to be a prerequisite of successful work in dynamic geometry software. Again, Laborde et al (2006) relate the understanding the dependence relationship in dynamic geometry software to the necessity for pupils to construct a relation between the graphic-spatial level and the theoretical level. Understanding the two types of dependencies can be impeded by the fact that in the dynamic geometry software, the relationships between objects can be changed during the solution of the problem.

Dynamic geometry software is a tool which supports concept development process in pupils and contributes to the discovery and proof of mathematical relationships (e.g., Marrades, Gutiérrez, 2000,; De Villiers, 2007; Zbiek et al., 2007; Baccaglini-Frank, Mariotti, 2011). The main tool for this process is *dragging tool*. Jones (2003) points out that especially at the beginning pupils consider it to be detrimental, a possible cause being that static geometry does not include the movement of geometric objects. Hölzl (2001) distinguishes two basic uses of this tool – for testing relationships and for discovering new relationships – and shows that the journey towards conscious use of dragging in any of the two contexts is a long term process and closely connected to theoretical geometric properties. Mariotti (2011) considers dragging to be suitable for solving open geometric problems in which pupils investigate invariants of construction. It is, however, necessary to lead pupils towards distinguishing assumptions and conclusions. Much research has shown that pupils do not often feel the need to justify the observed relationships and consider them as correct implicitly (Gawlick, 2002; Gillis, 200; De Villiers, 2007). When dragging an object and observing the changes, the pupil has to transform observed data into geometric assertions. That is, he/she has to make connections between graphic-spatical and theoretical spaces.

4.2 Specific skills connected to dynamic plane geometry software

The illustrations below are constructed in *Cabri* or *GeoGebra*. Tasks 10 to 16 are suitable for lower secondary pupils while tasks 17 and 18 for upper secondary pupils.

[2] Vaníček (2009) says that for this reason the record $k \ni B$ would be appropriate.

4.2.1 Theoretical space and computer-graphic space

Understanding differences between theoretical-geometric and computer-graphic worlds can be enhanced by tasks in which an object of certain properties is constructed in such a way that the properties are kept when the location of an element of the geometric figure is changed.

Task 10: Construct an isosceles triangle.

A beginner often constructs a triangle, e.g., ABC, measures the lengths of two sides and by changing the location of a vertex, e.g., vertex C, adjusts the same lengths of the sides (Fig. 17, left). If any of the vertices is moved, this property ceases to exist (Fig. 18, left). The mathematically proper, robust, construction requires that when constructing isosceles triangle DEF a point F is placed on the axis of side DE (Fig. 17, right), as in this case, the triangle remains an isosceles one when any of the vertices is moved (Fig. 18, right). We observed a soft construction of an isosceles triangle even in (student) mathematics teachers' work.

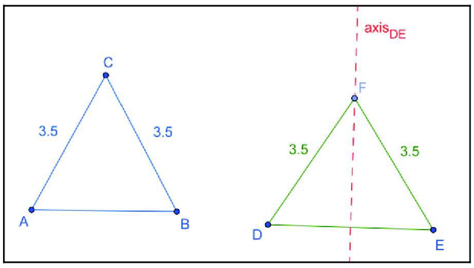

Figure 17. *GeoGebra*, isosceles triangles.

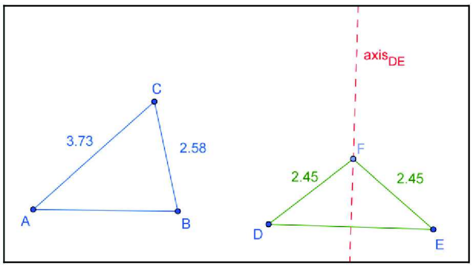

Figure 18. *GeoGebra*, moving points C and F.

Another typical mistake caused by the different philosophy of theoretical and computer worlds is illustrated by task 11.

Task 11: Construct an inscribed circle to a triangle.

On many occasions in our teaching, we observed that a pupil (or future mathematics teacher) constructed triangle ABC, the axes of two inner angles and their intersect S. Next, he/she used the command for constructing a circle given by a centre (i.e., point S) and another point (e.g., R), which he/she placed on the side of the triangle in such a way that the circle was 'inscribed' (Fig. 19, left). However, by dragging it was found out that the circle had not been constructed properly (Fig. 20, left). Robust construction of the inscribed circle for triangle DEF requires not only finding the centre O as an intersect of axes of angles but also determining its radius ρ with the help of point P, which is the foot of the perpendicular line through point O to the side of the triangle (Fig. 19, right). If the place of a vertex is changed, the circle remains inscribed (Fig. 20, right).

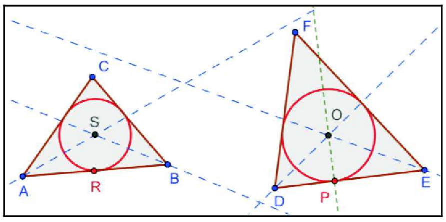

Figure 19. *GeoGebra*, inscribed circles.

