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Institutional practices and conceptual development – a case study of college level calculus courses

Abstract: In this paper, I discuss the influence that institutional practices can have on students' modes of thinking about limits of functions. Institutional practices and students' practices are characterized and analyzed from the perspective of the Anthropological Theory of the Didactic. Students' modes of thinking are described and assessed from the perspective of Vygotsky's theory of concept development. Based on 28 interviews carried out with college students, I argue that the absence of a theoretical discourse in institutional practices related to the topic of limits of functions may inhibit students' development of the limit concept.

1 Introduction

The main goal of this paper is to investigate the influence of *institutional practices* related to the topic of limits of functions – as presented at college level – on *students' practices* related to limits and their *modes of thinking* about limits. I am particularly interested in teaching and learning practices that occur *outside* of the classroom such as those related to preparing the course outline, choosing a textbook, preparing the final examination by a committee of instructors, studying for the final examination by students, etc.

To characterize institutional and students' practices I consider the epistemological framework proposed by the Anthropological Theory of the Didactic (abbreviated as ATD; Chevallard, 1999; 2002). In particular, I consider the notion of mathematical organization or praxeological organization of mathematical nature, also called mathematical *praxeology* – developed within the ATD framework – that provides a structure to characterize mathematical activity as it is carried out within institutions. A praxeology characterizes a practice in terms of the type of tasks involved in the practice, the techniques to accomplish these tasks, and the explanatory discourses about the techniques.

To analyze the influence of institutional practices on students' modes of thinking about limits I conducted 28 task-based interviews with college level students who have successfully passed the Calculus course. The methodology of task-based interviews, described in Goldin (1997), has been recently used in studies concerning the teaching and learning of limits (e.g. Lithner, 2000; Hähkiöniemi, 2006; Hah Roh, 2008).

The interviews were made of six parts, each based on a different task. In a previous paper (Hardy, 2009), I have focused on one of these parts, addressing mainly the question of students' and instructors' perceptions of the knowledge to be learned about limits on a North American college. In the present paper, I present an analysis of the first part of the interview, looking at the cognitive level of students' engagement with limit finding tasks. The previous paper (Hardy, 2009), used a combination of ATD and a framework for institutional analysis. In the present paper, I bring together ATD and Vygotsky's theory of concept development. The two papers intend to contribute to an understanding of the influence of the institutional practices that are carried out outside of the classroom on students' acquisition and consolidation of mathematical knowledge.

It was while analyzing students' behavior in these interviews, from the perspective of the ATD framework – this is, while characterizing students' praxeologies and contrasting them with the institutional praxeologies – that I arrived at the conjecture that institutional practices were, in some sense, inhibiting students' *development* of mathematical thinking. This idea is related to cognition and so I turned to theories of conceptual development, looking for analytical tools that could help in framing my conjecture. Vygotsky's theory of concept development (1987) seemed appropriate because it focuses on the development of scientific (and not everyday) concepts, which is certainly the case of limits as they are presented in college level Calculus courses, and (unlike the Piagetian theory) it attributes a primary role to the socio-cultural factors in the development of scientific concepts. According to Vygotsky, the development of *scientific* concepts does not happen naturally, but must be *pulled* by instruction. In the case I am studying, the notion of limit is introduced to students explicitly by name, definitions and properties, and not through spontaneous interactions with an environment. The level of cognitive sophistication at which students' learn this concept depends on the tasks they are challenged to engage with.

Accounts of Vygotsky's theory of concept development and its position relative to other classical frameworks in the mathematics education domain are given, for example, in Sierpinska (1998) and Lerman (2001). Sierpinska provides an interpretation of this theory and applies it to discussing the developmental constraints of understanding in mathematics (1994) and students' practical vs. theoretical thinking in linear algebra (2000). Schmittau and Morris report on the achievements made by students following Davydov's elementary algebra curriculum based on Vygotsky and Luria's work (Schmittau and Morris, 2004).

The paper is organized as follows. Section 2 contains a brief description of the ATD concepts used in this paper, an outline of Vygotsky's theory of concept development, and an explanation of how these two frameworks have been combined in my research. Relevant features of the college institution and the Calculus course studied in this paper are also presented in this section. Section 3 describes my research methodology. Section 4 gives an account of the results. The paper closes with a discussion of the results and some conclusions.

2 Theoretical framework

The theoretical framework used in this paper stands on two main pillars: the concept of *praxeology* as a model of institutional practice, and a distinction of *modes of thinking* based on Vygotsky's theory of concept development. The concept of praxeology as applied in my research has already been described in detail in Hardy (2009) and I will therefore only briefly summarize it here.

2.1 Institutional practices

The Anthropological Theory of the Didactic (ATD; Chevallard, 1999; 2002) provides an epistemological framework for describing mathematical knowledge as one human activity among others. The model proposes that any mathematical knowledge can be described in terms of a mathematical organization, also called mathematical praxeology. A mathematical praxeology is a particular case of a praxeology of any kind of practice. A praxeology is defined as a system made of four main components:

- **a collection T of types of tasks** which define (more or less directly) the nature and goals of the practice;
- **a corresponding collection τ of techniques** available to accomplish each type of tasks;
- **a technology θ** that justifies these techniques; and
- **a theory Θ** that justifies the technology.

The term "technology" is understood as the *logos* or the discourse about the techniques, which allows the practitioners to think of, about, and out the techniques (Bosch, Chevallard, and Gascón, 2005). The theory Θ provides a system in which these concepts are defined and rules and procedures are justified. The subsystem [T, τ] is called the *practical block* of the praxeology and corresponds to the know-how; the subsystem [θ, Θ] is called the *theoretical block* and describes, explains and justifies the practical block. It is the theoretical block that makes it possible to preserve the activity as a practice and to communicate it to others, so that they, too, can participate in it. This suggests that there is a didactic intention in any cultural practice; if there are no means to teach and therefore perpetuate an activity, it cannot become part of a practice.

In this research, I am looking at an educational system where there is an intermediate institution between high school and university, called "college". I am particularly interested in one of the subjects taught in college, namely Calculus. The course is usually taught as a multi-section course, with a common course outline, textbook and final examination. There are no common lectures. The course is taught by individual instructors to groups ("sections") of 25-35 students. The section instructors are not free, however, to teach what and how they like. Their practice is strictly regulated by the detailed course outline, the textbook and the final examination that are collectively designed, chosen, written and coordinated.

