ANNALS OF THE POLISH MATHEMATICAL SOCIETY
5TH SERIES: DIDACTICA MATHEMATICAE 34 (2012)

Klaus Hasemann
Leibniz University of Hanover, Germany

Mental representations of mathematical objects and relations in the first grades

Abstract: The importance of adequate external and internal (mental) representations both for mathematical understanding and for generalization is shown with examples taken from early mathematics in pre-school education and from primary mathematics in schools. Regarding relations between numbers and special aspects of addition and subtraction, in the main part it is discussed whether or to which extent referring to actions with concrete materials and to children's every-day life experience might be a learning obstacle and not helpful for children's insight and for their ability to generalize mathematical concepts. In addition, alternative ways for classroom practice are discussed.

Concrete and action-oriented thinking in primary school mathematics

It seems to be common sense not only in primary mathematics education that to proceed "from the concrete to the abstract" is the best – or even: the only – option, i.e. to invite learners to carry out actions with concrete objects or to present them situations taken from real life in such a way that they can grasp the "intended" mathematical concepts and procedures. However, action-related thinking obviously becomes inadequate if the task – as, e.g., a word problem – cannot be simulated by actions.

In this article it will be discussed which alternatives exist, and it will be shown that – especially with less competent students – in some aspects and situations another way is more promising, namely to focus their attention to the meaning of mathematical symbols and to help them to get insight in the way how mathematics is done by using words and signs.

An explanation for the dominance of the way "from the concrete to the abstract" might be found in Piaget's work and his – in fact – very important findings and ideas that might be summarized by his terms “abstraction à partir de l’action” or “abstraction réflèchissante” (see, e.g., Aebli, 1980, p. 217). It should be remarked that in this abstraction process Piaget focusses on the *reflection* on mental actions on mental objects.

One of the most convincing ideas promoting students’ learning processes in using real situations was created by researchers of the Freudenthal Institute when they established the concept “realistisch rekenonderwijs” (realistic mathematics; see, e.g., Treffers, 1987; or Streefland, 1991, who, however, also referred to the van Hiele levels). In their concept of “ongoing schematisation” in the focus are both transformations of real world problems into mathematical problems and processes within the mathematical system (see Treffers, 1987, p. 247).

An introduction of mathematical concepts based on situations the students are familiar with from every-day life might also be founded on Greeno’s situated perspective on cognition and learning and his discussion of generative knowledge (see, e.g., Greeno, 1989; Stern, 1998; Caluori, 2004, pp. 86ff).

There is no doubt that for many learners and many concepts it is very useful and important to start with real situations which are constructed in such a way that mathematical concepts and procedure can be applied. But it should also be taken into account that this method also includes risks: To grasp the intended mathematical concepts and procedures as useful and universal *mental* tools, it is necessary for the learners to pass abstraction processes. It seems that a lot of children have serious problems with these abstraction processes. For example Gray, Pitta and Tall (1997) in their study on the numerical concepts with primary school children underlined: “It is our contention that different perceptions of these objects, whether mental or physical, are the heart of different cognitive styles that lead to success and failure in elementary arithmetic” (ibid., p. 117). Rowlands and Carson put it even more strongly from their “critical review of ethnomathematics” (2002, p. 98): “Independent of good intentions, ethnomathematics runs the risk of attempting to equalise everything down to the poverty of the ’builders and well-diggers and shack-raisers in the slums” (this means: There is the risk that children remain on a very low level of knowledge and of experience. Obviously, ethnomathematics is an extreme example of action orientation; therefore this statement is just meant as a warning about exaggeration).

Action oriented mathematical thinking might be sufficient in many aspects of primary school, but it is not in higher grades. Reflection, generalization and abstraction are processes that start early and obviously do not only occur in higher grades. In the following sections, some examples and empirical findings will be presented to support this opinion.

Empirical findings from pre-school and primary mathematics

Findings from pre-school education

Mathematical thinking obviously does not start with formal instructions in school, so Early Mathematics has become an important field in research. The intention of the examples presented here is to show that the recognition of structures and pattern (together with the collecting of experience with real actions and concrete objects) is an important basis for the construction of mental representations and the development of mathematical thinking.

