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The process of solving a problem at different levels of mathematical knowledge and experience[1]

1 Introduction

The research presented in the book is connected with one of the three key questions which F. K. Lester (1985) has pointed out as central to research on solving a problem. The question is:

What does the individual do, both correctly and incorrectly (efficiently and inefficiently) during solving a problem?

Lester states that:

> Far too little is known about the nature of the problem-solving abilities of individuals. Models of competent problem solvers can serve not only to indicate what good problem solvers do, but also to point out the deficiencies in the problem-solving behavior of novice or poor problem solvers [...] Before useful models of instruction

[1]Summary of the book (Ciosek, 2005) presented on the 6th of November 2009 in Adam Mickiewicz University in Poznań, Faculty of Mathematics and Computer Science as the habilitation thesis. Proceedings were conducted by prof. Jerzy Kąkol from Adam Mickiewicz University, the referees were: prof. Zbigniew Semadeni from the University of Warsaw, prof. Ryszard Pawlak from the University of Łódź, prof. Maria Korcz from Adam Mickiewicz University, and prof. Michał Szurek from the University of Warsaw.

> can be developed we must have useful models of how individuals solve problems without instruction. Efforts in this direction are promising (Lester, 1985, p. 44).

The author of the dissertation agrees with the above opinion. A prerequisite of any discussion and research into the development of heuristic type abilities is a profound understanding of the process of prob1em solving itself, the specific nature of the process, the reasons of difficulties, and the conditions of success or failure. Although many new findings have been obtained in this area of research since 1985 (see for example A. H. Schoenfeld, 1985, 1987, 1994), Lester's words, as quoted above, retain their importance. We need to carry out further research 'to know more about the nature of the problem-solving abilities of individuals'.

The general goal of the investigation presented here was revealing ways of searching for the solution of an open problem by persons at various levels of mathematical knowledge and experience, as well as drawing conclusions concerning specificity of the solving process. In particular, the objective of the research was to obtain an answer, let it be approximate, to the following questions:

1. What features of the process of solving open mathematical problems are common for solvers at different levels of knowledge; what features differentiate those levels?
2. What heuristic strategies of solving open problems do apply experts, i.e. mathematicians, and what students?
3. What level of understanding the methodology of mathematics was exposed by the examined students?

   In fact, what was interest was whether the examined students understand the mathematical proof as a general argument (in opposite to the confidence based on checking several particular cases).
4. What kinds of errors occur in the reasoning of the examined persons?

2 Methodology

Four levels of mathematical knowledge and experience were considered. They were represented by:

- pupils in the last form of lower secondary school (14-15 years old),
- upper secondary school pupils (16-19 years old),

- students of mathematics at the tertiary level,
- mathematicians – teachers of mathematics at the tertiary level.

2.1 The choice of problems

A special set of seven mathematics problems was constructed for the investigation. They were selected so that

- the content of each one could be understood and remembered easily even by a lower secondary school pupil,
- the solution did not require advanced mathematical apparatus (the ways of solving each one are kept within the range of knowledge taught in the lower secondary school).
- mone of them can be solved by using a well-known scheme; each one requires a certain creative act from a pupil and from a mathematician. It is the third (highest) type of a problem in Krygowska's classification (Krygowska, 1977). According to another classyfication (see Kilpatrick, 1987) some of the after-mentioned problems (nb. 1, 3, 5) are well-structured requiring productive thinking, the others (nb. 2, 4 6, 7) are ill-structured problems.
- each one is an open problem in the sense that it allows for many different approaches and procedures during its solution.

Here are the problems:

Problem 1. You are given real numbers a, b, c, d, each of which lies between 0 and 1. Prove the inequality:

$$(1-a)(1-b)(1-c)(1-d) > 1-a-b-c-d.$$

Problem 2. For any coplanar segments a and b find the set of centers of the line segments XY such that X and Y belong to a and b respectively.

Problem 3. A is a set of three-digit numbers, the digits being different non-zero constants a, b, c. Show that in A one can find two distinct numbers whose difference is divisible by 4.

Problem 4. (Not for mathematicians.) Find the greatest possible number of triples of natural numbers satisfying the equation

$$x^2 + y^2 = z^2.$$

Problem 5. Let n be a natural odd number greater than 3. Show that the ten's digit of number n^2 is even.

Problem 6. Is the following sentence true?

If a natural number n is the sum of squares of two natural numbers then number $2n$ has the same property.