These regulatory mechanisms imply the existence of certain institutional practices in relation to the teaching of Calculus. A discussion and analysis of these practices from the perspective of a framework for institutional analysis can be found in Hardy (2009). Suffice it to say here that the teaching of Calculus in this college institution is an institution in itself, which I call *College-Calculus institution*. The classroom and the committees preparing the outline or the final examination, for example, are sub-institutions of this College-Calculus institution. In particular, the common final examinations define a core of the *knowledge to be learned*[1], as defined by the College-Calculus institution. In terms of these concepts, the goal of the present paper can be stated as: To investigate the influence of the College-Calculus institution's practices related to limits of functions on students' modes of thinking about limits.

Taking into account the existence of a common final examination – that indeed defines a practice since its general structure has not changed over the years (constants in the questions change but *types* of questions do not change)

[1]In this research, this type of knowledge is understood as that which students have to explicitly show that they have learned (Hardy, 2009).

– and that this final examination is the only practice enforced by the College-Calculus institution (midterms, assignments and practices *in* the classroom are the section instructors' business), I could characterize the College-Calculus institution's practices about limits by looking at the types of tasks related to limits of functions proposed over the last seven years (2002 to 2008) and their model solutions available in the textbooks and in the so-called "correction sheets" for the final examinations, written by teachers and made available to students. The model solutions contain no theoretical justifications or proofs of validity. Instructors are free to provide theoretical justifications of the techniques in their lectures, but they cannot require students to provide such justifications in the final examinations. Therefore, while instructors' classroom practices could be modeled with complete mathematical praxeologies, containing both the practical and the theoretical blocks, the final examination practice, which defines, for students, the knowledge to be learned, must be modeled with praxeologies containing only practical blocks: types of tasks and techniques for solving them.

The study revealed that, in relation to the topic of limits of functions, students could expect only three types of tasks, to be solved using the techniques presented in the model solutions. This practice can be modeled by the practical blocks ("PBs"), i.e. types of tasks and techniques, of the mathematical praxeologies MP1, MP2 and MP3 described in Hardy (2009: section 4). For the sake of brevity, I list only the types of tasks and techniques here, i.e. the practical blocks, referring the reader to Hardy (2009) for the description of the complete mathematical praxeologies MP1-3.

PB1

TASK TYPE T_1: Evaluate the following limit: $\lim_{x\to c}\frac{P(x)}{Q(x)}$

Description: c is a fixed constant; $P(x)$ and $Q(x)$ are polynomials such that the factor $x-c$ occurs in both $P(x)$ and $Q(x)$; $x-c$ has degree one in $Q(x)$.

TECHNIQUE τ_1: Substitute c for x and recognize the indetermination $0/0$[2] Factor $P(x)$ and $Q(x)$ and cancel common factors. Substitute c for x. The obtained value is the limit.

PB2

TASK TYPE T_2: Evaluate the following limit: $\lim_{x\to c}\frac{\sqrt{P(x)}-Q(x)}{R(x)}$

Description: $P(x)$, $Q(x)$ and $R(x)$ are polynomials such that $\sqrt{P(c)}-Q(c)=0$, $R(c)=0$ and the factor $P(x)-[Q(x)]^2$ has degree one in $R(x)$.

[2] The first step in τ_1 appears in the textbooks when strategies of calculating limits are described in general. However, this step is omitted in most worked out examples in the textbooks and in solutions written by teachers and made available to students. The same is true for τ_2.

TECHNIQUE τ_2**:** Substitute c for x and recognize the indetermination $0/0$. Multiply and divide by the conjugate of $\sqrt{P(x)} - Q(x)$. Factor out from $R(x)$. Simplify and substitute c for x. The obtained value is the limit.

PB3

TASK TYPE T_3**:** Evaluate the following limit: $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$.

Description: $P(x)$ and $Q(x)$ are polynomials such that m, the degree of $P(x)$, is less or equal to n, the degree of $Q(x)$.

TECHNIQUE τ_3**:** Divide both $P(x)$ and $Q(x)$ by x^n. Simplify each term and then use the algebraic properties of limits and the fact that the limit of a constant over a positive power of x, as $x \to \infty$, is 0.

Identifying and characterizing these practical blocks was essential in the methodologies used to design the interviews and then analyze the data. As can be seen above, types of tasks appearing in final examinations and related to limits are exclusively limit-finding tasks; this is why the focus of analysis of the interviews was on students' praxeologies and modes of thinking about *finding limits.*

In the next section, I define, following Vygotsky, the categories of *modes of thinking* that I will be using to describe the interviewed students' thinking about limits.

2.2 Modes of thinking

Vygotsky's theory of cognitive development describes the genesis of concepts from early childhood to adolescence. It distinguishes several stages in the development of concepts, each characterized by a specific *mode of thinking* (Vygotsky, 1987, pp. 134-166). This theory functioned as a structuring framework in my analysis of students' cognitive behavior in the interviews: I would describe a student's way of thinking about various tasks about limits in terms of the Vygotskyan modes of thinking. In discussing the role of institutional practices in students' conceptual development, another Vygotskyan concept became quite useful, namely the concept of "zone of proximal development" (ZPD; Vygotsky, 1987, p. 212). ZPD is the domain of potential development, or the range of tasks the learner can perform with some help of the teacher. In an often-quoted fragment, Vygotsky is saying,

> Instruction is only useful when it moves ahead of development. When it does, it impels or wakens a whole series of functions that are in a stage of maturation lying in the zone of proximal development. This is the major role of instruction in development.
>
> (Vygotsky, 1987, p. 212)

In mathematics education, researchers taking the socio-cultural perspective on the processes of learning and teaching assume that the general pattern of learning a new mathematical concept or domain is similar to the Vygotskyan pattern of stages of conceptual development. They also assume that instruction, to be effective, must go ahead of this development, stretching, as it were, the boundaries of the learner's zone of proximal development (Sierpinska, 1994, p. 143). However, when Vygotsky's theory of stages of development is applied to studying a mathematical concept in secondary school students and older, the model will apply only to the stages of development of this particular concept, and not to the general cognitive abilities of the students. Students may be capable of conceptual thinking at the highest stage in one domain, but not in another domain that they only start to study.

In the development of conceptual thinking from early childhood to adolescence, Vygotsky distinguished three stages: syncretic images, complexes and concepts (1987, pp. 134-166), each with several phases. The most mature phase of complexes is pseudoconcepts. Each stage is characterized by a *mode of thinking*, which "leads to the formation of connections, [...] relationships among different concrete impressions, the unification and generalization of separate objects, and the ordering and systematization of the whole of child's experience" (Vygotsky, 1987, p. 135). At each stage, the basis on which these connections are made, is of quite different nature. Here is a selection of excerpts from Vygotsky's explanations of the nature of these stages.