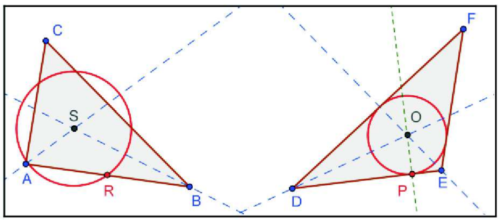

Figure 20. *GeoGebra*, verification by changing the location of points C and F.

The tool of *dragging* is an important tool for verification of robust constructions, thus it is advisable to encourage pupils to use this tool often to check their constructions. The skill of making robust construction is an important prerequisite for discovering invariants of figures and creating hypotheses (see also Healy, 2000; Laborde, 2001).

4.2.2 Identification, dependence and congruence of objects

From the above, it is clear that pupils must be able to see certain differences between computer-dynamic and traditional-static geometry. One skill which is based on these differences is the skill to determine uniquely the location of a geometric object by mouse. If objects are placed one on top of another or if they are next to one another, a pupil can use an incorrect object for the next construction or command.

Another skill in this area consists of understanding the definition of geometric objects and their dependence in dynamic geometry software. The way an object is defined, and thus constructed, influences the function of some tools. For example, when constructing a polygon with line segments instead of the polygon tool, the software does not interpret the polygon as a whole but as separate line segments. Thus, this polygon cannot be coloured or its area found. Task 12 illustrates mistakes caused by an imprecise identification of the object in connection with the definition of a geometric shape.

Task 12: Construct triangle ABC in which $|AC| = 6$ cm, $\sphericalangle ACB = 60°$, $\sphericalangle ABC = 75°$. Decide against which side of the triangle the biggest inner angle lies.

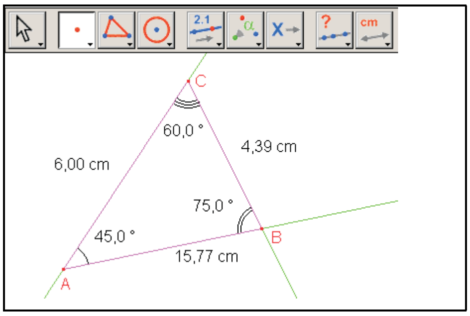

Figure 21. *Cabri*, command *Length*.

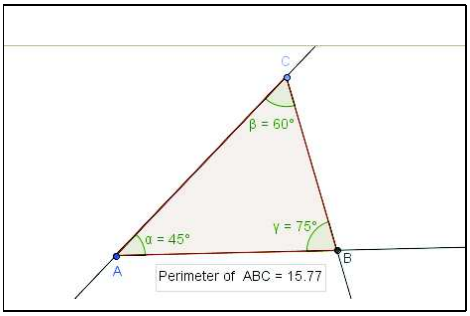

Figure 22. *GeoGebra*, command *Length*.

We often witnessed the following solution. Pupils determined the size of the angle at vertex A as they knew the sizes of the two remaining inner angles and constructed line segment AC and rays – sides of the angles at vertices A and C. Point B was made as an intersect of the rays. Finally, the triangle was highlighted with the command *Triangle* (in *Cabri*). If the pupils next activated command *Length* and moved the cursor closer to line segment AB, which they had constructed earlier, software identified both a line segment and a triangle at this place. If they did not specify the geometric shape (i.e., line segment) and chose the triangle offered, the tool of *Length* returned its perimeter, not the length of the side (Fig. 21, '15.77 cm'). This led to errors. Note that in *GeoGebra*, the user is notified by a supplementary text what was actually measured (Fig. 22, 'Perimeter of ABC').

Another important difference between static and dynamic geometry lies in the view of the identity of two points. In dynamic geometry software, several constructed points can lie at the same location. Vaníček (2009) illustrates this situation by the following task.

Task 13: Given line o and point A, $A \notin o$. Construct the image A' of point A in line symmetry with axis o, then the image A'' of point A' in the same line symmetry. Determine the relationship between points A'' and A. Next, construct line p parallel with axis o and going through A''.

The solution is in Fig. 23. At first sight, a pupil can see that points A'' are A coincident. If he/she checks this conclusion by *dragging* and moves point A, he/she can observe the changes of its coordinates in the algebraic window and see that coordinates of A'' and A are the same. However, in the dynamic geometry software, any of the points A, A'' can be used for further constructions. From the point of view of the computer-graphic world it is important which point is used for the construction of line p. If the pupil constructs the parallel line through A'' and later erases axis of symmetry o, point A'' ceases to exist and line p disappears, too.