Even kids in kindergarten show extremely big differences in their kind of thinking: Some recognize visually presented pattern and structures and are able to use them flexibly to solve mathematical problems, others can use only counting procedures they are familiar with. Referring to Linchevski and Livneh (1999) and Mulligan and Mitchelmore (2009) Lüken (2010) called this ability to recognize pattern and structures "early structure sense". With this term she indicates the "ability to see any predictable regularity or ordered entity and the relationships between parts in such a pattern" (ibid., p. 241).

According to Lüken a child can show his or her ability of pattern recognition in two different ways (see Lüken, 2011, p. 3; 2012, pp. 136f & 193ff): The child can recognize a pattern he or she is familiar with (as, for instance, the die-pattern of the six) as a *picture*, or he or she is able to detect the *regularity* included in the pattern. In an experiment with children in grade 1 (at their very beginning of school) Lüken (2011) presented the flash card in fig. 1 for a short moment to the children and asked them to determine the quantity from memory.

1 I: How many counters did you see?
2 S: ... There was the six. And one is missing.

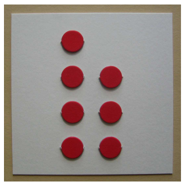


Figure 1. Flash card with seven dots.

The pupil was a low achieving child, regarding the summary of her results in all the tests which were carried out with the children. However, she seems to have recognized the die-pattern of the six as a regularity and not just as a picture: Either she takes eight dots for "the six" ($7 = 8 - 1$), or she refers to the six with one extra on top (in this case, "one is missing" might mean:

There are not two extra on top of the two rows of three but just one). In any case, to some extent she is able to recognize familiar pattern and to use them in a more complex situation. Usually, this ability to use regularities is characteristic of higher achieving children.

In her longitudinal study Lüken found out that there is a correlation between children's ability to recognize visual pattern and structure at the very beginning of school and their mathematical competences at the end of grade 2 (2010, p. 246): Children who already have this structure sense at the beginning of school are very likely to be the higher achievers at the end of grade 2, and vice versa, those how have no such sense tend to be the lower achievers.

In addition, Lüken discussed the question what the cognitive milestones in the development of an early structure sense are (2011, p. 2). From an analysis of video-taped interviews with children just starting school she concluded that the lower achievers, for example, do interpret a pattern of dots which are arranged as the die-five as *one* number (namely "*the* five") whereas the high achievers are able to interpret this pattern *in addition* as a partition of this number (4+1 or 2+2+1): "High achievers have an awareness of the spatial structure and function of particular configurations" (Lüken, 2011, p. 5). It follows that "a learner has to organize the perception of things in a particular, mathematical way, for instance learn to relate geometric clues to numerical matters, and flexibly decompose them to related substructures" and "intentionally reframe the structures of a pattern". Most learners cannot do this process by themselves, "they have to be instructionally supported with" (p. 7).

Figure 2. Pattern with six dots.

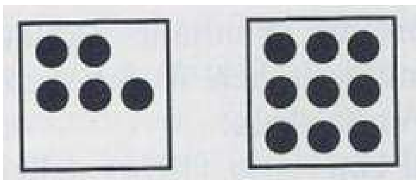


Figure 3. Pattern with 5 and 9 dots.

A similar conclusion was drawn by Scherer (2005, see also Scherer & Moser Opitz, 2010, p. 97) who emphasized that other numerical pattern as those on die (as, e.g., those in fig. 2) have to be worked out explicitly with the children. In the Australian "Early Numeracy Research Project" (see Clarke, Clarke, & Cheeseman, 2006; or Clarke, Clarke, Grüssing, & Peter-Koop, 2008) the recognition of pattern of 2, 3, 4, 5 and 9 (all arranged not as those on the die) was investigated with 5 years old children. These authors found out that between 95% and 100% of these children identified an arrangement of two dots, but only 43 % were able to identify the five dots and 9% to identify the nine dots in fig. 3.

In one of our own projects we focussed on the question how children operate with patterns or how they use them in special tasks. About 70 children in their last year of kindergarten the item in fig. 4 was presented in interviews (for more details see Hasemann, 2006, pp. 74f; 2007, pp. 34ff).