> **Thinking in syncretic heaps or images**: "Faced with a task that an adult would generally solve through the formation of a new concept, the child... isolates an unordered heap of objects... that are unified without sufficient internal foundation and without sufficient internal kinship or relationships.... [The objects] are externally connected in the [subjective] impression they have had on the child but are not unified internally among themselves."
>
> (Vygotsky, 1987, p. 134)

> **Thinking in complexes (or complexive thinking):** "[In contrast with syncretic thinking] generalizations created on the basis of this mode of thinking are complexes of distinct, concrete objects or things that are united on the basis of objective connections, connections that actually exist among the objects involved.... Complexive thinking is thinking that is both connected and objective.... At this stage..., word meanings are best characterized as family names of objects that are united in complexes or groups. What distinguishes the construction of the complex is that it is based on connections among the individual elements that constitute it as opposed to abstract logical connections. It is not possible to decide whether a given individual belonging to the Smith family can properly

> be called by this name if our judgment must be based solely on logical relationships among individuals. . . . The foundation of the complex lies in empirical connections that emerge in the individual's immediate experience. A complex is first and foremost a concrete unification of a group of objects based on empirical similarity of separate objects to one another." (Vygotsky, 1987, pp. 136-7) "In accordance with some associative feature, the object is included in the complex as a particular, concrete object which retains all its features rather than as the carrier of a single feature which defines the object's membership in the complex. No single feature abstracted from others plays a unique role. The significance of the feature that is selected is essentially functional in nature. It is an equal among equals, one feature among others that define the object."
> (ibid., p. 140)

> **Thinking in pseudoconcepts**: " [T]he adult cannot transfer his own mode of thinking to the child. Children acquire word meanings from adults, but they are obliged to represent these meanings as concrete objects and complexes. . . . [They are] obtained through entirely different intellectual operations. This is what we call a pseudoconcept. In its external form, it appears to correspond for all practical purposes with adult word meanings. However, it is profoundly different from these word meanings in its internal nature [which is closer to a complex than to a concept]."
> (ibid., p. 143)

Vygotsky strongly advocated the use of artificial, experimental situations, where children are given classification tasks, to study their cognitive development: "The experiment uncovers the real activity of the child in forming generalizations, activity that is generally masked from casual observation" (Vygotsky, 1987, p. 143). This encouraged me to use a classification task in my research as well.

In this paper, Sierpinska's (1994, pp. 142-159) interpretation of Vygotsky's theory of concept development will be followed. Thus, it will be assumed that syncretic thinking is characterized by loose criteria; objects are placed together on the basis of subjective, often affective, impressions of contiguity or closeness. In the complexive mode of thinking, impressions of kinship are replaced by connections that actually exist between the objects. In a concept, these relations are, logically, of the same type. In a complex, these connections are factual. Any connection between the object and the model suffices to include the former into the complex. A symptom of complexive thinking is that, in classifying objects, there is a lack of a homogeneous set of criteria. Objects are put together in classes based on some resemblances that may differ from one class to another. Another strongly discriminating characteristic of complexive thinking is that no feature plays a unique role. While a concept is based on a hierarchy of connections and a hierarchy of relations between features that create a new object, which is more than the union of its elements, a complex is a conglomerate of its elements and relations with other conglomerates are not relevant.

The transition from complexes to concepts takes the form of pseudoconcepts. In an experimental situation of classification, pseudoconceptual thinking and conceptual thinking produce the same classes. The difference lies in the type of criteria used to decide whether an object belongs to a certain class. While conceptual thinking is guided by abstract and logically coherent criteria, pseudoconceptual thinking is guided by concrete factual features and connections, as in complexive thinking. The referential meaning (the name of the class) is that of a concept, but the categorical meaning (the criterion to decide whether an object belongs to a class or not) is that of a complex.

2.3 Combining the institutional and the psychological frameworks

As mentioned before, the general pattern of the genesis of concepts in a child, from early childhood to adolescence, described by Vygotsky, seems to be recapitulated each time a person embarks on the project of understanding or building a mathematical concept (Sierpinska, 1994, p. 143). This application of a theory of *cognitive development* to the study of *concept development* in students learning particular mathematical concepts requires some explanation.

A theory of cognitive development speaks about *developmental stages* of an individual from birth to maturity. A theory of concept development speaks about *levels of thinking* an individual (mature or not) goes through when learning a particular concept. Thus, in a theory of concept development based on Vygotsky's theory of cognitive development, it is assumed that, when learning a new mathematical concept, the learner would go from thinking about it in syncretic images, to complexive thinking, to thinking in pseudoconcepts, and finally – in concepts.

This theory suggests some kind of linear progress in concept development: once an individual thinks of a concept at the level of, say, pseudoconcepts, he or she cannot go back to syncretic images, but must either stay at this level or go forward to conceptual thinking. The interviewed students' performance, however, appeared to contradict this presupposition of linear progress. Instead of thinking at some *level*, they seemed to be using the different *modes* of thinking (characteristic of each level) with respect to the same concept, showing one mode of thinking at one moment and another immediately after.

How could this phenomenon be explained? An analysis of students' responses to various questions in the interview suggests an explanation that takes into account the institutional context in which students were interpreting the mathematical tasks given to them. I realized that the mode of thinking that a student was using when approaching a task, or answering a particular question about a task, depended on what was required from him or her with respect to the concepts associated with that task. Thus, for example, in the language of ATD, when justifying the use of a particular technique, a student might use the complexive mode of thinking, while when supporting a technology, he or she might use the syncretic mode. A further explanation was needed at this point: why would students do this kind of switches between different modes of thinking in relation with moving between the different spheres of their praxeologies? The possibility of such incongruence between modes of thinking in the different spheres of a praxeology has been predicted in the ATD theory.

In explaining the difference between "technology" and "theory", Chevallard (1999, p. 228), gives an example of different modes of thinking with respect to these two explanatory discourses:

> Consider the induction principle: $P \subseteq N \wedge 0 \in P \wedge \forall n(n \in P \Rightarrow n+1 \in P) \Rightarrow P = N$. To justify this technological component, essential in proofs by induction, we can, among other possibilities, refer, like Henri Poincaré did, to 'the power of the human mind, capable of conceiving the infinite repetition of the same act once this act can be realized at least once' (Poincaré, 1902), or accept as an axiom that every non-empty subset of N has a first element, and then prove that the induction principle follows.
> (Chevallard, 1999, p. 228)[3]

Using Vygotskyan categories of modes of thinking, we can say that a student, who justifies the use of induction as a proving technique by stating the Principle of Mathematical Induction, gives evidence of thinking at the conceptual level, at the technology level of explanatory discourse. The same student, however, might justify this element of the technology – the Principle of Mathematical Induction – either by stating the axiom mentioned in the above citation or, as Henri Poincaré did, by stating a belief in the capacity of the human mind to conceive of an infinite repetition of the same act, once this act can be realized at least once. In the first case, we would assess the student's thinking as conceptual at the theory level of explanatory discourse; in the second case, we would say that, at the theory level, he or she thinks in syncretic images.

