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DOCTORAL THESES

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The role of multistage tasks in developing creative activities among mathematics teachers[1]

1 Introduction

The issues addressed in this thesis are focused on creative mathematical activities and on a conception of their development among in-service and pre-service teachers. Mathematics education depends mainly on mathematical and didactical knowledge of the teachers, therefore I decided to be knowledgeable about the knowledge and the skills related to the creative mathematical activities of the teachers and then design the workshops around multistage tasks which would develop that knowledge and skills.

On the one hand contemporary mathematics education follows in the direction of mathematical activities. In many professions creativity, ingeniousness and a creative attitude to the problems are required even from a young person who just entered the field. These expectations are reflected in many official documents – from European documents (like national curricula (Ministère de l'Education Nationale, de la Recherche et de la Technologie, 1997; DFE, 1997;

[1]The paper is a modified version of the author's lecture presented on 27th of October 2009 at the Pedagogical University of Krakow during her doctoral defense. Prof. Maciej Klakla (Pedagogical University of Krakow) was the supervisor of the PhD thesis. Reviews were made by: Prof. Ewa Swoboda (University of Rzeszow) and Prof. Eugeniusz Wachnicki (Pedagogical University of Krakow).

Ministério da Educaçao, 1997; NCTM, 1991, 1998; Podstawa programowa dla matematyki, 2007); "Education and training in Europe: diverse systems, shared goals for 2010" (2002); the principles of the PISA exams (Sułowska, Marciniak, 2004)) to the postulates in didactical literature (Krygowska, 1958, 1986; Polya, 1975, 1993; Mnich, 1980; Klakla, 1982, 2002; Mason, Burton, Stacey, 2005; Ponte, 2001, 2008).

At the same time, that creative side of education is almost absent at school. Mathematics teaching very often has an imitative and reproductive character. It is focused on elementary activities and skills and the students learn some schematic behaviours.

> In general, mathematics teaching pays little attention to the more advanced aspects of mathematical activity such as the formulation and resolution of problems, the formulation and testing of conjectures, the pursuit of investigations and mathematical proofs, and the argumentation and critique of results. While these are fundamental and current themes of mathematics education expressed in many curriculum documents across the world (..) they still find very little emphasis in classroom practice, both in our country and elsewhere (Lerman, 1989; Silver, 1993), (Oliveira, Ponte, Segurado, Cunha, 1997, p. 135).

This is because the teachers are not sufficiently prepared to promote creativity in mathematics among their students. They do not have sufficient knowledge, skills, experience and didactical tools to develop creative mathematical activities among their students (Klakla, 2008; Maj, 2006). The essential condition for the development of the skills needed for different kinds of creative mathematical activities among students is the deep understanding of those issues by the mathematics teachers. Only then the teachers would effectively form and develop these activities in their work with the students. According to Nęcka (2005):

> ...a creative teacher will educate creative pupils, a not much creative teacher will rather discourage pupils from unconventional thinking (p. 201).

The year 2009 was announced the European Year of Creativity and Innovation. This underlined the importance of creating the necessary conditions for the people to develop and fully utilize their talents and potentials; it also stressed the importance of forming the attitude of openness and curiosity of the world. By this thesis I wanted to contribute to promoting and supporting creativity.

2 Theoretical background

Mathematics is a specific domain of knowledge (Klakla, 2002), thus it requires a specific attitude to its teaching and learning. From the view that knowledge is not passively received but actively built individually by the learner, which is connected with constructivism (von Glasersfeld, 1995), we have moved to theories which additionally consider the importance of social interactions in constructing mathematical meanings (Ernest, 1991). These social interactions are seen as an essential element of learning through mathematical inquiry (Wilkins, 2008, Jaworski, 1994). Moreover, the researchers who promote the use of mathematical investigations believe that mathematics is a social activity, in which discussion, justification, arguing and negotiating constitute a base of mathematical interactions between the students, and between the students and the teacher (Cobb, McClain, 2006; Baroody, Coslick, 1998; Borasi, 1992; Yackel, Cobb, 1996; Jaworski, 1994; Lampert, 1990). At the same time, being involved in a society of learners is highly influenced by personal factors, such as emotions and attitudes towards mathematics:

> Enthusiasm as a manifestation of intrinsic motivation may promote teachers' active involvement in their work and is likely reflected in high quality instructional behavior (Kunter, et al., 2008, p. 469).