Problem 7. (Not for lower secondary school pupils.) How many scalene triangles whose sides' lengths are natural numbers $1, 2, 3, \ldots, n$ can be constructed?

2.2 The method of data collecting

Two methods of data collecting have been made use of in the investigation. One of them is the method of 'thinking aloud'. The subject gets a problem to solve and is asked to inform the investigator aloud about any thought occurring to him in connection with solving the problem. The observer watches and records (tape, taking notes) how the subject works, which is displayed by his/her comments aloud, written work, and drawings. What is especially important and noted, e.g. in problems on geometry, is the order in which various elements of the drawing appear. It is important to notice the moments when the problem solver comes back to his drawing and adds new elements, which sometimes may change the initial answer.

The method of 'thinking aloud' was enriched in the reported research by a certain element introduced to obtain psychological investigation of the thinking itself. In order to diminish one of the difficulties of solving a problem, which is often the lack of necessary knowledge (Puszkin, 1970), the solver could ask the observer questions referring to any mathematical fact which 'has been done' at school, which he or she does not remember at the moment, and which he/she considers essential at a certain moment. E.g. the solver might ask whether such and such a theorem is true.

The second method used is a certain kind of introspection. It followed after the solver declared that his/her work was finished. The researcher reconstructed aloud the solver's way of reasoning, reminding him/her of the stages of the process of solving the problem. Sometimes he/she asked questions, answering which the solver explained why at that particular moment he/she decided to change the apparatus (e.g. from synthetic geometry to analytical geometry), why he/she assumed such and such hypothesis, etc. In this second stage the solver sometimes spontaneously informed the researcher of his/her thoughts and feelings, not disclosed during the first stage. Sometimes he/she expressed as well his/her opinion about the problem just solved.

The method presented above is a certain variation of the method known in the literature on mathematics education as 'mutual observation' (Brink, 1990).

Some elements of introspection also occurred at the introductory stage, before the investigation proper began. Before starting the observation of other people, the researcher solved herself a great number of open problems. Each time she analyzed her own way of reasoning, trying to record moments important for the solution. This introductory stage was treated as a kind of training before commencing the observation itself of other persons. It was found to facilitate both observation and analyzing the collected data. It also influenced the choice of problems used in the course of the investigation.

2.3 Data analysis

The analysis of the material collected by means of observation was carried out in several stages. Firstly, a detailed observation record was prepared for each examined person. It included:

- the answer sheet of the subject with all his/her notes and drawings executed while solving the problem,
- 'thinking aloud' remarks of the subject (reconstructed from the researcher's notes or tape recorded),
- information about the order in which particular elements of the solution, drawings or elements within the drawings appeared.

The material collected in this way became the basis of a detailed description of the subject's activities from the moment of his/her getting the problem to solve, till the moment he/she announced that work was finished.

The second stage of analysis consisted in studying the observation record of each subject's work separately. At this stage it was possible to tentatively answer the following questions

- what had helped in working out the solution?
- what had hindered the progress?
- what elements of reasoning (empirical, intuitive, formal) occurred in the thinking process of the solver?

As the result of such a study, an 'observation chart' – separate for every subject – containing the tentative answers to the above questions, was worked out. Each was analysed from various points of view such as correctness of the solution, applied strategies, the sort of mathematical reasoning, sketched but abandoned ideas, the core of the erros made.

The third stage, which can be called the stage of comparative analysis, consisted in finding similarities and differences in the reasoning of the subjects within the same level of mathematical experience, as well as looking for similarities and differences in the solvers' activities at different levels of mathematical experience.

It was hoped that the above analysis would allow to outline some general approaches and heuristic behaviors, both those which facilitate and those which hinder solution of the problem.

3 Findings

It seems reasonable to conclude the following:

1. The reasoning of both students and mathematicians is not a sequence of logical inferences, but rather of hypotheses and questions the answers to which are obtained empirically, intuitively, or through formal reasoning. This is evidenced in frequent use by the examinee expressions like 'it seems to me', 'I feel', 'I'm not sure', 'perhaps', 'no, it won't be so', 'what could it be?'.
2. The exemenee demonstrated a variety of attitudes and behaviours while looking for a solution of each of the problms. This variety could be seen, among others, in: diverse approaches to the problems, making use of various mathematical tools, diverse ways of making use of a special case, diferent strategies applied.