[3] my translation

As Chevallard (1999, p. 227) pointed out, we can imagine an infinite regression of explanatory discourses that starts with technology and theory, and goes on forever: the theory of the theory of the theory...[4] This means that what can be one level of justification for one individual, can be another level of justification for another individual. Each of these justifying discourses can reveal different modes of thinking in an individual. This seems to apply to the college level Calculus students whose thinking I have studied. A question that arises from this analysis is *what is the part played by institutions and their practices in the modes of thinking that a student uses at the different levels of explanatory discourses?*

3 Methodology

3.1 Procedures

As already mentioned, the research was conducted using task-based interviews. Subjects were interviewed individually. The interview consisted in giving the student a task, asking him or her to think aloud when solving it and signal to the interviewer when finished. The interviewer would then ask the student to explain what he or she did and why. The process would be repeated with several tasks. I was the interviewer in all cases.

The first task was a classification task. Students were given twenty cards. Each card contained a written expression of the type $\lim_{x\to c} f(x)$, where c was either a constant or ∞, and $f(x)$ was a constant, a polynomial, a function involving a radical, a rational function, a quotient of functions involving radicals, or a function involving a trigonometric function. Students were asked to classify these twenty cards according to a rule of their choice. They were not asked to make the rule explicit before doing the classification. Once they had formed the classes, they were asked to "explain the rule" they had used for the classification. As I show in the results section, most students could not state a unique rule applying to all the classes they had formed. Rather, students offered short phrases describing each of these classes. After this "naming process", I challenged the membership of some of the objects placed in this or that class.

[4] Chevallard (1999, p. 227) claims that technology and theory suffice as a theoretical block to describe a mathematical praxeology. I interpret this in the sense that the theory level of justification is at the axiomatic level; anything beyond that does not necessarily correspond to mathematical concepts anymore.

3.2 The subjects

Twenty eight (28) students were interviewed; subjects were recruited from one North American college, among students enrolled in the college level Calculus II course in the winter semester of 2008. All subjects had successfully completed a Calculus I course in the previous semester.

Subjects were selected to represent a vast spectrum of the sections of the Calculus I course, taught by different teachers in the fall of 2007. In that semester, there were 19 sections taught by 14 different teachers; the sample of interviewed subjects covered at least 12 of these 14 teachers. For details on interviewed students' grades distribution and per teacher distribution see Hardy (2009).

3.3 The research instrument

Table 1 presents the twenty expressions written on the cards in the classification task. In the table, the expressions are numbered, for reference purposes. They were not numbered when presented to the students.

1. $\lim_{x\to 3} \frac{x^3-x^2-3x-9}{2x^2-4x-6}$	2. $\lim_{x\to 1} \frac{x^2-1}{(x-1)(x-3)}$	3. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$	4. $\lim_{x\to\infty} \frac{9x^3-x+2}{3x^3+1}$
5. $\lim_{x\to\infty} \frac{x^2-25}{x^3-1}$	6. $\lim_{x\to\infty} 7$	7. $\lim_{x\to 5} 3$	8. $\lim_{x\to 1} 4x^3+7x-9$
9. $\lim_{x\to 2} \frac{x^2-4}{x^2-3x-4}$	10. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x+4}$	11. $\lim_{x\to\infty} \frac{x^3+2x+1}{x^2+7x-1}$	12. $\lim_{x\to\infty} \frac{x^2-1}{x-1}$
13. $\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$	14. $\lim_{x\to\infty} \sqrt{x^2+1}-x$	15. $\lim_{x\to 1} \frac{x^2-x-2}{x-1}$	16. $\lim_{x\to 1} \frac{x^2+6x+19}{x^3-3x+2}$
17. $\lim_{x\to 5} \frac{x^2-4}{x^2-25}$	18. $\lim_{x\to 5} \frac{\sqrt{x+20}-\sqrt{5}}{5-x}$	19. $\lim_{x\to 0} \frac{\sin(x)}{x}$	20. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Tabela 1. Expressions given to students in the first part of the interview. The cards and the expressions were not numbered when presented to the students.

Expressions 1 to 4 are routine expressions in the sense of being instances of types of tasks appearing in final examinations. Expressions 1 and 2 are examples of type of tasks T_1. Expression 1 was chosen because exercises consisting in finding limits in which the factorization of a cubic expression is needed are common on final examinations. Students might feel, however, that factoring a quadratic polynomial is easier than factoring a cubic one, especially when the quadratic is a difference of squares that they usually recognize right away. To see how students react to these two versions of type of tasks T_1, I chose also to have expression 2.

Expression 3 is an example of a limit that can be found by rationalization – type of tasks T_2.

Expressions 4 and 5 are examples of limits of rational functions taken at infinity. Expression 4 is similar to those appearing in final examinations. Expression 5, on the other hand, contains the rational function $\frac{x^2-25}{x^3-1}$, whose numerator and denominator can be factored into polynomials of lesser degree with integer coefficients, using standard algebraic identities (difference of squares, difference of cubes). In the final examinations I looked at, such easily factorable polynomials would appear only in tasks of type T_1, never in tasks of type T_3. Thus, expression 5 is not a *routine* expression.

Expressions 5 to 20 are non-routine in the sense that they do not belong to any of the three types of tasks T_1, T_2 or T_3. Expressions 6 and 7 were chosen because, although they look very simple, in both, finding the limit requires some conceptual understanding or at least good memory (remembering by heart that the limit of a constant is the constant itself). The main idea in choosing expressions 8 to 18 was to have examples, some of which would look algebraically similar to routine expressions, and some would not, but, for all of them, the corresponding techniques would appear in the outline of the course. These techniques would be:

- direct substitution (expressions 8, 9, and 10);
- dividing every term in the numerator and the denominator by the highest power of x in the rational function, then cancelling out, and using the fact that, if c is a real number and r is a positive integer then $\lim_{x\to\pm\infty} \frac{c}{x^r} = 0$ (expressions 5 and 11 to 13);
- rationalizing technique (expression 14); and
- finding limits by inspection (expressions 15 to 18).

As for expressions 19 and 20, they contain trigonometric expressions. Students might not be familiar with techniques of finding limits of this kind because the topic is listed as optional in the course outline. They were chosen to see how students would classify expressions that were very different from the routine ones.