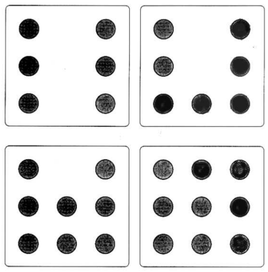


Figure 4. Point to the square with seven dots.

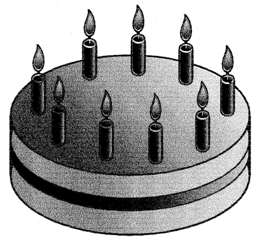


Figure 5. Cake with nine candles burning.

Nearly all the children could solve the problem in fig. 4, but the time needed to complete the task was extremely different: Some counted all the dots and needed minutes to find the correct square; others were ready in seconds as they had realized immediately that in this square there are six dots (arranged like those on a die) plus one dot, or they saw two times three dots and one dot. The picture in fig. 5 was presented to the children together with a story: *It is Kate's birthday anniversary, she is nine years old now, and the nine candles at the cake are burning. Kate is asked to blow out the candles; after the first blow five candles are still burning. How many candles Kate has blown out with her first blow?*

Some children in the experiment in their answers referred to the figures given in the problem (5 or 9), some tried to separate the (still) burning candles from the others and to count the (fictive) non-burning ones by using their fingers or a pencil. Most interesting, however, are those children who proceeded like Arnika:

1 I: ... and how many candles has Kate blown out?
2 S: (with great effort she looks at the picture) four.
3 I: How did you calculate this?
4 S: I have used my fingers.
5 I: You did it under the table ... please, show me!
6 S: First I did the nine (she shows nine fingers) ... then I have put away the five, and then I counted the rest.

This spontaneous use of fingers is most interesting for our insight in the development of mathematical thinking: The girl provided herself with representations for objects she wants to count, although they are not directly to be seen in the picture. The use of representations in this way seems to be an important step on the path to more general methods for the solving of problems like this, and a mathematical solution is now possible without having directly (or pictorial) access to the real objects. The next steps on this path are the construction of pure mental representations of the situations, and their symbolic representation in formal calculations.

Observations and findings in the first grades of school

The question is how to support the learners. In a teaching experiment in grade 2, Hasemann and Stern (2002) tried to find out which arrangements in the classroom might be more likely to support weaker students' ability to grasp numerical relationships (for details see the next section). As a starting point interviews on word problems were conducted at the beginning of grade 3. The following transcript is taken from an interview with an eight-years-old girl who was asked to solve this problem:

Jan has got 7 rabbits. He has got 4 rabbits more than Thomas. How many rabbits do both boys have together?

I: Please, read the text.
S: (reads the text) . . . 7 + 4.
I: How did you do that?
S: Because there is 'how many rabbits do both boys have together' . . . 7 + 4 equals 11.
I: Why is it 7 + 4?. . . (16 sec). . . How many rabbits has Jan got?
S: . . . 7.
I: And how many has Thomas got?
S: 4.
I: 4? (The girl nods her head). Where is that in the text?
S: points to the text.
I: Please, read the text.
S: He has got 4 rabbits more than Thomas.
I: . . . Who is 'he'?
S: Jan.
I: Fine. This means, Jan has got 4 rabbits more than Thomas . . . and how many rabbits has got Thomas?
S: 4.
I: 4?
S: nods.

This kind of dealing with the numbers in a word problem is widespread, and it might be interpreted in different ways: For example, it might be concluded that problems like this were ignored in the class yet; or the girl didn't pay enough attention or has weak understanding of this special kind of problems; or she doesn't like mathematics at all. Even if these conclusions were more or less correct (this girl was seen by her teacher as a rather bright learner in language, but not so good in mathematics), they do not reach to the heart of the matter.