That type of enthusiasm in didactical work favours active learning and engagement of the students. According to Meissner (2008):

> a mathematics teaching which furthers creative thinking needs specific environments (...) like motivation, curiosity, self-confidence, flexibility, engagement, humor, imagination, happiness, acceptance of oneself and others, satisfaction, success, ... (p. 166).

The notion of creativity is ambiguous, since it may appear in many contexts. It is also an object of interest for many sciences. Each one interprets that phenomenon from its own perspective, that is why there is not any general theory of creativity that grasps comprehensively that notion and is universally accepted. Creativity in mathematics is included in the general notion of creativity, thus we can notice some similarities in that how mathematicians see, define and recognize creativity. Likewise in psychological perspective creativity in mathematics is seen differently depending on the opinions of the authors. However, as a shared viewpoint we can recognize that creativity plays an essential role in mathematical thinking both in mathematics as the science and school mathematics.

According to Ervynck (1991) mathematical creativity is the ability of problem solving and/or developing structural thinking by taking into consideration the logical and deductive nature of mathematics. He gives some examples of creativity in mathematics: the ability to formulate a valuable definition by using the concepts which ensure the usefulness of the defined object in a later theory; the mathematization of basic ideas taken from a real context which were initially the base of a mathematical problem. Therefore, according to him mathematical creativity is both the ability of creating mathematical objects and of discovering mutual relations between them. Ernest (2008) sees creative mathematical activity as solving problems, using mathematical methods or performing investigative work; this usually includes the formulation of more complex tasks. It requires a novelty and insight into the choice of transformations which we use and also the choice of which elements of a task we would use to create a sequence of such transformations. For Lakatos (2005) mathematics is not predicable, you cannot create it step by step in a specific direction. You can rather compare it with discovering a new territory during which the 'trip' could be pointless. It is similar with mathematical thinking, which in opposite to ready-made knowledge is a creative activity which allows the possibility of mistakes.

From the above we can conclude that mathematical creativity is related to different kinds of creative mathematical activities. A conception of forming creative mathematical activities was worked out by Klakla (2002). He distinguishes particular kinds of creative mathematical activities, which are present in an essential way in activities of mathematicians. These are:

(a) hypotheses' formulation and verification;
(b) transfer of a method (of reasoning or solutions of the problem onto similar, analogous, general, received through elevation of dimension, special or border case issue);
(c) creative receiving, processing and using mathematical information;
(d) discipline of thinking and critical thinking;
(e) problems' generation in the process of the method transfer;
(f) problems' prolonging;
(g) placing the problems in open situations.

The second element of that conception refers to the multistage tasks which consist of series of tasks, problems and didactic situations which have a specific structure. They are based on problematic situations and connect different kinds of creative mathematical activity with each other in complex and rich mathematical-didactic situations. We can say that they provide a specific laboratory of creative mathematical activity for the students.

According to Meissner (2008) a mathematics teaching which promotes creativity

> needs "challenging problems". They must be fascinating, interesting, exciting, thrilling, important, provoking, ... Open-ended problems are [also] welcome or challenging problems with surprising contexts and results (p. 166).

Specific mathematical tasks can make the solver start analyzing more consciously a mathematical situation from the task based on his/her metacognitive knowledge (Efklides, 2006). Cognitive awareness is studied in mathematics education in the frame of metacognition (Flavell, 1976). The multistage tasks used in my research are chosen in such a way to make the solvers being aware of the creative mathematical activities which they undertake (particularly, five kinds of activities: hypotheses' formulation, hypotheses' verification, transfer of a method, problems' prolonging and discipline of thinking and critical thinking) as well as the aims of solving them, the particular stages' strategies and their importance in the learning process. Additionally, solving the tasks should be connected with a feeling of satisfaction.