3.4 Criteria to assess students' modes of thinking

To decide which mode of thinking a student is using at a particular moment of the interview, I operationalized the descriptions of the four modes of thinking described in section 2.2 in the form of sets of clear criteria. The following criteria were used:

Syncretic images: The subject classifies objects according to an affective relation he or she has with these objects.

Complexive thinking: The subject does not describe his or her classification in terms of a key feature or criterion that discriminates between the classes nor is he or she concerned about finding such key feature. Most importantly, the classes the subject forms are such that the classification rule cannot be, even theoretically, described using a unifying, key feature. There is no logical structure in the classification as a whole. Each class has its own rule or rules and the "name" of a class or the criterion to decide about the membership applies perhaps to some of the objects in the class but not necessarily to all of them. The hierarchy of features of an object changes from one class to another. Therefore, based on the subject's description of the class, another person may have trouble deciding whether an object different from those the subject has him or herself put in a class belongs to this class or not.

Conceptual thinking: The subject consciously searches for a classification key which, when found, is consistently applied in forming classes. The subject is not happy with the classification key unless it allows him or her unambiguously to decide whether a given object belongs to a class or not. There is a stable hierarchy in the features of classified objects.

Pseudoconcepts: The subject forms classes that could be produced using conceptual level. The criteria the subject gives for putting an object in a class do not qualify, however, as based on conceptual thinking: the subject does not give a unified classification key in his or her explanations, and his or her criteria may sound like those given at the complexive thinking stage. Theoretically, however, a conceptual classification key can be construed for the classes as a whole.

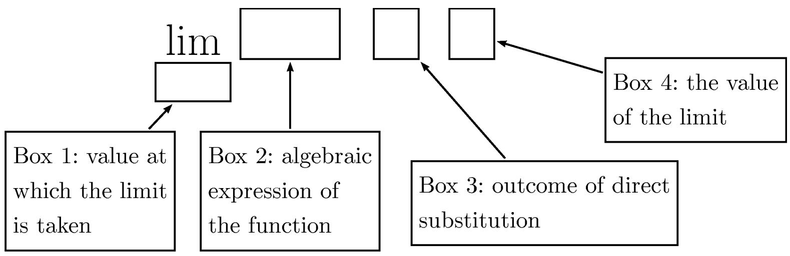
3.5 Means used to increase the credibility and objectivity of data analysis

In assessing students' modes of thinking a triangulation process was used. I asked two people, researchers in mathematics education, to read transcripts of the classification part of the interviews and independently decide which modes of thinking students' classifications and explanations appear to be representing. The two researchers were given the above criteria (section 3.3) and examples of their application. In the few cases in which there was no full agreement about how to interpret the behavior of a student, the final decision was reached through discussion.

The analysis of students' performance in the classification task from the perspective of the ATD and Vygotsky's theory of concept development generates two levels of analysis that are brought together to build a model of students' behavior in front of the task of finding limits of functions. Using the notion of praxeology, I identified each student's technique to accomplish the classification task, and his or her corresponding technology and theory. Next, based on Vygotsky's theory of concept development, I characterized each student's mode of thinking in relation to the two levels of justification. Based on each student's techniques, technology and theory related to the classification task, I reconstructed the student's praxeology in relation to the task of finding limits.

Based on the student's classification and explanatory discourse, I tried to understand what features of the limit expressions he or she was looking at to guide his or her classification. For this, I considered a scheme (see Figure 1) in which I represented the expressions in the cards and some additional information. The scheme has four "boxes". The symbol "lim" and boxes 1 and 2 represent the expressions that students could see on the cards. The third box corresponds to the arithmetic outcome of direct substitution (in $\mathbb{R} \cup \{+\infty,-\infty\}$). The fourth box corresponds to the value of the limit.

In terms of praxeology, the features a student is considering and *how* he or she is considering them correspond to his or her technique to accomplish the classification task. These features could be of four kinds: arithmetic, algebraic, belonging to Calculus, or analytic (i.e. belonging to Mathematical Analysis). For example, a student who focused only on box 3 and made a classification forming four classes, named them "a number over zero", "infinity over infinity", "zero over a number", and "a number", would be characterized as using a technique that belongs to arithmetic (in $\mathbb{R} \cup \{+\infty,-\infty\}$).



Rysunek 1. A scheme of limit expression.

A student who focused only on box 3 but made the classes: "infinity", "indetermination zero over zero", "indetermination infinity over infinity", and "a constant", would be described as using a technique that belongs to Calculus. A student who focused only on box 2 and made the classes: "rational functions", "expressions with radicals", "constants", and "trigonometric functions", would be characterized as using a technique that belongs to Algebra. I assumed that the features that are so relevant to the student that he or she chooses them as a basis to accomplish the classification task are also those that the student is likely to consider when he or she has to choose a technique to find the limit of a function.

I characterized the student's supporting discourses – the technology and the theory in the sense of ATD – based on the phrases he or she used to describe the classification. In the analysis, I have been consistent in considering as technology the discourse that directly justifies the technique, the *logos* about the technique. Hence, this level of justification corresponds to phrases that a student used to "name" or to describe the classes. Whenever the student provided enough information, I conjectured what his or her theory – the discourse supporting the technology – could be. From these explanatory discourses, I inferred students' technologies and theories in the praxeological organization corresponding to the task of finding limits.

An important observation to understand the analysis of Vygotskyan modes of thinking within the framework of the ATD is the following. If we accept that an infinite sequence of explanatory discourses is conceivable (Chevallard, 1999, p. 227), it might be the case that affect – characteristic of thinking in syncretic images – always plays a role at some level of the chain. For my research, it is essential to distinguish, whenever a student made a statement that seemed to reflect a syncretic mode of thinking, whether the student was thinking in syncretic images at the technology level, at the theoretical level, or at none of these. I considered that an individual is thinking syncretically at the technology level if, for classifying an object, he or she is considering features that are internal to him or her – but not intrinsic to the classified object. In the other modes of thinking, the features considered relevant for the classification are external to the classifier. Thus, to decide whether the first level of justification, the technology level, is or is not syncretic, I analyzed *how* affect influenced the classification. Whenever affect influenced an individual's attention, but was not the classifying feature, I did not consider that he or she was thinking syncretically, although it might be the case that he or she was thinking in syncretic images at the theory level.

4 Results

My analysis relied on the information that I could extract from the interviews; this information varied considerably from one student to another, and so did my level of analysis. In particular, for some students, I was able to make conjectures about their theories, that is, the second level of justification. For many students, however, I did not have sufficient ground for conjecturing about their theories, and then I could only discuss the first level of justification, the level of technology. In this section, I present the analysis of one student's (S1) behavior in front of the classification task. On this example, I explain the way in which I have interpreted students' behaviors and discourses to infer their praxeologies in front of the task of finding limits and their mode(s) of thinking. Table 2 shows student S1's classification of the twenty limit expressions.