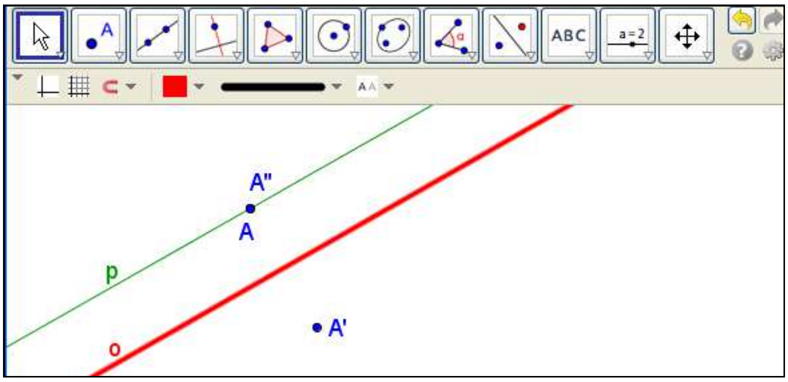

Figure 23. *GeoGebra*, congruence and dependence of objects.

Our experience from the education of future mathematics teachers and practising secondary mathematics teachers (see also Gergelitsová, 2011) shows that in order to prevent the above errors or, in other words, to develop necessary specific skills, it is not enough just to demonstrate them to pupils. It was more suitable if pupils manipulated objects and observed the influence of the definition of objects and their dependence. The teacher guided this manipulation.

4.2.3 Representation of real numbers, interpretation of numerical results

From the point of view of metric tools in dynamic geometry software, it is necessary to develop the skill to understand the representation of real numbers and interpretations of numerical results.

Task 14: Triangle $A'B'C'$ is an image of triangle ABC in homothety with centre T and coefficient $k = -2$. Find the relationship between the areas of the similar triangles.

The situation is in Fig. 24. After determining areas $S = 6.00\ \text{cm}^2$ and $S' = 24.01\ \text{cm}^2$ of the two triangles with the tool *Area,* some future mathematics teachers got numerical results which did not reflect the theoretical relationship: if the coefficient of similarity is $|k| = 2$, the ratio of areas of similar triangles is $k^2 = 4$. It is caused by the fact that the displayed numerical values are given with the precision to two decimal places, while internally the software works with a higher precision as can be seen from the result of the calculation $\frac{S'}{S}$ (down middle of Fig. 24). So it can happen that the displayed values do not correspond to precise results. It points to the importance of justifying the hypothesis made on the basis of observing the situation on the computer screen. The example given above can be used for this purpose. Numerical computations in this environment cannot be used as a substitute for proof (De Villiers, 2007).

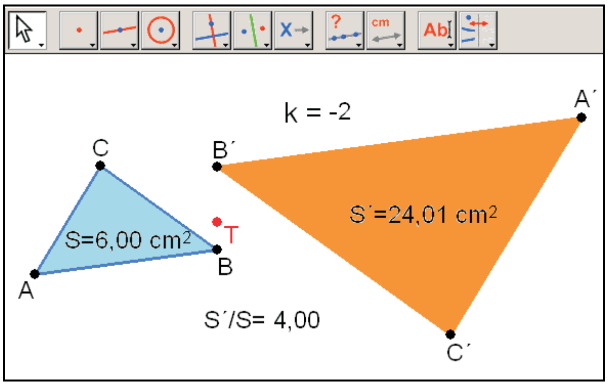

Figure 24. *Cabri II*, numerical computations and rounding.

4.2.4 Experimenting, hypothesis and proof

One of the main contributions of the dynamic geometry software in mathematics lies in its potential to support pupils' experimental work. Through manipulating with geometric objects in this environment, pupils can 'discover', formulate and check hypotheses about properties of and relationships among geometric objects. Pupils, however, should not stay on the level of checking hypotheses, they should be led to justifying – proving the relationships. It is important that the pupils not only manipulate the objects in a dynamic figure but they also describe in words what they see and what conclusions they have reached (Johnston-Wilder, Mason, 2006).

An important prerequisite for making a suitable hypothesis is a skill to make robust constructions. Thus, a pupil can manipulate with figures, notice invariants of the geometric situation and formulate a hypothesis. Tasks 15 and 17 are simple tasks in which pupils have enough opportunities to experiment and discover. Tasks 16 and 18 are connecting problems and illustrate 'limitation' of pupils' independent experimental work.

Task 15: Construct any parallelogram $ABCD$ and centres of its sides E, F, G, H. What are the properties of quadrilateral $EFGH$? Check the properties by changing the location of some vertices of parallelogram $ABCD$. Justify your observations.

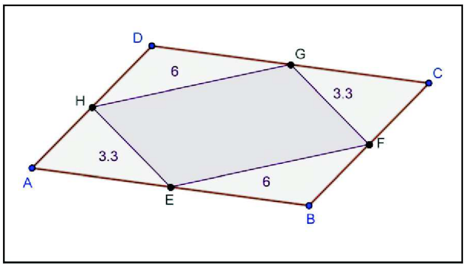

Figure 25. *GeoGebra*, lengths of sides of a quadrilateral.