3 Multistage tasks used in the research

Multistage tasks is a structured set of tasks, problems and didactical situations, which can be illustrated by some maps (Klakla, 2002). The initial situations of three multistage tasks which were used during the main phase of the research will be presented. The workshops were designed and conducted around these tasks. It is worth underlining that the mathematical level of the tasks does not go beyond the curriculum of the secondary school. This fact eliminates a number of difficulties which are connected with the new knowledge, and the attention of a solver is focused on mathematical activities undertook during the solving process. When the problem is an open-ended and not trivial and known concepts are in new situations, it can become a challenge.

Task 1 – 'Lengthening the sides of a triangle' (Gruber, 2000).

An acute triangle ABC is given. Construct the triangle $A'B'C'$ by lengthening each side of the given triangle ABC by its own length in the same manner of circulation. Compare the area of the triangles $A'B'C'$ and ABC. Give the ratio of their areas.

- What happens with the ratio of the areas when you make 2-times, 3-times, n-times ($n \in N_+$) longer the sides of the triangle ABC?
- Consider an analogous situation for a convex quadrilateral $ABCD$.

Task 2 – **'Butterfly'** (Czernecka, 2000).

An acute triangle ABC is given. Through any point P which belongs to the inside of the triangle, three lines parallel to every side of the triangle are drawn. The lines divide the triangle ABC of the area S into six parts. Three of them are triangles of areas S_1, S_2, S_3. The figure which is a sum of these three triangles with the common vertex P is called 'butterfly'. The three triangles are called the 'wings' of the 'butterfly' (Figure 1).

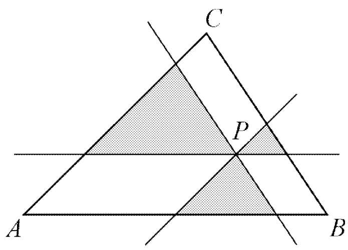


Figure 1. The initial situation of the task 'Butterfly'

Without any suggestions on the direction of enquiry the teachers were asked by the instructor: "What questions would you like to ask to that situation?"

Task 3 – **'Averages'** (Klakla, 2002).

Positive numbers a, b are given ($a > 0, b > 0$).

$S_1(a,b) = \frac{a+b}{2}$

$S_2(a,b) = \sqrt{ab}$

$S_3(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}$

Without any suggestions on the direction of enquiry the teachers were asked by the instructor: "What questions would you like to ask to that situation?".

4 Methodology

The pilot study (Maj, 2006) was the first attempt to recognize the knowledge, skills and awareness related to creative mathematical activities of teachers and students of mathematics (pre-service teachers). The main phase of research was two-staged. In the first stage the aim was the recognition of those issues among in-service and pre-service teachers. The second stage was related to developing that knowledge, skills and awareness. In the first stage I used the tool which was worked out during the pilot study, in the second – I designed some workshops with the use of multistage tasks.

Therefore the aims of the main phase of the research can be described as follows:

1. Recognition in the research group of the:

   A) awareness of the need of developing CMA (creative mathematical activities);

   B) current state of the skills of undertaking CMA.

2. Verification of the hypothesis, that multistage tasks can be used as a tool:

   A) to **introduce** the teachers (in-service and pre-service) into a particular kind of creative mathematical activity (an **introductive tool**);

   B) to **develop** the **skills** of undertaking CMA among the teachers (a **developing tool**);

   C) to **raise** among the teachers the **awareness** of what are CMA and of the need of developing CMA among the students (an **awareness-raising tool**);

   D) to **diagnose** the **skills** of undertaking a particular kind of CMA among the teachers (a **diagnostic tool**).

The research was conducted in two groups: mathematics teachers and mathematics students (pre-service teachers). In both groups they had a similar course but partially they had different aims. Because of the size of the group of the teachers (7 persons) it was possible to make a detailed analysis of the process of solving the tasks and treating the results as a source of the research hypotheses. Whereas the research in the much larger group of the mathematics students (three groups, in total 39 persons) served as a supplementation of the results and as an initial verification of the formulated hypotheses.