   Seven solution strategies have been identified:

   I. Strategy of estimating the values of algebraic expressions;
   II. Strategy of estimating an expression with another expression;
   III. "Regular route" strategy;
   IV. Strategy of postulating the solution;
   V. Strategy of a geometric interpretation of the given problem;
   VI. Strategy "Solve a related problem";
   VII. Examining special cases strategy.

   One of the factors which caused this variety was the interest of the subjects of in a problem. It was expressed by the will to solve it, in spite of the difficulties encountered in the chosen way of investigation; the subjects did not give up easily.

3. The significant common feature of the solving process, observed in the work of all subjects, both students and mathematicians, is rambling:

   - formulating false hypotheses,
   - chaotic empirical checking,
   - implementing an idea that might be rejected after a reflection,
   - unnecessary generalization and formalization of the given conditions,
   - shortening algebraic calculations, causing errors,
   - multiple coming back to steps already done,
   - abandoning ways that were not exploited sufficiently.

   It means that ideas are taken then abandoned making room for new ones, then sometimes the previous ones are re-taken. This phenomenon contradicts Schoenfeld's (1979) claim that the conduct of mathematicians while solving a problem differs dramatically of that of a beginner.

4. Another feature, common for all four levels of mathematical experience, is making use of the latter strategy. In other researches, as by G. Polya (1964, 1975) and A. Schoenfeld (1985) it was shown that considering several special cases may lead to finding the general solution. Here, it was evidenced that the solution can be arrived at as the result of considering only one object fulfilling the problem's conditions. Three sub-strategies of this general strategy were distinguished:

   (a) guessing a generic property for the general solution,
   (b) searching for a transformation that would conserve the case's properties,
   (c) deductively oriented analysis of the special case.

   A solver following strategy (a) starts with taking an object fulfilling the condition of the problem and treats it as the general solution. Then he/she analyzes it trying to find a property such that would generate the general solution. The initial example is the source of hypotheses on the shape of general solution. In strategy (b) the solver puts the following question: What transformation of the particular object might produce other objects fulfilling the problem's conditions? And in strategy (c) a certain property of the special case is noticed and then the question is asked: Why this example possesses such property? Answering this question is a deductive procedure, so it can be generalized onto other cases.

   Strategies (a) and (b) were used by representatives of various competence levels, (c) – by mathematicians.

5. An interesting difference was found in handling hypotheses. All solvers rejected a hypotheses if a counterexample was found. But mathematicians and students differed in their attitude when several examples confirming the hypothesis were found. A mathematician, supposing the hypothesis is true, tried to find an argument to convince him that the conditions are fulfilled in other objects in the considered domain (not necessarily a formal proof). When such an argument was found the mathematician took the hypothesis as a statement; if he could not find a convincing argument and no counterexample had been encountered the hypothesis retained its hypothetical status. To the contrary, numerous students and a few university students of mathematics verified their hypotheses empirically, i.e. substituting numbers for variables or accepting properties of a drawing.
6. Another essential difference between mathematicians and some students was found in their formulation of hypotheses based on an example. It can be characterized as 'carefulness' of mathematicians and 'carelessness' of students who tend to too widely generalize.
7. A difference in experience also appeared in the behavior after having successfully solved a problem. None of the teenagers continued by Polya's 'looking back', for example showing interest in another way of solution or formulating and solving similar problems. Such a reflection was present in the case of a few university students and some mathematicians.
8. Analysis of the reasonings revealed errors of different type, dony by those solving open problems:
   - errors in the merit (as a result of uncaucious analogy – done by university students),
   - methodological errors (verifying conjectures with the help of examples or drowings only – done mainly by school children and also by some students of mathematics),
   - heuristic ("strategic") errors (choosing tools which are unsuitable to solve a poblem – done by pupils).

4 Examples

Those similarities and differences of the solving process presented by school pupils and professional mathematicians will be illustrated by outlines of works of four examined persons.