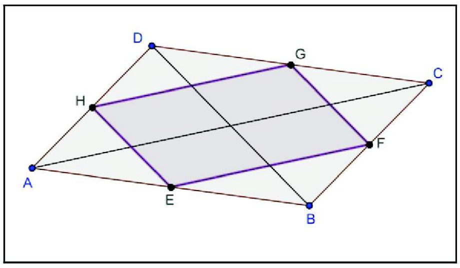

Figure 26. *GeoGebra*, 'justification' of parallelism of sides.

Pupils mostly constructed both parallelogram $ABCD$ and quadrilateral $EFGH$ first and measured the lengths of sides of inscribed quadrilateral $EFGH$ (Fig. 25). The lengths of its opposite sides are the same thus they reached a conclusion that it was a parallelogram. By *dragging*, they checked that it indeed was a parallelogram even when the vertices of the original parallelogram $ABCD$ were moved. Some pupils then verified the parallelism of the opposite sides of quadrilateral $EFGH$. These observations confirmed that the hypothesis about parallelogram was correct, however, it is not a proof. Justifying the correctness of the hypothesis is a demanding task for many pupils and according to our experience, it requires the teacher's help. The pupils were advised to construct diagonals AC and BD (Fig. 26) and asked questions about the relationship of line segments EF and AC (midline and side of triangle ABC). They could thus justify the parallelism of EF and AC, and HG and AC (analogically for EH and BD, and FG and BD).

Task 15 can be widened.

Task 16: Find the areas of parallelograms $ABCD$ and $EFGH$ from Task 15. What is the relationship between the areas? Check your observations by changing the location of some vertices of parallelogram $ABCD$. Justify your observations.

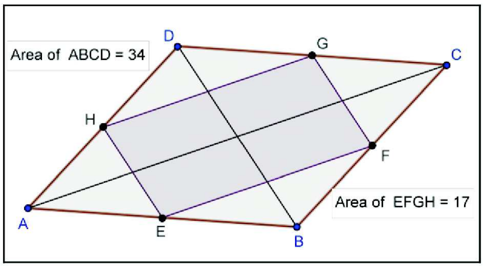

Figure 27. *GeoGebra*, areas of parallelograms.

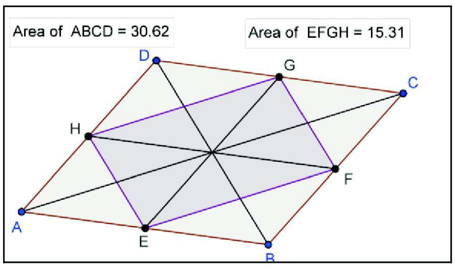

Figure 28. *GeoGebra*, highlighting congruent triangles.

Dynamic geometry software includes tools for calculating areas, thus pupils could quickly find out that the area of parallelogram $EFGH$ was a half of the area of parallelogram $ABCD$ (Fig. 27). They checked this by dragging some vertices of $ABCD$. However, as already stated for Task 14, this calculation cannot be considered the proof. Again, the pupils were provided with a hint to draw diagonals EG and FH of the inner parallelogram (Fig. 28). Pupils noticed that parallelogram $ABCD$ consisted of sixteen congruent triangles, while parallelogram $EFGH$ included eight of these triangles.

Task 17: Construct circle k with centre S and radius r. Place points K, L, M, N on circle k. Construct quadrilateral $KLMN$ and measure the sizes of its inner angles. What is the relationship between the sizes of opposite inner angles of the quadrilateral? Check your observation by dragging vertices of the quadrilateral. Justify.

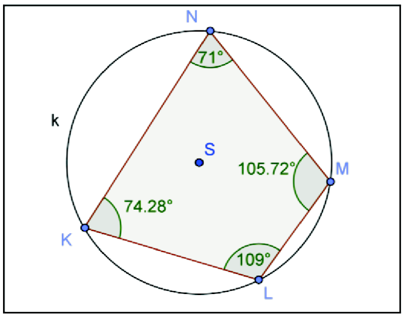

Figure 29. *GeoGebra*, cyclic quadrilateral.

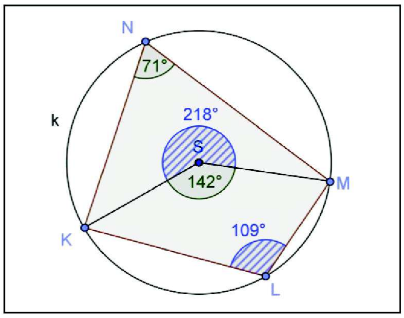

Figure 30. *GeoGebra*, 'justification' of the relationship.

