ANNALES OF THE POLISH MATHEMATICAL SOCIETY
5TH SERIES: DIDACTICA MATHEMATICAE 31 (2009)

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On Adult Students Learning Fractions at a Community College[1]

Abstract: This paper deals with the process of learning fractions by adult students attending courses of remedial mathematics at community colleges in the Bronx, New York City. It investigates the schema of fractions with the help of the model created by (Charalambos and Pitta-Pantazi, 2007), which is based on the work of Kiernan and the assumption that the fraction schema is composed of five subconstructs, a fundamental part whole subconstruct and four subordinate subconstracts: ratio, quotient, operator and measure. As (Charalambos and Pitta-Pantazi, 2007), conducted their research among young children, its application to the case of adult student brings interesting comparative knowledge about the fraction schema of children and adults.

For adult students, we were able to find partial support for the Kiernan fraction schema because all the subordinate sub-constructs except ratio (figure 4, table 3) correlated significantly with the part-whole subconstruct. It was determined that, while the adults engaged in mathematical reasoning about ratios and the abstract operator concept with approximately the same level of proficiency as the children however, the adult students were not able to assimilate this information into their fraction schema nearly as efficiently as the children.

[1] **Authors' Note.** This teaching research project was motivated and partially supported by the CUNY Collaborative Grant "Fractions Grid, Fraction Domino: Investigating Effectiveness at Community Colleges of the Bronx", 2007-2008 awarded to Vrunda Prabhu of Bronx Community College and Bronislaw Czarnocha of Hostos Community College.

It is of significance that adult students had understood ratio as a separate concept from part-whole. The ratio sub-construct is typically used to represent a comparative process between quantities and the part whole sub-construct represents a process of taking a fraction part out of a total.

The lack of flexible thinking in understanding the subtle relationship between the fraction as a numerical quantity corresponding to the number of parts taken out of a partitioned whole and as a comparative process in the ratio sub-construct is evidenced by a common addition error. When a students add: 1/3 +2/5 and obtain 3/8 they are incorrectly employing addition of ratios as a natural extension of the addition of whole numbers. The fact that the ratio is separate from the rest of their fraction schema, and that the operation of addition of ratios is a natural extension of whole number addition suggests changes in the curriculum.

Perhaps the ratio concept needs to be more extensively taught including its natural operations while highlighting the subtle relationships between the ratio and part-whole sub-constructs.

1 Introduction

Mathematical instruction in arithmetic for adult students frequently referred to as remedial education is common in the United States and other countries in the world. "The United Nations 1997 has recognized that one of the most important spaces today is education for adults and the research derived from the same, which have been put aside in the international setting for multiple reasons" (Valdemoros, M. E., 2004). According to the American Mathematical Association of Two-Year Colleges about half the students in community colleges in the United States take remedial mathematics, and research indicates that these students exhibit a much lower graduation rate than students placed in a college level mathematics course (Blair, R. 2006; Adelman, C. 2004). The lowest level of remedial mathematics is pre-algebra or basic arithmetic skills. Research at the City University of New York (CUNY) indicates that students who are placed at this level of remediation have about a 50% chance of passing the course.

Adult remedial course material is developed under the assumption that students need a quick review of basic mathematical material before proceeding into college mathematics. Thus, the mathematics curriculum of middle school or early high school and that of remedial mathematics courses are relatively close. However, the experience of instructors of remedial mathematics and the low passing rates in these courses, suggests that many of these students have unresolved issues with mathematics that transcend the ability of a refresher course. One particularly difficult subject for students in pre-algebra mathematics is fractions. However, because much of the research on fractions is done with children, the relevant question for educators of remedial pre-algebra is whether there are differences in the way children and remedial students understand fractions that should be reflected in the remedial curriculum.