During the research several methods, techniques and different research tools were used. The various research aims led to different research methods: observation, comparison and measurement (Juszczyk, 1998). Diagnostic survey, analysis, assessment and interpretation of the products and observation, interview and conversation (Juszczyk, 1998, Łobocki, 2006) were used. This decision had an influence on the choice of research technique – analysis of the documents (Łobocki, 2006). The research tools were adapted to the methods. The multistage tasks which were used in the research were treated in two ways – firstly, they were the main tool for stimulating the development of CMA, secondly, the subject of the research was the influence of these tasks on the realization of the aims of developing CMA. To verify if and in what degree the development of CMA occurred, some additional tools were needed: a *starting test*, a *set of two open-ended tasks* (Maj, 2007) and a *diagnostic questionnaire*. Additionally, observations and analyses of the lessons conducted by the in-service teachers (Maj, 2008; 2009) and analyses of the scenarios of the lessons of the pre-service teachers were done. In Table 1 the aims, methods and the research tools related to the research done among in-service teachers are presented.

Aims		Methods	Research tools
1AB	Recognition of the skills and awareness related to CMA (initial diagnostic research)	Diagnostic survey; analysis, assessment and interpretation of the teachers' work; conversation	*Starting test*
1B	Recognition of the skills of undertaking CMA (initial diagnostic research)	Analysis, assessment and interpretation of teachers' solutions	*The set of two open-ended tasks*
2ABCD	Developing the skills and awareness related to CMA	Non-participant observation; analysis, assessment and interpretation of transcripts of the audio recording of the workshops	*Multistage tasks*
2BD	Second recognition of the skills of undertaking CMA (final diagnostic research)	Analysis, assessment and interpretation of teachers' solutions	*The set of two open-ended tasks*
2C	Second recognition of the awareness related to CMA, assessment of the workshops and the multistage tasks as the tool (final diagnostic research)	Diagnostic survey; analysis, assessment and interpretation of the teachers' work; conversation	*Diagnostic questionnaire*
2BC	Verification of the skills of provoking and developing CMA among the pupils, recognition the awareness related to CMA	Non-participant observation; analysis, assessment and interpretation of transcripts of the audio recording of lessons conducted by the teachers	*The sets of tasks constructed by the teachers*

Table 1. Aims, methods and research tools used in the research among in-service teachers

The main part of the research was the conducted workshops. The in-service teachers (of gymnasium and high schools) took part in a series of workshops from March to September 2006 (in sum 24 hours (45 min.)). They were organized as part of the Professional Development of Teachers Researchers (PDTR) during the mathematics course. The mathematics students participated in the workshops within the course ‘Didactic of Mathematics’ (in winter semester 2006/2007) and their duration was as follows: group [1]: 24 hours, group [2]: 24 hours and group [3]: 15 hours.

The analysis of the workshops’ transcripts was conducted in the direction of answering the following questions:

- Did the teachers gladly engage in working on multistage tasks (emotional aspect)?
- Did the teachers notice and name some kinds of mathematical activities which they were undertaking during solving a task?
- Did the teachers formulate some problems, put questions, hypotheses and did they verify them? Were the formulated problems non-trivial?
- Were discipline of thinking and critical thinking evident in the way of solving the problems and verifying the hypotheses?
  - Did the teachers invoke known theorems and definitions during the verifying of the hypotheses (in case there was such a need)?
  - Did the teachers evaluate their own reasoning, were they able to find and correct their errors? Did the teachers evaluate the others’ reasoning, were they able to react, find and correct the errors?
  - Did the teachers have the ability to communicate in precise mathematical language?
- Did the teachers prolong the tasks?
- Did the teachers use the transfer of a method or the transfer of a method with modifications?
- What roles did the instructor – who was acting as a model of a teacher working on developing CMA – play?
- Was the undertaking of some activities caused by the stimulation by the instructor or by the problem itself?

5 Results

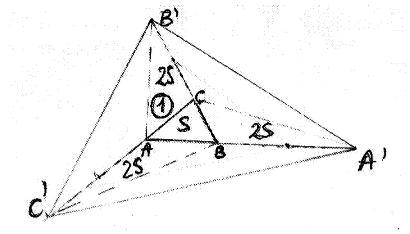
Part of the results presented in the thesis consisted of the fragments of thirty four dialogs from the workshops conducted among the teachers. In this paragraph five of them together with their analyses will be shown. The dialogs were realised either between the instructor (I) conducting the workshops and the teachers (T) or between the teachers.