In a study with children from kindergarten to grade 3 Riley and Greeno (1988, see also Hasemann, 2007, pp. 196f) found 14 types of word problems with extremely different levels of difficulty. Most items of the "compare" type (as for instance: "Mary has got 4 marbles. She has got 3 marbles more than John. How many marbles has John got?") are rather difficult for younger children. The level of difficulty of an item, above all, depends on the difficulty children have to transfer the real situation given in the word problem into the mathematical language, i.e. it depends on the fact how easy or how difficult it is to represent the situation in mind, to connect this situation with available knowledge, and to deduce adequate calculations.

Following the path "abstraction from realistic situations" sequences of symbols like "$4 = \square + 3$" or "$\square + 4 = 7$" only make sense for children if they have learnt to connect these sequences with different situations in such a way that they can transfer it also to new situations. A step in this mental process from situations to sequences of symbols (and back from symbols to situations) might be pictures (or diagrams) which do not just reproduce the situations but represent the relevant mathematical relations without irrelevant details. As an example we take the task

There are 9 children on the red bus. There are 6 children more on the green bus than on the red bus. How many children are on the green bus?

The pictures in fig. 6 were drawn by a student in grade 3 who reproduced the situation with a lot of (irrelevant) details whereas the diagram in fig. 7 (which is taken from the work in the classroom [see the next section]) represents quantities and the relevant relationship between these quantities:

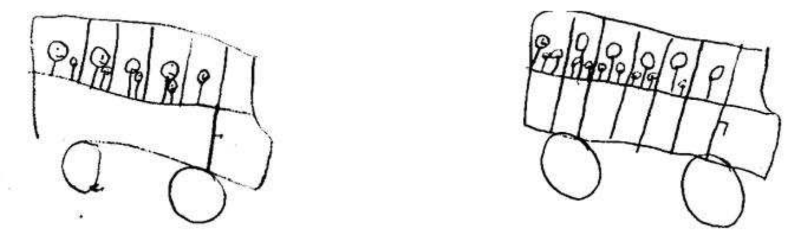


Figure 6. The red bus and the green bus.

Figure 7. The red bus and the green bus.

As mentioned above, action-related thinking becomes inadequate when the situations cannot directly be simulated by actions. In fact, a word problem becomes nearly insoluble for a lot of children when a relation between quantities has to be recognised; the girl in the interview (at the beginning of this section) had this problem: She ignored the relation in the relevant statement ("he has got 4 rabbits *more than* Thomas"), but referred to a cardinal number (in the sense of "he has got 4 rabbits") and did an obvious calculation ($7 + 4 = 11$).

Most lower achievers in mathematics are not able to detach their thinking from concrete objects and real actions: "The properties by which the physical objects are described and classified need to be ignored; and attention is focused on the actions on the objects which have the potential to create an 'object of the mind', which has new properties associated with new classifications and new relationships. For some there may be a cognitive shift from concrete to abstract in which the concept of number becomes conceived as a construct that can be manipulated in the mind. For others, however, meaning remains at an enactive level; elementary arithmetic remains a matter of performing or representing actions" (Gray, Pitta, & Tall, 1997, pp. 115f). These authors' evidence is based on responses to a range of elementary context-free addition and subtraction problems given by children at ages from 7 to 11: "'Low achievers' tended to highlight the descriptive qualities of the items in strongly personalised terms, ... there was a tendency to associate these items with a story in the sense that they were seen as pictures that required colour, detail and a realistic content. In contrast, 'high achievers' concentrated on the more abstract qualities within (the) series of items. Though they initially focused on core concepts, they could traverse at will a hierarchial network of knowledge from which they abstracted these notions or representational features" (p. 23).

The next examples are taken from our work with mathematically gifted children, aged 5 to 8 (Hasemann, Leonhardt and Szambien, 2006). Confronted with the item

To finish a special work 4 machines need 25 days. Unfortunately, after 7 days one machine breaks down and the work is finished with only 3 machines. How many days the work is delayed?

a boy produced as an answer the diagram in fig. 8 (the reader is hearty invited to find out why the boy – rightly – regarded it as a solution of the problem):