1. **Konrad (15 year old) solving problem nb 3.** Konrad lists all the three-digit numbers with the digits 1, 2, 3 in a column in the increasing order. He calculates several differences and notices that the third one, $231 - 123 = 108$ satisfies the required condition. His first conjecture is that the point is in subtracting the first number in this sequence from the fourth. This is rejected in the next attempt – triple 1, 2, 4. Konrad returns to triples with digist 1, 2, 3 finding consecutively all positive differences with fixed minued and listing them out in separate lines. Each time he checks their divisibility by 4 and circles "good" ones. After rewriting the triples using letters a, b, c and some further experimentation Konrad focuses on the difference $614 - 164$, which is not divisible by 4, but, in his words, *the last digit of the difference is zero. It's always going to be so for such an arrangement of digits. For this difference to be divisible by 4 [...]* $c - a$ *has to be even.* He checks this idea with the triple 1, 2, 3 and says: *In the same way you may create differences for other choices of digits which include two digits listed here. And what do we do if the difference* $c - a$ *is odd?*. After a while Konrad rejoiced: *But if you have three different digits, there must be two among them which are even or two which are odd. You put these digits in the hundred's and ten's places and the third remaining digit at the end. From such a number you will subtract the number with the same unit's digit and with interchanged digits in the hundred's and ten's places. You may write this difference as I have already done before:* $cab - acb$. *But this time* c, a *are both even or both odd... and you have to add that* a *is less than* c. *This difference has to be divisible by 4 since* $a - c$ *is even.*
2. **Person B. (a mathematician) solving problem nb 3.** B. begins the reasoning in a fashion quite similar to the one of Konrad: he chooses digits 1, 2, 3 and lists out set A. In this set B. searches for a pair of numbers whose difference is divisible by 4 and such that the unit's digit is the same. He finds it: 312, 132, and directs his attention to the value of their difference: 180. B. asks *Is it just a coincidence that the difference equals 180?*. He switches to algebra saying: *The digits* a, b, c *are three consecutive digits:* a, $a+1$, $a+2$, $1 \leqslant a \leqslant 7$. *We take number* $(a+2) \cdot 100 + a \cdot 10 + (a+1)$ *and subtract from it the number with the same unit's digit but with the ten's and hundred's digits interchanged with each other, that is, the number* $a \cdot 100 + (a+2) \cdot 10 + (a+1)$. *As a result, we get number* $2 \cdot 90$, *which, obviously, is divisible by 4. The problem is solved for the case of the set* A *for three consecutive digits.* B. continues reasoning aloud: *What does the result* $2 \cdot 90$ *mean? Why two times ninety?* (B. stresses the word 'two' and circles number 2 in the

notes.) After a while B. answers the question: *Number 2 in this product results from the fact that the difference between the greatest and the least number among the chosen digits equals 2. W*e will get a similar result if we replace number 2 by 3, 4 etc., and generally k. There's no need to repeat the calculations, it will be entirely analogous. I'll just write down the result: Let $a < b < c,\ c = a + k$. Then

$$(c\,a\,b)_{10} - (a\,c\,b)_{10} = k \cdot 90.$$

Number $k \cdot 90$ will be divisible by 4 if only k is even. It implies that the problem has been solved in the case of set A for such triples that the difference between the greatest and the least of them is even, which means for $k = 2, 4, 6, 8$.

Now B. considers triples a, b, c, $a < b < c$ with $c = a + l$ analyzing separately cases for $l = 3$, 5, 7. It takes him a long way to go, which ends with this conclusion: *In each case there exist two numbers such that their difference, as an even multiple of 90, is divisible by 4.*

But differently than Konrad, mathematician B. does not finish with this solution. He says: *Although the problem has been solved, I am worried by the great number of these cases. I wonder if it would be possible to shorten this reasoning [...]. Why was it possible to satisfy the required condition for every choice of digits?* After a short consideration B. communicates with a noticeable relief and satisfaction: *But of course, it had to be so. Every time we were able to indicate two digits with an even difference since among any three arbitrary digits there must be two of the same parity of the digits.* Everything is clear now. He elaborates the new proof, then adds: *I would also like to notice that, as a matter of fact, we have solved not only the original problem, but also a series of related problems. We have shown that in set A there are two numbers whose difference is not only divisible by 4, but also by any other divisor of* 180.

The researcher asked B.: *Could you try to explain what made you think that there was something particular about number 180?* And here is his answer: *I suppose* – says B – *that what happened about 180 was this: The analysis of the example made me realize that the whole thing lies in permuting the digits of a three-digit number. I associated it with a similar problem on two-digit numbers I had come across. Namely, when a two-digit number is subtracted from the number with permuted digits the result obtained is a multiple of the 9. I thought – if we have multiples of 9 there, maybe in this case we would get multiples of 90. What remained to be done was just a simple algebraic calculation.*

3. **Person C. (a mathematician) solving problem nb 2.** C. considers the case of non-parallel, non-perpendicular line segments with a common end. *If the figure we are searching for is a line segment – of which I am sure – says C., then in this case it will be the bisector of triangle ABD, drawn from point A (Fig. 1.). This is the set of centres of line segments of the type considered, parallel to line segment BD*. After a while C. discovers: *No, it won't be a line segment, because if we take the second bisector as a line segment of the considered type, its center will not lie on the former bisector; the bisectors in a triangle intersect at the ratio of* 1 : 2. (Fig. 2.).