The authors followed an existing model developed by Charalambos and Pitta-Pantazi (2007) and employed the same fraction schema as a framework for the study. The use of the fraction schema as a framework for this study with remedial students is especially important in order to determine what conceptual relationships if any are missing or weak in their understanding. The model contains two parts; the first was developed by Kiernan (1976) and the second by Behr and others (1983). It contains five sub-constructs, the fundamental sub-construct of part-whole and four subordinate sub-constructs of: ratio, operator, quotient and measure which together make up the concept of fraction. Charalambos, Pitta-Pantazi used this model to study a more holistic view of how students understand fractions and to examine students' understanding between the foundational part-whole sub-construct and the subordinate four, as well any potential links to fraction operations and fraction equivalence. The value of this approach is that it allowed the researchers to examine each sub-construct individually and the relations between them. Thus, Kiernan's model allowed both Charalambos/Pitta-Pantazi and the present authors to determine if one can attribute the difficulties students have learning fractions to their inability to recognize different fraction representations. The first objective of this article is to replicate the work of Charalambos and Pitta-Pantazi in a setting with adult remedial students in order to investigate any cognitive differences between the fraction schema of children and of adult students.

The second objective of this article is to search for cognitive evidence in students' fraction schema of their capacity for flexible thinking which is defined as the ability to understand or relate the multiple representations of fractions (Bulgar, 2009). Although relationships between the subordinate sub-constructs are not part of the Kiernan model the notion of flexible thinking suggests that such relationships would be an important part of a students' fraction schemata.

2 Theoretical Foundation

Kiernan's model of fractions is based on the fact that a single concept governs how students learn fractions and he elected to study fractions as a set of interrelated constructs: part-whole, ratio, operator, quotient, and measure. Later on, Behr et al. further developed Kiernan's ideas and used a theoretical model shown in the third level of Figure 1 that formed links between the different interpretations of fractions to equivalence, multiplication, problem solving and addition.

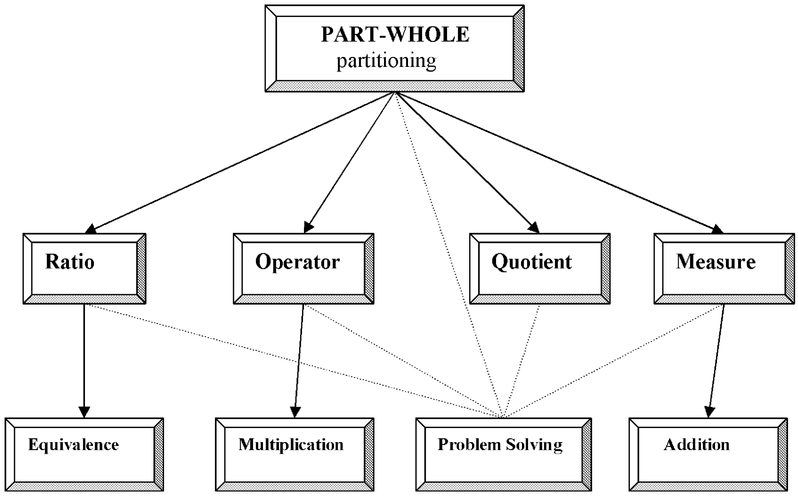

Figure 1. Charalambos Y.C., & Pitta-Pantazi, D., 2007, p. 295.

This model is taken to be represented hierarchically by causality. Thus, knowledge of part-whole implies knowledge of the other four sub-ordinate sub-constructs and problem solving, as indicated by the first level of arrows. In the same manner, knowledge of a given subordinate sub-construct implies knowledge of the associated procedural task and problem solving as indicated by the second level of arrows. Charalambos and Pitta-Pantazi (2007, p.294) note that although this model has been used by several researchers, it is not clear if it completely captures childrens' construction of fractional knowledge. Other researches have commented that the variety of representations of fractions is often a source of confusion for students (Wu, 2001 and Lamon, 1999). The variety of meaning given to the fraction symbol makes it a difficult topic for both children and adult remedial students.

The Kiernan model of fractions contains five different hypotheses. The reader may simply follow the arrows in Figure 1 in a top down fashion to see the hierarchical nature of this model. For example, the part-whole sub-construct is placed at the top of the model and the remaining four subordinate sub-constructs are considered to follow based upon knowledge of the part-whole concept. Thus, students must first understand the part-whole concept before they can move onto understanding ratio, operator, quotient or measure. The second layer of the Kiernan model shows how each sub-construct relates to fraction operations. For example, the operator sub-construct is a prerequisite to understanding multiplication and problem solving.