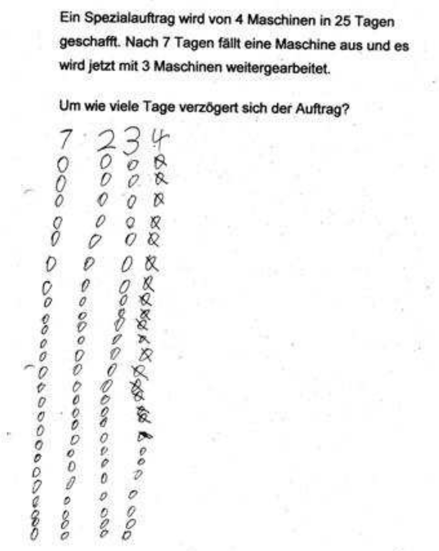
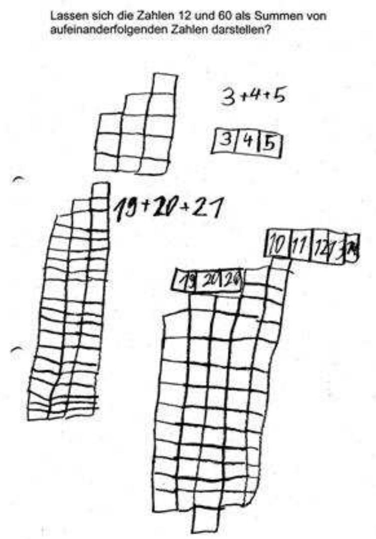


Figure 8.



Picture 9.

The diagrams in fig. 9 were produced by the same boy some days later to this item

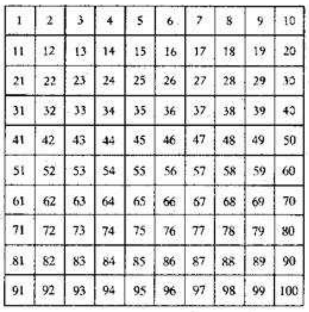
Is it possible to write 12 and 60, resp., as sums of consecutive numbers?

These diagrams, especially that one which is related to 60, highlight the role and the importance of external and internal representation in the process of generalization.

A teaching experiment in grade 2

Having in mind the behaviour of students as presented in the interview in the previous section, Hasemann and Stern (2002) carried out a 12 lessons intervention study on word problems in nine classes at the end of grade 2 in the Hanover area. At that time word problems were well-known to the children. Two different additional training programmes for the solution of word problems were developed, each tested in three classes; in addition, there were three control classes. One of the programmes focussed on students' real-life action-related behaviour. In this programme the teachers' instructions followed the scheme "from the concrete to the abstract" (and were guided by the idea of "ongoing schematisation" developed in the Utrecht project mentioned above) while the other programme was based on abstract and symbolic activities.

The "abstract-symbolic" training-programme was conducted in three classes. The mathematical relations and structures that are particular difficult for children were topic and made explicit in these lessons, and specific help to overcome the obstacles were provided. This programme was not "abstract" in the sense that just formal calculations were carried out; instead this programme was also action-related and included a lot of "games" appropriate for children at grade 2. However, in order to visualise relationships between numbers, not real objects were used, but tools which represent arithmetical relationships on a symbolic level such as the 100 square and the number line (fig. 10 and 11), this means, the representation was mathematically structured.



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 10. 100 square.

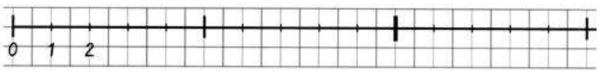


Figure 11. The beginning of the number line.

Exercises with the 100 square: The children sat in a semicircle in front of a big 100 square-poster and followed a route on it given by the teacher:

1 At the beginning I stood on the 7.
2 Then I walked a step downwards.
3 After that I walked 26 steps forwards.
4 Then I walked 3 steps upwards and one to the left.
5 Where I am?

After some "Where I am" – games the students were encouraged to follow the route with blindfolded eyes.

At the number line a game called "Mister X" was played: An empty number line was drawn on the board and the teacher (or a student) wrote a number ("Mister X") between 0 and 100 on the back of the board. The players tried to guess this number by narrowing down the numeral range, it was only announced whether the number was too small or too big; the players had 10 attempts at most.

The children also played "brain-games" like: "I imagine two numbers. One is bigger by 5 than the other. Which numbers could it be?" In the training in these classes mainly relations between numbers were emphasised, and then more and more used by the students to solve word problems.