The 'Lengthening the sides of a triangle' task was the first to be considered at the workshops. During the work on the task three methods of solving were used: proper 'cuttings', use of a particular formula of the triangle's area and use of triangles of the same areas. Then the teachers were asked to prolong the initial situation by changing the times of lengthening on 2-times, 3-times and n-times:

Fragment 1

1 I: So we have the task: we are lengthening 2-times the triangle's sides...

2 T2: Lengthening 2-times... (on the blackboard):



The height is the same (ACB' and ABC) and the base is 3-times longer (the triangle ABB')

3 T1: 2-times.

4 T4: But it depends on which triangle we are talking about...

5 T2: Exactly, maybe we're talking about this one (he is showing the triangle ACB') and here the base is 2-times longer.

6 I: So write the area.

7 T1: $2S$.

8 I: And now what?

9 T2: Now...

10 T3: Where else is $2S$?

11 T2: And now here... Here it will be the same (the triangle ABC'), won't it be?

12 T1: Yes.

13 T2: Because there is the base AC and the height here (he is showing). And the third connection (he is drawing a triangle CBA'), here is again the same, I mean $2S$. Now, what about these ones? (the other triangles: $CA'B'$, $AB'C'$, $BC'A'$)

14 I: Yes, what about them?

15 T1, T3: It is 2-times bigger (the triangle $AB'C'$).
16 I: When I told you about this student, what I said about this and this? (she is showing the triangles ACB' and $AB'C'$)
17 T2: I must have missed it... what is happening here? (he is thinking)
18 T1: I also... I must have missed it too.
19 I: He compared this one with this one (ABC and ACB') and then this with this one (ACB' and $AB'C'$).
20 T4: And then he turned this situation and there he had the same heights (ACB' and $AB'C'$).
21 T2: So he compared this with this, the heights are the same, so it has to be $4S$.
22 T1: Once again!
23 T2: It will be $2 \cdot 2S$ – because of that (he is showing on the drawing)
24 I: He is now comparing this triangle and this triangle (she is showing)
25 T2: The height is the same (showing) and the base is 2-times longer, so the area is 2-times bigger than this one ($2S$), so $4S$.
26 T1: I see now!
27 T2: The same is here $4S$ and $4S$ (he is writing the areas for the triangles which are left).
28 I: So the previous ratio was $\frac{1}{7}$, what is now?
29 T2: Now we have 3 times 4 and 3 times 2, so 18 plus one, so $\frac{1}{19}$.

In spite of the fact that the third way of solution ('triangles of the same areas') was introduced by the instructor (by giving some clues) as an alternative way, the teachers liked that way the most. Probably it was because of its elegance and simplicity. And they chose that method for solving the next problems. Although that method seemed that it does not require advanced knowledge, when the teachers wanted to transfer it on the analogous task they had difficulties. The instructor reminded them that method (by the story of a 12-year old student who solved that problem) but she did not tell them what the areas are. She only suggested what triangles the teachers should look at.

The example of that task shows that by solving an analogous problem you can develop an important creative mathematical activity – transfer of the method. In spite of the fact that the method was quite easy, using it in the next problem was challenging for them. They started asking questions by themselves, explaining themselves and after such discussions we can say that they really familiarised with this method and understood it. The important fact was that work in groups gave them opportunities for such common discussions.

The other prolonging of the task was considering a quadrangle instead of a triangle:

Fragment 2

30 I: Let's work on the next task: a quadrangle.

The teachers are working silently; they are drawing, cutting the figure into triangles. Then they are discussing in pairs (or triads) trying to use the method of triangles of the same areas:

31 T7: These ones are for sure equal, these also, but I have too many of these Ps...

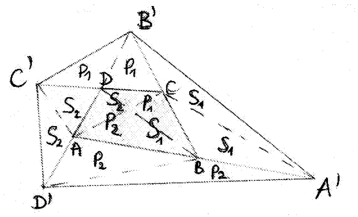
32 T4, and T1: (they obtained the result $\frac{1}{5}$ and are happy like children): I HAVE IT, HURREY!!!