3 Research Questions

The first objective of this article is to follow the work of Charalambos and Pitta-Pantazi. in an effort to determine if there are any cognitive similarities between the fraction schemas of children and adult students. The second objective of this article is to investigate the nature of the adult fraction schema and in particular to search for evidence of relationships between the four subordinate sub-constructs.

The first objective leads to five research questions which simply replicate the investigation of Charalambos and Pitta-Pantazi (2007, p. 301) with adult learners instead of children.

Research question #1: Are there five well defined conceptual entities: part-whole, ratio, operator, measure and quotient? Is the students' performance on these five sub-constructs different?

Research question #2: Does students' performance on the sub-constructs reflect the hierarchical structure of the model, i.e. are the other four sub-constructs subordinate to part-whole?

Research Question #3: Is the ratio sub-construct the necessary path to proficiency with fractional equivalence?

Research Question #4: Is the operator sub-construct the necessary path to proficiency with multiplication of fractions?

Research Question #5: Is the measure sub-construct the necessary path to proficiency with addition of fractions?

The second objective of this study is to investigate the nature of the adults' fraction schema and in particular to determine whether there are any relationships between the four sub-constructs in the fraction schema. Charalambos and Pitta-Pantazi examined students' performance in a top-down, hierarchical manner but did not look for any evidence of a lateral relationship between the different sub-constructs.

Research Question #6: What is the adult fraction schema? Are there significant correlations between the four subordinate sub-constructs?

4 Setting

The following study is based on the statistical analysis of data involving fraction concepts collected over several semesters by adult students taking developmental mathematics at two urban community colleges in the CUNY system. Unlike the report of Charalambos and Pitta-Pantazi, which involved children, the subjects in this study were adult students who tend to struggle with fractions. The student body at Hostos Community College is predominately female (80%) and minority (95%). It is important to note that the student body used in this study is taken from the mathematically weakest group of students applying to the CUNY system. These students failed both the algebra and pre-algebra placement exams in mathematics. Thus, in the interpretation of the results one must take into account the fact that the adult learners are seeing the material on fractions for the second time, whereas the children in Charalambos and Pitta-Pantazi were learning the material for the first time. Adult students in remedial courses frequently did not do well in mathematics when they learned it as children. It is also possible that they had forgotten much of what they learned since many non-traditional students have not been in school for several years.

At these community colleges this pre-algebra course lasts 14 weeks and contains: whole numbers, fractions, decimals, ratios, rates and proportions, percents, measurements, integers, and scientific notation thus each topic receives 2 to 3 weeks. In this time frame, for each topic the instructor will typically give a very brief review of any necessary foundational conceptual knowledge and then focus on procedural proficiency. In particular, at the introduction of fractions the instructor (following the textbook) will cover the part-whole interpretation of fractions, and although it is not stressed in most textbooks in the syllabus placement of fractions and mixed numbers on the number line is included and most instructors consider this an important topic. The quotient interpretation of fractions will not be covered with fractions but instead in decimals to the extent that the fraction bar can be used to represent division in the process of converting fractions to decimals. The quotient concept of dividing a whole in equal parts is also covered in problem solving in the first three sections. Likewise, the ratio interpretation of fractions will not be covered within fractions but instead will be covered in the ratio section as one possible notation for a ratio.

5 Methodology

Five mathematics instructors at two urban community colleges CUNY gave 95 pre-algebra students sets of exercises. These exercises included questions on the five sub-constructs; part-whole, ratio, operator, quotient and measure as well as three related areas of procedural proficiency; fractional equivalence, multiplication and addition. The exercises used in this study are identical to and used with the permission of the original authors, Charalambos and Pitta-Pantazi. (See Appendix A)

In contrast to the original study, in which these exercises were given to children during class time, the adult students in this study were given these exercises as a take home extra credit. Thus, students had sufficient time to answer exercises. Students were instructed to answer all exercises and told that blanks would count against them: both incorrect and unanswered exercises were scored as a zero. This method of grading was used also by Charalambos and Pitta-Pantazi and repeated here to preserve the validity of the comparison between the different groups of students.