Before and after the training-programme a test was carried out in the nine classes taking part in one of the two training programmes mentioned above or in the control classes. During the evaluation a considerable improvement was becoming obvious, especially with the low-achieving children, not only in the correct solution of word problems but also regarding their ability to solve other arithmetical tasks. An increase of efficiency was to be expected because there is always a correlation between time of lessons and progress in learning. The main surprise was, however, that the programme focussed on pupils' real-life action-related behaviour had the lowest success, while the "abstract-symbolic" programme achieved the most increase of efficiency with the lower-achieving children (i.e. the weaker half of the children). Regarding all children in the tests on the one hand and for the lower-achieving children on the other hand, on average, for the 16 word problems in the pre- and in the post-tests on word problems the results were the following:

There is an increase of correct answers in the "abstract-symbolic" programme classes from of 6.5 to 9.8 (i.e. +3.3) for all children, and +4.5 for the lower-achieving children; in the action-related programme classes from 6.3 to 8.7 (i.e. +2.4), and +2.8 for the lower-achieving children; in the control classes from 6.4 to 9.0 (i.e. +2.6), and +3,2 for the lower-achieving children, cf. Hasemann and Stern, 2002, pp. 236ff; Hasemann, 2005).

Conclusions

These findings are not really surprising. It is even plausible that especially the less competent children are best aided by helping them to recognise relations, patterns and structures which they – in contrast to the more competent children – are not able to find by themselves in the concrete and obvious. This recognition evidently stands in contradiction to a popular way of acting (see, e.g., Gellert 1999, pp. 114/131). Most of the teachers seem to believe that especially with the less competent children the only way of acting is "from the concrete to the abstract", or – the worst – they believe that the only way of teaching is to remain in the obvious and the concrete.

The difficulties numerous children have with mathematics, not only in primary but also in secondary schools, are partly due to the use of numbers exclusively as cardinal numbers (quantities) and in rather simple arithmetics. This leads to a restricted mathematical understanding and makes generalization difficult (or even impossible). In the first grades it is possible to solve most arithmetical tasks only by the conception of concrete actions. This thinking is insufficient in higher classes (and – among other problems – leads to the well-known difficulties with fractional arithmetic), children should learn to shape relations between numbers already in primary school. In addition, the procedure "from concrete to abstract" is not sufficient enough to help low-achievers to detach themselves from the concrete and obvious and to recognise the relations and structures in the actual situation. As a matter of course it is necessary in mathematical lessons to start with concrete actions and a practical context which is directly comprehensible for children; however, it is important to go carefully directed (and not only implicit) into relations and structures. If they are not misunderstood as counting-tools, materials like the 100 square and the number line (with their pre-forms as abacus, 20 number grid and calculation chain) are excellent fields of experience and practise especially for less competent children to create mental models of situations where mathematical relations are represented. Our study has shown that it is possible to encourage low-achieving primary school children through carefully directed abstract-symbolic activities to insights in mathematical relations. Materials for instruction and methodological suggestions for such lessons are available for a long time past.

References

A e b l i, H.: 1980, *Denken: Das Ordnen des Tuns*, Klett-Cotta, Stuttgart.

C a l u o r i, F.: 2004, *Die numerische Kompetenz von Vorschulkindern*, Kovač, Hamburg.

G e l l e r t, U.: 1999, Vorstellungen angehender Grundschullehrerinnen von Schülerorien tierung, *Journal für Mathematik-Didaktik*, **20**, 113-137.

C l a r k e, B., C l a r k e, D., C h e e s e m a n n, J.: 2006, The mathematical knowledge and understanding young children bring to school, *Mathematics Education Research Journal*, **18(1)**, 78-103.

C l a r k e, B., C l a r k e, D., Grüßing, M. & Peter-Koop, A.: 2008, Mathematische Kompetenzen von Vorschulkindern: Ergebnisse eines Ländervergleichs zwischen Australien und Deutschland, *Journal für Mathematik-Didaktik*, **29**, 259-286.