Class	Members of class (*labels refer to Table 1*).	Phrases used by the student in response to the question "what was the rule of your choice?"
1	2, 5, 9, 12, 17	Difference of squares.
2	6, 7	Constants.
3	19, 20	With trigs. That confuses me.
4	3, 10, 13, 14, 18	With square roots.
5	1, 4, 8, 11, 15, 16	Polynomials.

Tabela 2. Classification made by student S1.

In describing class 1, the student used the phrase "difference of squares" to refer to rational functions that contained a difference of squares. When she said "with trigs" to describe class 3, she was referring to expressions containing the sine function. When she said "with square roots" to describe class 4, she was referring to rational expressions containing square radicals. Finally, she used the phrase "polynomials" to refer to rational functions.

Technique. The student focused on box 2 (I refer to Figure 1) to make her classification. In particular, she considered some features of the algebraic form of the function to decide about the membership of an expression in one class or another. She was considering features that were purely algebraic and not related to the algebra of Calculus. Hence, I concluded that her technique belonged to the domain of Algebra.

Technology. The justification of the technique, i.e., the phrases that the student used to describe each class, evoked typical topics of school Algebra textbooks: constants, difference of squares, trigonometric functions, rational functions, rational expressions with radicals. The phrases that she used to "name" the classes constitute themselves the immediate explanatory discourse about her technique, i.e., her technology. I interpret the discourse as, "I placed expression 19 in class 5 because it has a trigonometric function in it"; or "I placed expression 13 in class 3 because there's a square root in it". I do *not* interpret it as, for example, "class 5 corresponds to quotients of polynomials such that none of them is a difference of squares". Although one might believe that this is what she meant, one can only infer it from looking at the members of the classes, not from the phrases or "names" she used; one would be changing the "name" or description of the class. Moreover, one might want to go further in completing the ellipsis in her discourse, saying, for example, that she meant, "class 5 corresponds to quotients of polynomials such that the numerator is not a constant and none of the polynomials is a difference of squares". However, the "name" or description is an essential component in Vygotsky's definition of stages of concept development and modes of thinking. The mode of thinking depends not only on how an individual sorts a certain collection of objects, but also on his or her criteria of this classification, and these criteria can be only inferred from his or her justifications. Another person given the same objects and asked to classify them based only on student S1's descriptions would not necessarily be able to complete the elliptic discourse and might struggle to decide on the membership of some of the objects. In the particular case of the classification done by student S1, another person might start by trying to classify expression 8 and place it in class 5 because it is a polynomial. Then this person might take expression 17 and place it in class 1 because both the numerator and the denominator are differences of squares. However, when considering, for example, expression 9 he or she might hesitate in prioritizing the feature "numerator is a difference of squares" over the feature "both numerator and denominator are polynomials". These considerations support the claim that student S1's technology is based on complexive thinking.

Theory. While explaining what her classes were and why she had placed this or that object in a class, the student said, in reference to class 3: "Because that [trigonometric functions] confuses me, I put them together". I analyze the levels of her explanatory discourse as follows. The sentence "these are with trigs" is an answer to, "what is the description of the class containing objects 19 and 20?". This explanation belongs to the level of technology. The sentence "because that confuses me, I put them together" is an answer to "why did you consider the feature 'trigonometric function' as a key to build a class?". This is an explanatory discourse at the next level of justification, the level of theory. Her statement is of syncretic nature, because it refers to an affective relation that she has with this particular feature. Her short statement about trigonometric functions could also point to a reason why she put the rational functions with a “difference of squares” apart from her “polynomials”. First, the “difference of squares” is a topic given a special emphasis in school Algebra textbooks and is treated separately from the chapter on polynomials in general and even separately from the chapter on quadratic functions. Moreover, if one considers these expressions as tasks to be performed, and the operation to be done is factoring, then this student could have taken into account the fact that a difference of squares is easier to factor than an arbitrary trinomial or polynomial. All of this points to an affective context of learning. At the level of theory, she justified her behavior based on her affective relation with the tasks to be performed. Hence, I claim that student S1 was thinking in syncretic images at the theory level of justification.

Based on this analysis, I have constructed a model of student S1’s praxeology in front of the task of finding limits of functions. I propose that the student’s technique belongs to Algebra. This implies, in particular, that she would decide which technique to apply in a problem exclusively based on the algebraic form of the function. Next, I propose that the student’s technology, or her explanatory discourse about the technique, is based on her previous algebraic knowledge with a vocabulary and system of concepts typical of high school Algebra. As for her theory, it appears to be based on her affective rapport with the different problems she has had to do as a student. It is important to notice that this praxeology does not qualify as a mathematical praxeology because the theoretical block is not of mathematical nature.

The above model is summarized in Table 3.

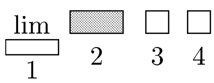
Technique	Technology	Theory
	The technology evokes school Algebra textbook categories – this might be reinforced by the PBs.	Affective context of learning.
The technique belongs to Algebra.	The justification of the technique is based on complexive thinking.	The justification of the technology is based on syncretic thinking.

Tabela 3. A model of student S1’s praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

This ends the example.

In assessing students’ behavior in the interview, it is important to take into account the fact that students were told – when they were recruited – that the interview would take about one hour, but they were not told at any point, before or during the interview, how many tasks there would be. Each task was given to them as if it was the last one. Hence, there is no ground to believe that students felt time-pressured while doing the classification task. The students' decision to calculate or not the limits in the cards they were classifying was a choice based on their own criteria to carry out the classification. The only student who calculated each of the 20 limits – some mentally and some with pen and paper – was student S28. Other students made references to the fact that they were considering the value of the limit for some expressions (for example S20); other students referred to the techniques they would use to find the limits (for example S10); and there were students who did not seem to be concerned at all with finding the limits (for example S9).

Of the twenty eight interviewed students, twelve focused exclusively on the algebraic features of the functions (box 2) to classify the twenty items. Seven students considered only the Calculus features (in one way or another they considered information that is meaningful from the point of view of Calculus: the value at which the limit is being taken, the type of indetermination as a tool to decide which technique to apply, the actual value of the limit). Two students based their classification on arithmetic computations in $\mathbb{R}\cup\{+\infty,-\infty\}$. Only one student's classification applied criteria characteristic of mathematical analysis. Six students used a combination of two approaches: two combined Arithmetic and Calculus, and four of them – Algebra and Calculus.