The Part-Whole sub-construct is measured by several different problems involving the relationship between different part-whole geometric exercises, as well as exercises that test the relationship between geometric constructs and fractional $\frac{\text{part}}{\text{whole}}$ notation. In part I, the part-whole sub-construct was evaluated by five separate exercises that measured the ability to relate or translate geometric pictures into fractional notation. In part II, the part-whole sub-construct was evaluated by 7 exercises that required the students to relate geometric pictures to fractional notation as in part I and two exercises required the student to relate a fraction such as $\frac{2}{3}$ to the concept of a whole divided into three parts and then taking two of them.

The ratio sub-construct questions asked students to compare two numbers and to understand fractions as ratios. This set of question also contained two parts. In part I the ratio sub-construct was tested by two different types of exercises. The first contained two exercises that require students to relate ratio concepts in fractional notation. The second contained three exercises that require the student to compare two given ratios. In part II, the ratio sub-construct was evaluated by three exercises that were similar to the exercises in part I, however, the exercises involved more abstract reasoning without any given numbers.

The operator sub-construct was also evaluated using two types of exercises. The first type in part I required students to view fraction multiplication as an input-output system. The three exercises in part II asked students to decide if given operations with fractions were equivalent to each other.

The quotient sub-construct was measured by problems that required knowledge of the concept of "fair shares" of a quantity. Part I problems included tasks about dividing quantities into equal parts. In part II, students were asked to represent these divisions as fractions.

The measure construct involves an understanding of the size of a fraction and students were asked to place numbers in the appropriate position on a number line. In part I, students were asked to place the whole number "1" on the line but were only given one reference point to use. They were also asked to find a fraction located in between two other fractions without a number line. In part II, the students only had one reference point on the number line and were asked to place different numbers on a number line that was scaled to units of $\frac{1}{4}$.

6 Results and Analysis

The basic structure of Kiernan model, shown in Figure 1, indicates that the fraction concept contains five sub-constructs which may be viewed as five well defined conceptual entities. "Well defined" means that all the sets of exercises or components used to evaluate any given sub-constructs correlate with one another at a significance level of 0.05 as well as with the variable that results when all these components are combined into one variable or factor. The authors of the original study state the correlations between the individual exercises and the combined variable or factor used to evaluate any given sub-construct, (Charalambos and Pitta-Pantazi., 2007, p. 306). For this reason, the criterion "minimally well defined" will mean that all components correlate significantly with the combined variable or factor but not with one another. Any exercise that did not correlate significantly with the combined factor was discarded and not used in this report.

A "partially well defined" sub-construct is one in which all components correlate significantly with the combined variable or factor and there exists at least one component that all the others significantly correlate with at the 0.05 level.

(Note: In visual examples all lines represent correlations significant at the 0.05 level.)

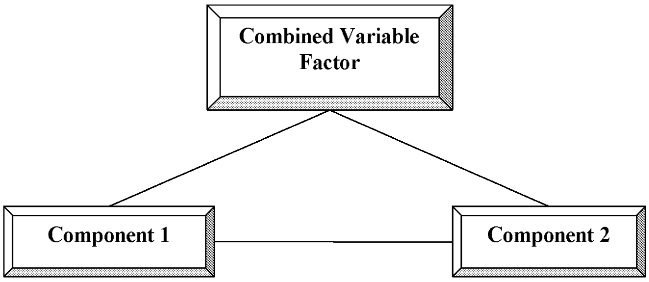

Figure 2a. Well-defined.
Significant correlation between all components as well as between components and combined variable or factor

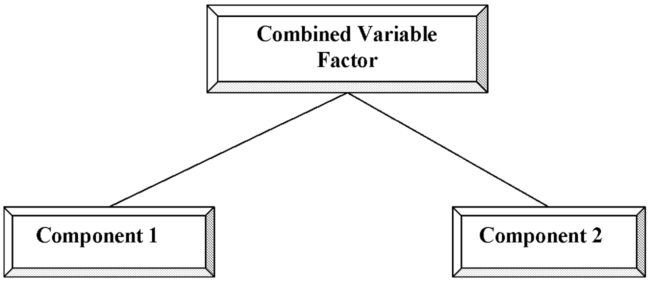

Figure 2b. Minimally well defined.
Significant correlation between each component and the combined variable or factor but no between the individual components