When constructing the objects, pupils quickly discovered that the sum of the sizes of opposite inner angles of $KLMN$ was 180° (Fig. 29). They checked this relationship by dragging vertices of the quadrilateral along circle k. In order to justify it, they had to realise that angle KNM and convex angle KSM (Fig. 30) were inscribed and central angles belonging to the same arc of the circle (similarly for angle KLM and non-convex angle MSK). From the relationship between inscribed and central angles and from the sum of the sizes of the two central angles (which is 360°) it followed that the sum of the sizes of opposite angles in a quadrilateral was 180°. Our experience from the classroom shows that the justification of the relationship is demanding for most pupils and cannot be done without the teacher's help.

Task 18 (Henn, Jock, 1994, quoted in Hölzl, 2001) follows Task 17.

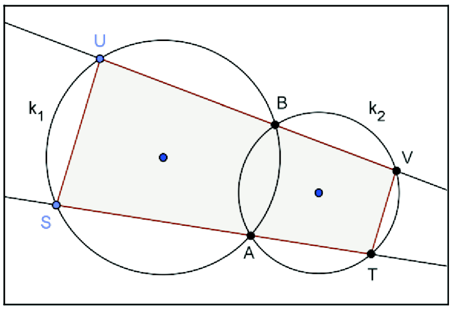

Figure 31. *GeoGebra*, Task 18.

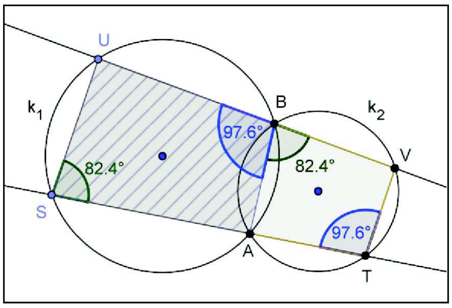

Figure 32. *GeoGebra*, 'justification' of the relationship.

Task 18: Draw two circles k_1 and k_2, that intersect in two points A and B. On k_1 choose two points S and U. Line SA intersects k_2 at T, line UB intersects k_2 at V. Points S, T, V, U are vertices of a quadrilateral (Fig. 31). What is the quadrilateral like? Move S or U on k_1 to come to an observation. Justify your observation.

The parallelism of sides SU, VT can easily be seen and checked by changing the location of point S or U. However, to justify it, one must realise, for example, that line segment AB determines two cyclic quadrilaterals $SABU$ and $ATVB$ in which the sum of sizes of the opposite angles is 180° (Fig. 32).

5 Dynamic space geometry software

5.1 Research on dynamic space geometry software

Similarly to plane geometry software, work in space geometry software is influenced by its philosophy. As will be seen from section 5.2, specific skills are needed in order to master the software. A problem particular to 3D software is connected to a pupil's insufficient space imagery as many authors pointed out that pupils orient themselves worse in a 3D situation mapping than in plane situations (Vaníček, 2009, Gergelitsová, 2011). 3D geometry software is much younger than plane geometry programs and thus research in this area is still sparse.

Hattermann (2008) investigated the influence of 3D dynamic geometry software integration in the education of future mathematics teachers and reached the conclusion that mastering this software was very demanding for the future teachers even if they had had previous experience with it. They only reluctantly used some tools including *dragging* (this corresponds to results for plane geometry software). It confirms the fact that mastering tools of 3D software is more difficult for pupils and students than mastering the tools of 2D software and again, that some specific skills must be carefully developed.

Some studies indirectly pointed to the necessity for pupils having specific skills, one of them being using different view angles when observing objects. Chino et al. (2007) looked into cognitive processes of pupils of lower secondary school when constructing spatial objects and observed a markedly positive influence of the integration of 3D dynamic geometry software on their study results. They say that the opportunity to investigate spatial objects from different view angles contributed to the development of pupils' cognitive processes. Similar positive conclusions from the point of view of the development of spatial imagery were reached by Güven and Kosa (2008) and also Subroto (2011). He found out that software *Cabri 3D* and its tools improved the ability of rotation, relation, and orientation, while the ability of perception and visualization did not improve significantly.

Next, basic skills which should be developed in connection with an effective use of 3D geometry software are described. The illustrations below are made in *Cabri 3D*. All tasks below are suitable for upper secondary school pupils. Task 19 and partially Task 20 can be solved by lower secondary school pupils.

5.2 Specific skills connected to dynamic space geometry software

5.2.1 Definition and dependence of objects

Similarly to plane geometry software, pupils have to understand the dependence of objects and their definitions. The basic skill consists of placing the point, or correct determining of the location of the point in the displayed situation.

Task 19: Construct a prism with the bottom base in the xy-plane.

Future mathematics teachers often placed vertex A of the top base 'over' the bottom base when constructing a prism (Fig. 33). Using the tool *Prism*, they constructed a 'prism' (Fig. 34), however, when rotating the view, it could be seen that it was not a prism but a plane object (Fig. 35). I was caused by the fact that the vertex of the upper base was placed without using a key which is used for controlling the placement of the point in the direction of axis z (in *Cabri 3D* it is the SHIFT key), so that the point is placed by eye 'over' the bottom base; in fact, the point lies in xy-plane.