Among the twelve students who based their classification exclusively on the algebraic features of the functions, for eight students (S1, S7, S8, S10, S12, S16, S17 and S18) the expressions in the cards triggered a behavior that could be associated with a path of developing concepts about limits. These eight students either made an explicit reference to the fact that they were classifying limit expressions, or their classes seemed to be influenced by the institution's mathematical praxeologies related to limit-finding tasks. This was not the case for the other four students (S9, S15, S24 and S26), who did not refer to limits in their descriptions or explanations and their classification features cannot be associated with the institutions' mathematical praxeologies.

Tables 4 and 5 summarize the analysis above.

Several observations can be made based on Table 5. Complexive thinking prevails in justifications of techniques (technology) embedded in all domains except Analysis. Conceptual thinking in justification of techniques occurs only if the techniques belong to Analysis. Syncretic thinking in justification of techniques is rare. It is more frequent in justification of technology (theory). However, offering a syncretic justification of technology could be seen as better performance than offering no justification at this level at all: 50% of all 28 students did not offer any justification at the theory level.

Students' labels	Technique in limits tasks belongs to the domain of:	Mode of thinking about Technology	Mode of thinking about Theory	Number of students in category
1, 10, 17, 18	Algebra	Complexive	Syncretic	4
7, 8, 16	Algebra	Complexive	[unable to say]	3
12	Algebra	Pseudoconceptual	Complexive	1
5	Algebra/Calculus	Syncretic	Syncretic	1
11, 21	Algebra/Calculus	Complexive	Syncretic	2
22	Algebra/Calculus	Complexive	Complexive	1
6	Arithmetic	Syncretic	[unable to say]	1
19	Arithmetic	Complexive	[unable to say]	1
25	Arithmetic/Calculus	Complexive	Syncretic	1
4	Arithmetic/Calculus	Complexive	[unable to say]	1
23	Calculus	Complexive	Syncretic	1
20	Calculus	Complexive	Complexive	1
2, 3, 14, 27	Calculus	Complexive	[unable to say]	4
13	Calculus	Complexive	Conceptual	1
28	Analysis	Conceptual	Conceptual	1
9, 15, 24, 26	[unable to say]	[unable to say]	[unable to say]	4
TOTAL	Alg - 8, Alg/Cal - 4, Ar - 2, Ar/Cal - 2, Cal - 7, An - 1	Syncretic - 2, Complexive - 20, Preseudo-conc. - 1, Conceptual - 1	Syncretic - 9, Complexive - 3, Preseudo-conc. - 0, Conceptual - 2	28

Tabela 4. Students classified according to the domain of their techniques and their modes of thinking.

The praxeologies of at least thirteen students seemed to be influenced by the PBs (S1, S2, S3, S5, S7, S8, S12, S16, S17, S18, S21, S22 and S23). In addition, at least eight students held a discourse that evoked topics or sections on college level Calculus textbooks (S2, S4, S13, S19, S20, S21, S25 and S27). The union of these two classes results in nineteen students, out of the twenty eight, whose explanatory discourses were not related to concepts but to *how* these concepts are presented by the institution (in the final examinations or in the textbook).

Finally, it is interesting to observe that eighteen students (S1, S2, S3, S5, S8, S9, S11, S12, S13, S15, S16, S17, S18, S21, S22, S23, S24, and S26) classified expressions 19 and 20 (involving the sine function) in a class of their own. I surmise that this behavior has two (related) sources. On the one hand, limits of trigonometric functions are treated in college level Calculus textbooks separately, in a different chapter than limits of polynomials, rational functions and expressions with radicals. On the other hand, limits of trigonometric functions do not belong to the institution's mathematical praxeologies. Thirteen students are in the intersection of the class of students who put the expressions 19 and 20 together, and the class of nineteen students mentioned in the previous paragraph. I surmise that, for students in this intersection, there are four types of tasks. Three types are defined by the techniques in PB1, PB2 and PB3. The fourth type consists of tasks that cannot be solved by any of these methods; the associated technology is based on the algebraic form of the function and evokes the typical examples in college level Calculus textbooks.

Domain of technique	No. of stud. & frequency	Mode of thinking about technique				Mode of thinking about theory				
		Syncr.	Compl.	Pseudo-concept	Concept	Syncr.	Compl.	Pseudo-concept	Concept	Unknown
Algebra	8 (33%)	0	7	1	0	4	1	0	0	3
Calculus	7 (29%)	0	7	0	0	1	1	0	1	4
Alg./Cal.	4 (17%)	1	3	0	0	3	1	0	0	0
Ar./Cal.	2 (8%)	0	2	0	0	1	0	0	0	1
Arithm.	2 (8%)	1	1	0	0	0	0	0	0	2
Analysis	1 (4%)	0	0	0	1	0	0	0	1	0
Totals	**24**	**2**	**20**	**1**	**1**	**9**	**3**	**0**	**2**	**10**
%($N=24$)	**100 %**	**8%**	**83%**	**4%**	**4%**	**38%**	**12%**	**0%**	**8%**	**42%**

Tabela 5. Modes of thinking in the technology and theory levels vs. the domain of the techniques.

5 Discussion and Conclusions

The classification task has shown that 83% (see Table 5) of the students operated in the complexive mode of thinking when justifying techniques (the technology part of their praxeology). These students were at the age and stage of their cognitive development that is propitious for the development of the conceptual mode of thinking. Yet, with respect to concepts relative to limits of functions, most of them were operating at the complexive level. Their attention would shift from one feature to another of a limit expression, and they would fail to identify features relevant from the perspective of Calculus. Students operating at the complexive level of thinking would not identify the abstract logical connections among the individual instances of the limit finding tasks; they would only see empirical connections emerging from their individual, immediate experiences. Tasks T_1, T_2 and T_3 could not challenge this mode of thinking. To solve these tasks, it suffices to identify algebraic features with which students are familiar: division of polynomials (T_1 and T_3), the reducibility ("factorability" in students' language) of the polynomials (if not "factorable", then it is an instance of T_3), and the presence of radicals (T_2). Therefore, institutional practices do not fulfill the role of *pulling* students' cognitive development beyond their immediate individual capabilities – that is, they fall short of awakening "a whole series of functions that are in a stage of maturation lying in the zone of proximal development" (Vygotsky, 1987, p. 212). It would not suffice, however, to simply replace these tasks by others; their *routinization* by the College-Calculus institution plays a significant role (Lithner, 2004; Selden, Selden, Hauk, and Mason, 1999). What might help is a change in some institutional educative habits; for example, by creating situations where students are given a chance to engage in creative, critical mathematical thinking.