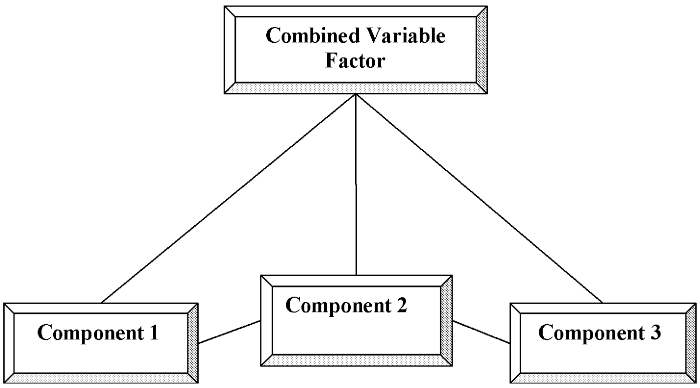

Figure 2c. Partially well defined

Results and Analysis: Research question #1

#1.1 The part-whole sub-construct is well defined

The part-whole factor contained two components, the five exercises in part I (part-whole 1) and seven in part II (part-whole 2). These were combined to form the factor or combined variable 'part-whole'.

- Correlation between part-whole 1 and part-whole is R = 0.78.
- Correlation between part-whole 2 and part-whole is R = 0.89.
- Correlation between part-whole 1 and part-whole 2 is R = 0.41.

All correlations were significant at the 0.01 level.

#1.2 The ratio sub-construct is minimally defined

The ratio factor contained three components. In part I, there was a set of two exercises that related the part-whole concept used in ratios with its use in fraction notation (ratio 1) with a mean of 0.59, and a set of three proportional reasoning exercises (ratio 2) with a mean of 0.67, and in part II there was a set of three abstract proportional reasoning exercises (ratio 3) with a mean of 0.37. None of these sets of ratio exercises correlated significantly with one another however, when these seven exercises were combined into one variable or factor 'ratio' each of the sub-components did correlate with this total or combined variable or factor.

- Correlation between ratio 1 and ratio is R = 0.58.
- Correlation between ratio 2 and ratio is R = 0.61.
- Correlation between ratio 3 and ratio is R = 0.59.

All correlations were significant at the 0.01 level.

#1.3 The operator sub-construct is well defined

The operator factor contained two components. In part I there was a set of two exercises that used input-output boxes to evaluate the operator sub-construct (operator 1), and a set of three exercises (operator 2) in part II.

- Correlation between operator 1 and operator is R = 0.73^{**}.
- Correlation between operator 2 and operator is R = 0.82^{**}.
- Correlation between operator 1 and operator 2 is R = 0.202^{*}.

* significant at the 0.05 level,
** significant at the 0.01 level.

#1.4 The quotient sub-construct is partially well defined

The quotient factor contained four components. In part I the first component was the pizza problem (quotient 1) and the second component was an application problem where students were given a total of 20 kilograms, the fractional amount required for one pie and asked to determine the number of pies (quotient 2). In part II, the third component was an application problem structurally similar to the second component but worded using pizzas (quotient 3) and the fourth evaluated students' recognition of fractional notation as division (quotient 4).

- The correlation between quotient 1 and quotient is R = 0.6,
- The correlation between quotient 2 and quotient is R = 0.7,
- The correlation between quotient 3 and quotient is R = 0.63,
- The correlation between quotient 4 and quotient is R = 0.57.

All correlations were significant at the 0.01 level.

The second component quotient 2 is central and each of the other components correlated significantly with it.

- The correlation between quotient 1 and quotient 2 is R = 0.204*,
- The correlation between quotient 3 and quotient 2 is R = 0.25*,
- The correlation between quotient 4 and quotient 2 is R = 0.40**.

* significant at the 0.05 level,
** significant at the 0.01 level.

#1.5 The measure sub-construct is partially well defined

The measure factor contained three components. In part I were two components, the first component (measure 1) asked students to place the unit number 1 on a number line that had marked one fraction or mixed number and the second component (measure 2) asked students to find a fraction between two existing fractions. In part III was a number line exercise (measure 3).