33 T3 asks T4: How to calculate this?

34 T4: You have to only notice that $S2 + S4 = S1 + S3$.

35 T3: How nicely you wrote it!

36 T2: (drawing):



37 T7: It can't be in 2, you have to do it in 4 (divide the quadrangle in triangles).

38 T5: You should do it in 4 from the beginning.

39 T1: In 4? Aha! I also did it in 2.

40 T2: $2P1 + 2P2 + 2S1 + 2S2 = 2P + 2P$ plus $P-$ that initial one, that is $5P$.

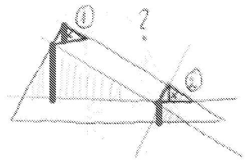
This time the teachers did not have any problems with the transfer of the assumptions of the task. They started working in small groups. Their strategy was to divide the initial quadrangle into triangles and it was done in two ways. In both cases the teachers were searching for triangles of the same areas with these which arose as the result of the division of a triangle. Therefore they modified the previous method of solving and they transferred it into the new problem. But not all managed to do it so easily (e.g. T7 in [31] or T3 in [33]). When T2 presented his solution on the blackboard, the discussion in how many triangles should be a quadrangle divided started.

In this fragment the teachers undertook a creative mathematical activity – transfer of the method of solution on an analogous task. They did it by themselves without any suggestions. They tried to transfer the method which they used in the case of triangles. When it showed that it cannot be transferred directly they modified it in such a way to use it also in the case of a quadrangle. It is worth noting the joy that the solution of this task gave them [32]. It is also important to underline the special role of the instructor who only initiated the problem and then she backed out and let the teachers decide by themselves how to solve it.

The next three fragments are related to the second multistage task called 'Butterflies'. Without any suggestions on the direction of enquiry the teachers were asked by the instructor: "What questions would you like to ask to that situation?". The teachers formulated some questions and answered them but they were still searching for other relations in the initial situation:

Fragment 3

41 T2: T3 has something.
42 I: T3, show us what you have.
43 T3: But I'm not sure if it is true, it's the question for now...
44 T2 Put hypothesis!
45 Everybody: Put hypothesis!
46 T3: But it's not the theorem!
47 T2: A bold hypothesis and a good hypothesis is precious by itself.
48 T4: I'm not sure if it can be related to the point P – that the sum of the heights of the small triangles equals the height of the big one.
49 I: T3, show us what you have.
50 T3: I don't know if it is so. For now it is only a conjecture (she draws on the blackboard):



For sure those heights (bold lines) give us the height of the triangle ABC. The question is if it is x – if that height and the height of that triangle (1 and 2) are equal!

In this fragment we can observe how the workshop's participants forced one of them to put a hypothesis. The reaction of the teachers showed us on the one hand the willingness of searching a hypothesis and on the other hand the joy of putting it [47]. It also proved that the participants of the workshop became aware of undertaking mathematical activities and their significance. Putting hypothesis resulted immediately in a trial of verifying. T3 knew that the sum of the heights equals the height of the triangle ABC, but she had doubts if the heights of the triangles 1 and 2 had the same length. In spite that the drawing suggested that it is so, the teacher was careful in formulating the theorem. Her reasoning was characterized by discipline of thinking.

The described fragment showed us that the teachers noticed that the work on the task has sense when you put other hypotheses and try to verify them. The teachers felt a need of prolonging the task and that is why when one of them made an observation, the others forced her to present it and then everybody was working on its verification (in the later part of the workshop). When they finished working on a previous problem they started searching for new relations and new ways of prolonging the task by trying to use whole potential of the given situation. In the short presented fragment many kinds of creative mathematical activities appeared: putting and verifying hypotheses, prolonging the task and discipline of thinking. Moreover, the work was accompanied by intense emotions.

Fragment 4

The instructor is talking with the teachers T6, T7 and T3:

51 I: But what would you like to prove? Tell me what you want to prove.
52 T7: That P is the point of intersection of the medians in the triangle ABC.
53 I: Wait, what is your assumption and what do you want to prove?
54 T3: If...
55 I: Exactly, what theorem do you want to prove?
56 T7: That... I mean, I understood that I have to say in what position is..., what is the position of the point P in order for the wings of the butterfly to be congruent to each other.
57 I: Ok.
58 T6: If the point P is the point of intersection of the medians then the newly-created triangles are congruent.