The absence of a theoretical block in the teaching of limits – or the dissociation between definitions and theoretical discourses from uses of techniques – has been reported by other researchers (e.g. Lithner, 2004; Raman, 2004; Barbé, Bosch, Espinoza, and Gascón, 2005). In this case, the absence of theoretical blocks in the problems that students are challenged to engaged with by the College-Calculus institution deprived many (26 of the 28 interviewed students – 93%) students of means to develop mathematical justifications at the conceptual level. Students had no mathematical theoretical resources to reflect on their own behavior and to justify it. Hence, their explanatory discourses about why they would use this or that technique, referred to social validations of the techniques, that is, to the institutional uses and not to why the techniques were mathematically valid in this or that situation. Analysis of students' behavior in another part of the interview supported this interpretation: many have explained their "factoring" behavior by a habit ("this is what we usually do in this case"), rather than by reference to mathematical reasons (see Hardy, 2009).

Furthermore, the absence of the theoretical blocks may be one of the causes of students switching from the complexive mode of thinking at the technology level to the syncretic mode at the theory level. This kind of behavior affected at least 30% of the students. Some fragments of the theoretical blocks do seem to belong to the knowledge to be taught, if we look at the knowledge implied in official documents such as the course outline, and perhaps to the knowledge to be learned as defined by some sub-institutions of the College-Calculus institution (it may be knowledge tested in assignments or class tests). The fact, however, that no part of the theoretical blocks belongs to the knowledge required on the final examination may lead students to neglect mathematical theory. They may conclude that theory is not their business. Hence, when required to provide deeper explanatory discourses, students would produce reasons based on affect (which represents thinking at the syncretic level).

My goal in this research was, on the one hand, to draw attention to the strong influence that practices involved in teaching and learning phenomena and carried out *outside* of the classroom may have on students' learning. On the other hand, I wanted to show an example of how the combination of two different theoretical perspectives (institutional-epistemological and psychological) can enhance our understanding of students' behavior in front of mathematical tasks. This research explains, to a certain extent, where North-American university students' difficulties with theoretical expositions of mathematics are coming from. It supports Sierpinska's conjecture that,

> [I]f, at the time of the development of conceptual thinking (usually around adolescence), the student is not given the opportunity and is not guided to engage in more formal reasonings, deductions and inferences in which the premises or reasons are made explicit and whose rules are agreed upon, he or she may never become able to develop the style and level of thinking necessary to understand and construct mathematical proofs [...] [A]t the university level, the propitious developmental moment would have passed, and it may be too late for the teaching intervention to have any effects".
>
> (1994, p. 140)

If one of the goals of teaching Mathematics at the college level is to contribute to students' understanding and appropriation of scientific thinking and if modern scientific thinking relies on theoretical thinking, then the findings of this research point to the necessity of changing some institutionalized (rooted in habit and tradition) teaching and learning practices; perhaps, (college level) institutional practices should be directed at creating a learning context where the participants are given enough tools to construct and challenge explanatory scientific discourses – thus pulling towards a conceptual mode of thinking.

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Praktyki instytucjonalne i rozwój pojęciowy – studium przypadku kursów *rachunku różniczkowego i całkowego* w północno-amerykańskim college'u

S t r e s z c z e n i e

Artykuł dotyczy nauczania matematyki w specyficznie północno-amerykańskiej instytucji, tzw. „college'u, na kursach rachunku różniczkowego i całkowego zwanych „Calculus", różniących się znacznie od uniwersyteckich kursów „Analysis", obowiązujących jedynie studentów specjalizujących się w matematyce. Kursy „Calculus" różnią się od kursów „Analysis" przede wszystkim skupieniem się na technikach obliczania granic, pochodnych i całek oraz brakiem teoretycznego uzasadnienia tych technik. Kursy „Calculus" reprezentują styl nauczania, zwany przez Amerykanów „a cookbook approach to teaching mathematics" (podejście w stylu książki kucharskiej), który trudno sobie wyobrazić, gdy się studiowało matematykę w Polsce. „Książka kucharska" jest metaforą bardzo dosłowną: materiał nauczania jest podzielony na wąskie typy zadań (∼ „dań") i przykłady ich rozwiązania (∼ „przyrządzania") dla konkretnych wartości parametrów charakteryzujących dany typ (∼ „dla konkretnej liczby osób"). Uogólnienia dla dowolnych wartości parametrów często nie ma, a uzasadnienie techniki jest częściej dydaktyczne niż matematyczne. W swoim artykule, na podstawie wywiadów z dwudziestoma ośmioma studentami, którzy taki kurs „Calculus" ukończyli z sukcesem, autorka przedstawia opłakane skutki, jakie to podejście ma dla rozwoju pojęcia granicy u studentów. Autorka tłumaczy także, jak struktura organizacyjna tych kursów – uświęcona tradycją praktyka instytucjonalna – wymusza niejako takie, a nie inne podejście do nauczania (i uczenia się) granicy funkcji. Indywidualny nauczyciel nie ma wielkiego wpływu na to, co studenci uważają za „wiedzę, którą powinni posiąść" („knowledge to be learned") i której się ostatecznie uczą.

Dla zbadania i opisu praktyk instytucjonalnych, autorka posłużyła się głównie pojęciem „prakseologii" zapożyczonym z teorii „Théorie Anthropologique du Didactique" rozwijanej przez Yves Chevallard'a i jego uczniów,

głównie we Francji i Hiszpanii. Teoria ta była już opisywana i wykorzystywana w badaniach z dydaktyki matematyki wcześniej i dlatego artykuł tylko pobieżnie definiuje pojęcie prakseologii, odsyłając zainteresowanego czytelnika do źródeł. Dla oceny poziomu rozwoju pojęcia granicy u studentów autorka zaadaptowała typologię sposobów myślenia, którą znajdujemy u L. S. Wygotskiego. Typologia ta jest mniej znana i wykorzystywana w dydaktyce, dlatego adaptacja jest dość szczegółowo opisana w artykule, łącznie z cytatami z L. S. Wygotskiego, na podstawie których adaptacja została dokonana.

W Polsce badania nad rozumieniem pojęć matematycznych przez studentów dokonywane są głównie z perspektywy psychologicznej lub epistemologicznej, abstrahując od czynników instytucjonalnych, które, jak wykazują badania autorki, mogą mieć nietrywialny wpływ na to, czego studenci uczą się na kursach matematyki. Artykuł zwraca uwagę na ramy teoretyczne, które mogą pomóc w systematycznym braniu pod uwagę kontekstu instytucjonalnego nauczania w badaniach w dydaktyce matematyki.