- The correlation between measure 1 and measure is R = 0.69,
- The correlation between measure 2 and measure is R = 0.324,
- The correlation between measure 3 and measure is R = 0.88.

All correlations are significant at the 0.01 level

The central component is measure 1, placing the unit 1 on the number line.

- The correlation between measure 2 and measure 1 is $R = 0.231^{*}$,
- The correlation between measure 3 and measure1 is $R = 0.282^{**}$.

$*$ significant at the 0.05 level,
$**$ significant at the 0.01 level.

Sub-construct	**Category**
Part-Whole	**well defined**
Ratio	**minimally well defined**
Operator	**well defined**
Quotient	**partially well defined**
Measure	**partially well defined**

Table 1. The category of each sub construct

In the original study a t-test was used to verify differences between mean scores for the individual pairs of sub-constructs. "The use of the t-criterion for paired samples revealed that all the differences between each pair of sub-constructs were significant at level $p < 0.01$" (Charalambos and Pitta-Pantazi, 2007, p. 306). Following their approach in this study, two sided t-tests were conducted on each pair of sub-constructs, all were different at significance level $p < 0.01$ with two exceptions. The two sided p-value for quotient and operator is $p = 0.33$ and the two sided p-value for operator and measure is $p = 0.16$.

The sub-constructs used were either well defined or partially well defined with an identifiable central component that correlated significantly with the other components except for the ratio sub-construct which was only minimally well defined. In particular, while the students did relatively well on the three proportional reasoning exercises (ratio 2) in part I which involved exact numerical values, they did poorly on the three proportional reasoning exercises (ratio 3) in part II which involved abstract comparative phrases. Furthermore, the correlation between these two components both of which can be viewed as proportional reasoning was basically zero $R = -0.02$. Thus, students did relatively well on the concrete proportional reasoning exercises of ratio 2 but poorly on the abstract reasoning of ratio 2 and did not see any connection between these sets of exercises. This indicates abstraction is difficult for these adult students.

Results and Analysis: Research question #2

The first working hypothesis of the original authors is, "The part-whole sub-construct is considered a fundamental for developing understanding of the four subordinate constructs of fractions" (Charalambos and Pitta-Pantazi, 2007, p. 295). The first hypothesis predicts that students will do better on the fundamental part-whole construct then on the subsequent sub-constructs and that there will be a strong correlation between the part whole sub-construct and the others.

Research question 2 will be evaluated in two parts: first, are the mean scores of the four sub-constructs – ratio, operator, quotient and measure less than the mean score of the part-whole sub-construct; and second, are the relationship (correlations) between the sub-constructs those predicted by the model?

To analyze the first component of the second research question the mean score of the students on each of the sub-constructs was compared for both the original study (table 2a) and the present study (table 2b). To analyze the second, correlations between the five sub-constructs are studied (figure 3b) and compared to the values obtained from the original study (figure 3a).

Sub-construct	Mean	SD
Part-Whole	0.75	0.20
Ratio	0.64	0.25
Operator	0.45	0.35
Quotient	0.55	0.29
Measure	0.25	0.33

Table 2a. Mean scores and standard deviation (SD) of each sub-construct of fractions-original study (Table II, Charalambos Y.C., & Pitta-Pantazi, D., 2007, p. 306)

Sub-construct	Mean	SD
Part-Whole	0.658	0.215
Ratio	0.571	0.204
Operator	0.437	0.308
Quotient	0.471	0.273
Measure	0.391	0.272

Table 2b. Mean scores and standard deviation of each sub-construct of fractions-present study

The part-whole is fundamental and the other sub-constructs are subordinate to it. While the adult students in this study tended to have slightly lower mean scores on the various sub-constructs tasks (table 2b) than the children (table 2a) the pattern is remarkably similar with both the children and the adults doing better in the part-whole sub-construct than the other four sub-constructs. Thus, data from both the original study working with children and the present study with adults support the hypothesis that the part-whole sub-construct is foundational conceptual knowledge and the other four sub-constructs are subordinate to the part-whole sub-construct.

To analyze part II of the second and sixth research questions the correlations between the sub-constructs are listed in table 3.