This part of the dialogue shows the teachers' difficulties in using mathematical language. This is surprising because as teachers they should be able to educate their students to communicate in mathematical language. For the question "what would you like to prove?" [51] the teacher replied only by giving the conclusion without mentioning the assumptions. Only when the second question appeared, the teacher T3 started her sentence with the word 'if'. The instructor wanted to enforce putting the hypothesis in the form ‘if..., then...’, but T7 was so involved in her reasoning, that she only explained what was the aim of the search, although she had found the solution before (in the previous talk). Then T6 formulated the correct theorem [58] so the instructor finally got the answer to her question. We can notice the special role of the instructor, whose aim was to develop creative mathematical activities, in which discipline of thinking and critical thinking has significant meaning. Without this interference, the teachers would prove a hypothesis in which the assumption was not clear and they would start solving the next problems. By focusing on the activity itself, the ‘determination’ of the instructor made them aware of the importance of using precise mathematical language.

Fragment 5

The talk is between three teachers T4, T6 and T1:

59 T4: But exactly what has to be proved? That point P belongs to the median? No, if these are medians then the ratio is 1 to 2. Is it enough?

60 T6: No, it’s not! For me not, because I would prove after all that this is the half of this side, that this side is divided into halves. It isn’t enough, that ratio, for me personally. I know that if it is a median then..., but I don’t know that if it is divided (in the ratio $2:1$) then it is a median.

61 T1: Yes, that theorem says that if these are medians then they are divided in the ratio 2 to 1. But it is not vice versa – if they are divided in that ratio then they are medians.

62 T6: A median for me is a ray which is drawn in a special way, which fulfils a condition, that when it is drawn from a vertex, it divides the opposite side into halves. So if I want to prove that this is a median, I think I should prove that.

63 T1: But it is a segment! (not a ray).

The talk is between three teachers. T4 was not sure that in order to prove that the segments in a triangle are medians, if it is enough to prove that they intersect in the ratio $2:1$. The teachers did not remember exactly the theorem about medians. They concluded that it is safer to prove that the segment divides the opposite side into halves. So, they referred to the definition of the median in a triangle. They tried to specify their talk and to use mathematical language – which was missing in the previous discussions. What is more – they enforced that formal language on themselves. T6 formulated: “I know that if it is a median then..., but I don’t know that if it is divided (in the ratio $2:1$)

then it is a median" [60]. T1 immediately specified: "that theorem says that if these are medians then they are divided in the ratio 2 to 1. But it is not vice versa – if they are divided in that ratio then they are medians" [61]. T6 then quoted the definition of the median of a triangle, but she made a mistake by saying that the median is a ray, which was noticed and corrected by T1 [62-63]. If the teachers would not use mathematical language this time, the essence of the problem could be missed. Whereas, using that language resulted in all of them talking and thinking about the same thing. The everyday language could be a source of misunderstanding. Precision of the talk helped them to understand the problem. This time the situation in which they were – and not the instructor – imposed on them the discipline of thinking and critical thinking.

In Fragment 5 we have observed how mathematically rich the problems which were considered during workshops are and how important is the interaction between the participants. If that problem was solved by one person, such consideration could not have appeared. When you solve the problem by yourself, you do not use formal language. In that case the teachers worked by stimulating each other and by considering some very important mathematical issues: how to understand a theorem, when to use it, if and when to refer to a definition and a theorem.

Apart from developing the knowledge and skills of undertaking CMA, an important role of multistage tasks was to raise awareness through experience. Awareness was related to what is creative mathematical activity, how to work with it, how it can be provoked, that it needs special tasks, tools and didactic methods to develop it.