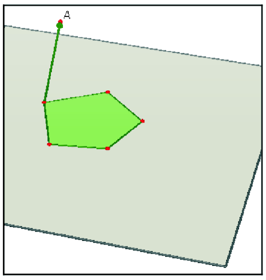

Figure 33.

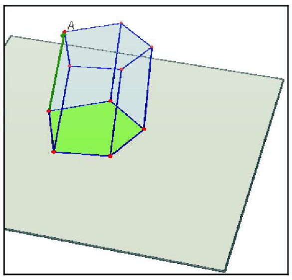

Figure 34.

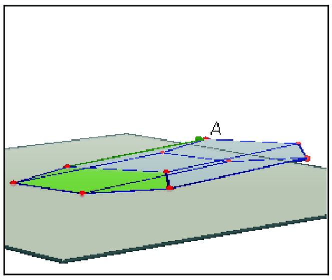

Figure 35.

5.2.2 Orientation in display, view of geometric objects

As observed by many authors, the above mistake manifests itself quite often (Vaníček, 2009, Gergelitsová, 2011). It also relates to the way of displaying infinite areas as closed shapes (e.g., in Fig. 34 above and Fig. 44 below, the planes are displayed as parallelograms). The skill to orientate oneself thus requires practice.

Another important skill consists of mastering views of geometric objects. When rotating the scene, the pupil can be 'lost in space' – he/she no longer knows where he/she as an observer is located. This problem is a serious one as in *Cabri 3D*, there is no tool to return the view to its original state.

In order to develop the ability to orientate oneself in 3D, in our courses we led future mathematics teachers to work with pre-prepared models of solids in which they made various objects such as points and lines and investigated, for example, relationships between the objects. It was found out, that an important prerequisite which made their orientation easier was labelling important points, lines and planes.

Task 20: Be given a cube. Label its vertices A through H. Construct line p which is given by points A and E and line q which goes through the centre of edge $EF$and the centre of face $CDHG$. Determine mutual relationship between lines p and q.

When labelling cube vertices whose faces are opaque, student teachers have to manipulate with the scene to label non-visible vertex D (Fig. 36, Fig. 37). Also when constructing lines p and q, they changed the view angle (Fig. 38), while they repeatedly returned to the view in which the front face $ABFE$ could be seen. Lines p and q seem to be concurrent in the situation in Fig. 38, however, after changing the view angle, they could see that they were skew lines.

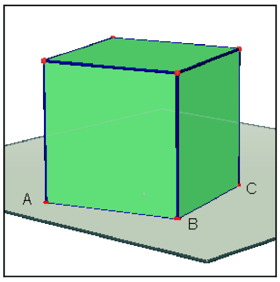

Figure 36.

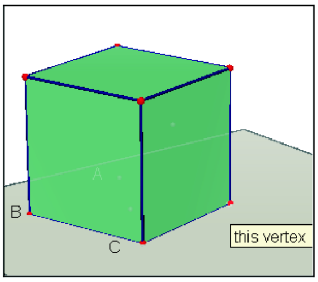

Figure 37.

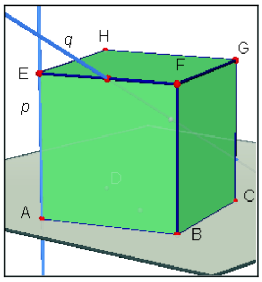

Figure 38.

A graphic differentiation of important objects helps to orientate oneself when changing the view angle, too. Different styles of lines and different colours can be used.

5.2.3 Numerical calculation skills

As is the case with other software, in 3D geometry software rational approximations of irrational numbers are used in calculations. Again, we can get results which do not correspond to theoretical results, which is caused by rounding numbers and by the way solids are constructed.

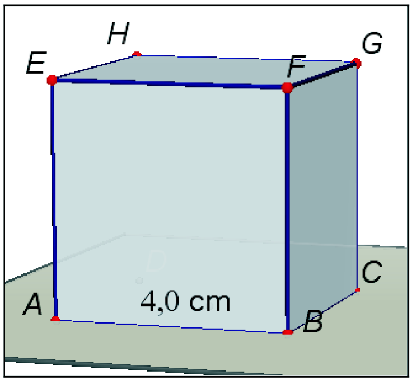

Figure 39. Calculation of length to 1 decimal place.

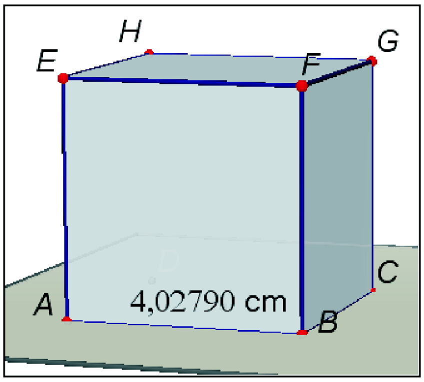

Figure 40. Calculation of length to 5 decimal places.