G r a y, E., P i t t a, D., T a l l, D.: 1997, The nature of the object as an integral component of numerical processes, in: *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, vol. 1, 115-130.

G r e e n o, J. G.: 1989, Situations, mental models, and the generative knowledge, in: Klahr, D. & Kotovsky, K., (eds), *Complexe information processing: The impact of Herbert A. Simon*, Erlbaum, Hilldale NJ, 285-318.

H a s e m a n n, K.: 2005, Word problems and mathematical understanding, *Zentralblatt für Didaktik der Mathematik*, **37**, 208-211.

H a s e m a n n, K.: 2006, Mathematische Einsichten von Kindern im Vorschulalter, in: Grüssing, M. & Peter-Koop, A, (eds), *Die Entwicklung mathematischen Denkens in Kindergarten und Vorschule*, Miltenberger, Offenburg, 67-79.

H a s e m a n n, K.: 2007, *Anfangsunterricht Mathematik*. Spektrum Akademischer Verlag, Heidelberg.

H a s e m a n n, K. Stern, E.: 2002, Die Förderung des mathematischen Verständnisses anhand von Textaufgaben – Ergebnisse einer Interventionsstudie in Klassen des 2. Schuljahres, *Journal für Mathematik-Didaktik*, **23**, 222-242.

H a s e m a n n, K., L e o n h a r d t, U. S z a m b i e n, H.: 2006, *Denkaufgaben für die 1. und 2. Klasse.* Cornelsen-Scriptor, Berlin.

L i n c h e v s k i, L., L i v n e h, D.: 1999, Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, **40**, 173-196.

L ü k e n, M.: 2010, The relation between early structure sense and mathematical development in primary school, in: *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Belo Horizonte, Brazil, vol. 3, 241-248.

L ü k e n, M.: 2011, *School starters' early structure sense*, Paper presented at the 35th Conference of the International Group for the Psychology of Mathematics Education, Ankara, Turkey.

L ü k e n, M.: 2012, *Muster und Strukturen im mathematischen Anfangsunterricht – Grundlegung und empirische Forschung zum Struktursinn von Schulanfängern*, Waxmann, Münster.

M u l l i g a n, J., M i t c h e l m o r e, M.: 2009, Awareness of Pattern and Structure in Early Mathematical Development. *Mathematics Education Research Journal*, **21**(2), 33-49.

R o w l a n d s, S., C a r s o n, R.: 2002, Where would formal, academic mathematics stand in a curriculum informed by ethnomathematics? A critical review of ethnomathematics, *Educational Studies in Mathematics*, **50**, 79-102.

Riley, M. S., Greeno, J. G.: 1988, Developmental analysis of understanding language about quantities and of solving problems, *Cognition and Instruction*, 5, 49-101.

Streefland, L.: 1991, *Fractions in Realistic Mathematics Education*, Springer, Heidelberg.

Stern, E.: 1998, *Die Entwicklung des mathematischen Verständnisses im Kindesalter*, Pabst Publisher, Lengerich.

Treffers, A.: 1987, *Three dimensions. A model of goal and theory description in Mathematics Instruction – The Wiskobas Project*, Reidel, Dordrecht.

Van Luit, J., van de Rijt, B., Hasemann, K.: 2001, *OTZ. Osnabrücker Test zur Zahlbegriffsentwicklung. Manual*, Hogrefe, Göttingen.

Reprezentacje myślowe obiektów i relacji matematycznych uczniów klas pierwszych

Streszczenie

W pracy pokazano, że adekwatna, zewnętrzna i wewnętrzna (myślowa) reprezentacja jest istotna tak dla matematycznego rozumienia, jak i uogólniania. Użyto w tym celu przykładów z edukacji matematycznej przedszkolnej i wczesnoszkolnej. W głównej części rozważa się kwestię, czy i w jakiej mierze posłużenie się konkretnymi materiałami i codziennym doświadczeniem dzieci może stanowić przeszkodę w poznawaniu przez nie relacji między liczbami i specyficznych aspektów dodawania i odejmowania oraz zdobywaniu umiejętności uogólniania pojęć matematycznych. Dodatkowo omówiono alternatywne rozwiązania dla szkolnej praktyki.