Before the workshops the teachers declared (in the *starting test*) that they do not consider with their pupils either the tasks which are prolonged or the open-ended ones (at all or very rarely). Already during the series of workshops they started using multistage tasks (or they declared that they want to use them) in their didactic work at schools. They admitted that it considerably broadened their knowledge and skills concerning CMA and that they started being aware that this kind of activity can and should be developed among the pupils. The participation in the workshops had a big influence on their work at school and their professional development. An example can be drawn from two fragments of the declaration of one of the teachers, who in the *starting test* claimed that:

> *In the classes with mathematics specialization I have 4 hours of lessons and lots of material to realise. The pupils have problems*

> *to learn elementary and basic mathematical skills, to solve typical tasks with the use of basic mathematical techniques, to use mathematical language. In that 4 hours we exercise mainly the elementary activities which without them it would be impossible to solve any open-ended tasks.*

The same teacher in the *diagnostic questionnaire* admitted:

> *At the typical tasks I look totally different. I want to prolong them, I often search for some analogy to known examples together with my pupils.*

6 Conclusions

By analyzing the process of the work on the multistage tasks it can be discerned that there was a hierarchy among the different kinds of creative mathematical activities. That hierarchy was related to the order of becoming aware of different areas of creativity in mathematics. It seems that the easiest to raise is the awareness of what multistage task is and how does the work on it look like (as a structure of the tasks, problems and didactical situations). In the next stage the skill of putting hypotheses and prolonging that tasks appeared. Then the teachers presented an awareness of what is verifying the hypotheses and the need of their verifying. The awareness of discipline of thinking and critical thinking proved to be the most difficult compared to the other creative mathematical activities. The skills of undertaking this activity were developed gradually; this was noticeable in the fragments of the dialogs which were presented. The teachers had the occasion to appreciate the value of undertaking that activity, thus they started to do it consciously.

The aim of my research was to propose a tool to develop knowledge, skills and awareness related to creative mathematical activities and to verify that tool. On the base of the analysis of all research data it can be stated that it is possible to develop creative mathematical activities among the teachers and mathematics students when the special tool is given and when the instructor knows how to use it. That 'effective' tool is a multistage task which can be used to:

- introduce the teachers into a particular kind of creative mathematical activity;
- develop the skills of undertaking CMA among the teachers;

- raise among the teachers the awareness of what CMA are and of the need of developing CMA among the students;
- diagnose the skills of undertaking a particular kind of CMA among the teachers.

Additionally, it can be concluded that:

1. Creative mathematical activities do not develop spontaneously; creativity requires conscious didactical methods and tools on every educational level.
2. Creative mathematical activity is possible to develop at least at the level of gymnasium, secondary school and higher education. However, apart from conscious didactical tools it also requires the conscious attitude of the teacher.
3. The conscious attitude of the teacher can be formed through the personal experience of different forms of such activities. It can be realised in the form of workshops during the process of education of future mathematics teachers and it should be continued as a part of some courses of professional development of the teachers.
4. Multistage tasks is a verified tool for the development of creative mathematical activities among mathematics teachers as well as among mathematics students. These tasks can be selected according to the different kinds of CMA. Among the activities which were examined during the research were: hypotheses' formulation, hypotheses' verification, transfer of a method, problems' prolonging and discipline of thinking and critical thinking. Additionally, the work on the multistage tasks supports the social character of mathematics learning and stimulates the exchange of thoughts and communication.
5. It turned out that it was relatively easy to make the teachers being more open on mathematical problems, more flexible in thinking, more able to see the problems in multifaceted way and able to create the connection of many different concepts (create the cognitive web). That can be obtained as a result of the work with multistage tasks. However, it is much more complicated to form the discipline of thinking and creative thinking. The multistage tasks used during the workshops support the development of discipline of thinking and critical thinking, but they are not sufficient. The discipline of thinking and critical thinking has to be considered comprehensively, it should be formed in a conscious way in every opportunity. Thus, the skills in that area depend on the teachers' previous knowledge and mathematical skills.

6. It became apparent that the problems of didactics of mathematics as a science are closely related to mathematics as a science. Didactics of mathematics is supported by the pedagogical and psychological knowledge, but solving the problems connected with mathematics teaching is deep-rooted in mathematics.
7. The research confirmed that in mathematics teachers education there is not enough attention on the aspect of creativity in mathematics. The mathematics students do not have sufficient knowledge and they are not properly prepared for working at school in the range of creative mathematical activities.

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