Task 21: Find the area of the cross section of cube $ABCDEFGH$ with the edge of $a = 4$ cm by plane $S_{CG}S_{EH}S_{AB}$.

Student teachers mostly constructed a cube, measured its edges and by moving the vertex of the cube modified the length of the edge into the required length (Fig. 39). However, when they increased the number of decimal places, they could see that the edge was not precisely 4 cm (Fig. 40), which, of course, influenced the resulting area of the cross section which is seen in Fig. 41. If the construction is properly made (the edge is constructed using the tool *Measurement transfer*, then the correct result is produced (Fig. 42). A similar problem manifests itself in numerical computations made by the tool *Calculator* (Fig. 43).

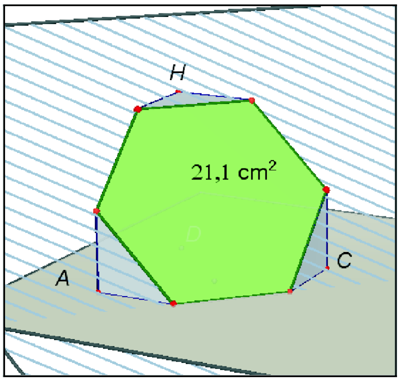

Figure 41. Approximate area of the cross section.

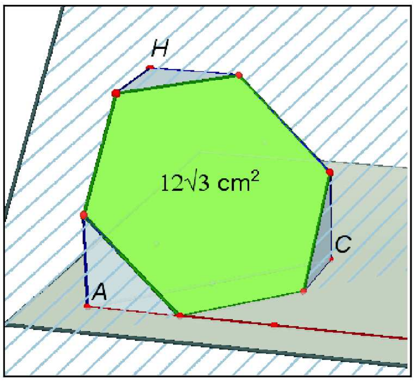

Figure 42. Precise area of the cross section.

In common with 2D, mastering the tool of *dragging* is an important means to test and discover relationships.

Task 22: Find the ratio of volumes of a circumscribed and an inscribed spheres to a cube.

Student teachers constructed a picture of the space situation with no problem. Using the tool of *Volume* and *Calculator,* the ratio was found (Fig. 43).

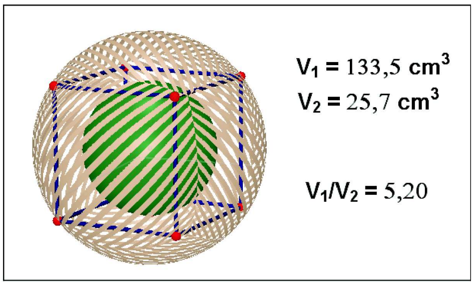

Figure 43. *Cabri 3D*, the ratio of volumes of circumscribed and inscribed spheres.

By changing the placement of some vertex of the cube, they found out that volumes of the two spheres changed with the change of the length of the edge and that the ratio remained constant. The ratio of volumes found with the tool of *Calculator* is, however, approximate which can be seen when the number of decimal places of the result is enhanced. This is a suitable moment for a theoretical justification of the discovered relationship. For the ratio of the volume of the circumscribed sphere V_1 and of the inscribed sphere V_2 to the cube with the edge a, it holds

$$\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \frac{r_1^3}{r_2^3} = \frac{\left(\frac{\sqrt{3}}{2}a\right)^3}{\left(\frac{a}{2}\right)^3} = 3\sqrt{3} \approx 5.20,$$

where r_1 is the radius of the circumscribed sphere and r_2 the radius of the inscribed sphere. This example illustrates a mix of skills needed for the solution of one problem: work with numerical values, using the tool of *dragging* to discover a geometric relationship, judging critically the acquired result and distinguishing between the hypothesis and proof.

5.2.4 Analytic description of lines and planes

Cabri 3D can also be used for teaching analytic geometry. The software can display a point of given coordinates and equations of straight lines (as an intersecting line of two planes) and of planes and coordinates of vectors. Thus, the skill to display lines and planes of a given analytic expression is needed. To construct a plane which is given by an equation or parametrically means that a pupil determines coordinates of three non-collinear points of the plane which are displayed and then a plane is drawn through them. Or a point and a line from the plane can be constructed first and then a plane drawn through them.

Task 23: Find the distance of point $Q[7;1;9]$ from line p which is given by point P and vector $\vec{u}$, for which $P[1;3;-1]$ and $\vec{u} = (4;1;3)$.