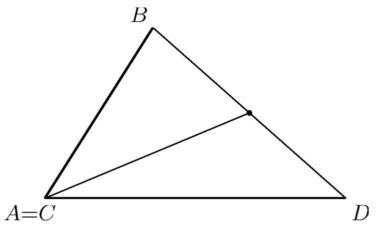

Figure 1.

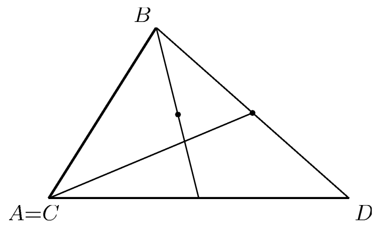

Figure 2.

In the next stage C. examines systematically the case which he considered, verifying the initial conjecture. He discovers the solution (starting with extreme points), and then reduces the general case to the one already investigated. He does it in the following way: he moves line segment AB in parallel, so as to have point A in point C (Fig. 3.). In this way he obtains a parallelogram and moves it back in parallel so thet its vertex reaches the center of segment AC. (Fig. 4.).

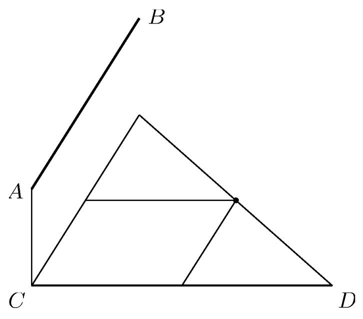

Figure 3.

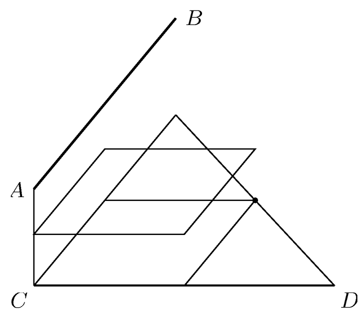

Figure 4.

4. **Witold (an upper-secondary student) solving problem nb 4.** Witold realizes that he knows a triple satisfying the equation, (3, 4, 5) and says: *I'll check whether we also obtain a root of the equation by rising each component of the triple (3, 4, 5) to the second power.* Computation gives the answer 'No'. Then he asks himself: *What if we multiply each component of the triple (3, 4, 5) by 2?* Having obtained equality $6^2 + 8^2 = 10^2$ the boy says: *Well, we have another good triple.* He applied the same operation to the last triple and said: *It is already a system. It must be a system. I will put to the test one example more.* After finding the squares of 24, 32 and 40 and noticing that the sum of the first two equals the third he states: *I am sure that it is a good system. I found an infinite set of positive integers satisfying the equation* $x^2 + y^2 = z^2$.

   Then the reasercher asked:
   R. (Researcher): What does your system consist in?
   W. (Witold): I started with the triple (3,4,5). I multipied the triple, then the new obtained triple by 2 and it turned out again that the result was another triple of those I was looking for. It was the moment I alreadz saw my system. Testing one more example confirmed my feeling that I was right. This way could be repeated to infinity. Take the next example: 48, 64, 80 (calculating the squares) – It is also good.
   R: esearcher: How do you know that the 100-th triple in your system is a root of the pytagorean equation?
   W: It must be.
   Now follows the additional interview.
   R: Why? You examined few examples only.
   W (after a while): I'm sure. Well... this system can be described somehow in general. (He looks carefully at the triples found and continues:) We have (3, 4, 5). First we multiplied it by 2, then... by 4, next by 8, 16, and so on. In general, we will obtain numbers $3 \cdot 2^n$, $4 \cdot 2^n$, $5 \cdot 2^n$. Now we can put them to the equation for x, y, z respectively. (Witold checks that $(3 \cdot 2)^n + (4 \cdot 2)^n = (5 \cdot 2)^n$ and declares:) Each triple of the type $(3 \cdot 2^n, 4 \cdot 2^n, 5 \cdot 2^n)$ where n is an arbitrary natural number satisfies the pythagorean equation.

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