Student teachers had to realise that the distance is measured on a perpendicular line. Thus it was necessary to find the foot R of the perpendicular line k drawn from point Q perpendicularly to line p. First, a parametric expression of line p: $X = P + t \cdot \vec{u}$ was found. From $(X - Q) \cdot \vec{u} = o$, the value of real parameter t for which $X = R$ was deduced and then the distance was found: $d = |pQ| = |RQ| = 6$.

Alternatively, a different strategy can be used in which foot R is found as a point of intersection of plane ρ, which goes through point Q and is perpendicular to line p, and line p. This strategy is based on the synthetic approach to finding the distance of a point from a line in space. The situation depicted in Fig. 44 is, however, unclear mainly due to a finite image of plane ρ and the fact that it is not clear that plane ρ goes through point Q; the situation is not improved by the choice of a view from a different angle.

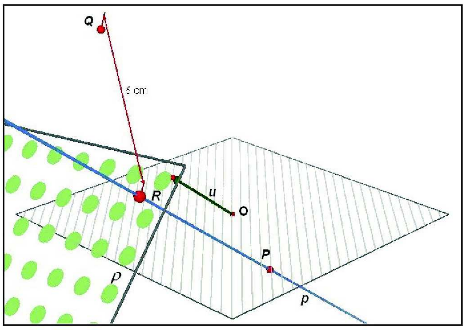

Figure 44. *Cabri 3D*, distance of a point from a line.

6 Conclusions

An important agent which determines whether technology is used to best advantage is, again, the teacher (Pittalis, Christou, 2011; Robová, 2012), as his/her attitudes to ICT itself and its role in the teaching/learning process markedly influences the efficiency of the ICT integration into teaching. Mishra and Koehler (2006) looked into the teacher's role more deeply and stress that in order for the teacher to teach successfully with technology, he/she has to have knowledge about content (C), pedagogy (P), and technology (T). Building on Shulman's work (1986), they introduce a model which expresses mutual relationships between and among these three phenomena. For example, Technological Content Knowledge is knowledge about the manner in which technology and content are related, i.e., how the content can be changed by the application of technology (e.g., dynamic software can visualise proofs making them accessible to more pupils). The central part of the model is Technological Pedagogical Content Knowledge (p. 1029): "Technological pedagogical content knowledge (TPCK) is [...] the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones."

The specific skills described in this article are, indeed, part of teachers' technological content knowledge. They themselves must be aware of specific 'mathematics' which is used by the calculator and software or, rather, concrete algorithms used by ICT tools. But the skills are, at the same time, part of the teachers' TPCK as they should devise ways in which these skills can be developed in their pupils. The tasks above can become part of this knowledge.

Some specific skills presented in the article such as rounding results, estimations, i.e., numerical skills, are relevant for both calculators and dynamic geometry software, while others such as adjusting the range of coordinates on the display or dependence and congruence of objects are particular to a concrete ICT tool. They have been drawn both from research results and the author's teaching and experience from future education of teachers. In total, 23 tasks were formulated and discussed in order to illustrate the specific skill as well as to show how the skill can be developed.

The focus for the tasks was mainly skills connected to technical mathematical activity (Zbiek et al., 2007) such as numerical calculations, constructing graphs, solving equations, etc. and at the same time tasks for which ICT tools are important. The tasks can play two roles: as a means to develop the skills and as a way to diagnose whether the pupils have the skills.

As consistently stated above, one of the merits of integrating ICT is an opportunity for pupils to experiment and discover relationships both on calculators and in dynamic geometry software. However, it is time consuming and might explain why teachers often formulate precise instructions of work in order to make the activity more effective. This, however, reduces pupils' independent work and does not contribute much to their learning and that is why an important role is played by the teacher's TPCK.

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Specyficzne umiejętności potrzebne do efektywnej pracy na lekcjach matematyki z wykorzystaniem teleinformatyki (ICT)

S t r e s z c z e n i e

Narzędzia teleinformatyki (ICT) wchodzą coraz częściej do nauczania matematyki a z drugiej strony ich zastosowanie w kształceniu jest przedmiotem badań dydaktycznych. Ogólnie wyniki badań wskazują, że ich wykorzystanie może przynieść wiele korzyści uczniom, z drugiej strony – wykorzystanie tych technik niesie z sobą wiele ograniczeń. Podkreśla się, że proces uczenia się nie będzie przebiegać efektywnie bez świadomego wsparcia płynącego od nauczyciela oraz bez odpowiedniego doboru zadań. Celem tego artykułu jest a) prezentacja wyników niektórych badań dotyczących wykorzystania kalkulatorów graficznych oraz oprogramowania do geometrii dynamicznej, głównie z punktu widzenia problemów, na jakie napotykają uczniowie korzystający z tych narzędzi, b) przedstawienie specyficznych umiejętności, które musimy rozwijać u uczniów, aby mogli oni przezwyciężać owe trudności c) zasugerowanie niektórych specyficznych zadań dla rozwijania tych umiejętności.