		Partwhole	Ratio	Operator	Quotient	Measure
partwhole	Pearson Correlation	1	.117	.264**	.409**	.486**
	Sig.(2-tailed)		.260	.010	.000	.000
	N	95	95	95	95	95
Ratio	Pearson Correlation	.117	1	.215*	.398**	.152
	Sig.(2-tailed)	.260		.036	.000	.143
	N	95	95	95	95	95
Operator	Pearson Correlation	.264**	.215*	1	.349**	.345**
	Sig.(2-tailed)	.010	.036		.001	.001
	N	95	95	95	95	95
Quotient	Pearson Correlation	.409**	.398**	.349**	1	.508**
	Sig.(2-tailed)	.000	.000	.001		.000
	N	95	95	95	95	95
Measure	Pearson Correlation	.486**	.152	.345**	.508**	1
	Sig.(2-tailed)	.000	.143	.001	.000	
	N	95	95	95	95	95

Table 3. Correlation between Sub-Constructs in Present Study
Correlations between sub-constructs: part-whole, ratio, operator, quotient and measure

Figure 3b has been constructed for comparison with figure 3a. However, it does not represent the entire fraction schema given by the data in table 3, because there were significant correlations between the four subordinate sub-constructs. The fraction schema based upon the significant correlations at the 0.01 level from table 3 is given in figure 4 and will be used to answer research question 6. The model given in figure 1 uses arrows to represent relationships of causality. In keeping with this model, arrows in figure 4 will represent correlations significant at the 0.01 level between sub-constructs predicted by the model. Lines in figure 4 represent correlations significant at the 0.01 level between the subordinate sub-constructs not suggested by the model.

According to figure 3a in the original study the part-whole sub-construct correlated well with all the other sub-constructs, except perhaps the measure sub-construct, thus the children drew connections between the tasks in each of these areas and their conceptual knowledge of part-whole. The results of figure 3b show that in this study with adults the part-whole sub-constructs correlated (at 0.01 level of significance) with all sub-constructs except the ratio sub-construct. The correlation between part-whole and ratio ($R = 0.12$) is not significant at even the 0.05 level.

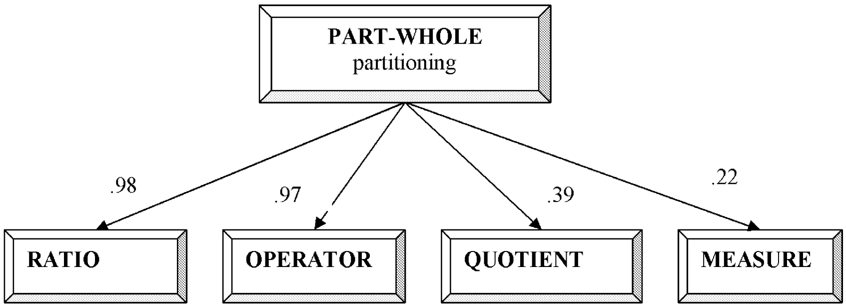

Figure 3a. Correlation between sub-constructs R-values: Original Study (Excerpted from Figure 2, Charalambos and Pitta-Pantazi, 2007, p. 306)

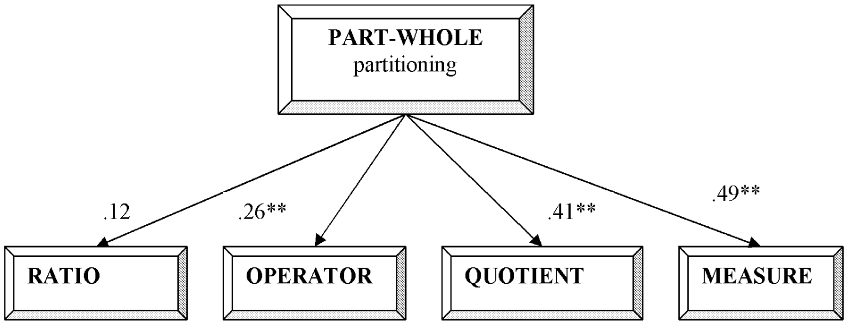

Figure 3a. Correlation between sub-constructs R-values: Present Study Data from table 3

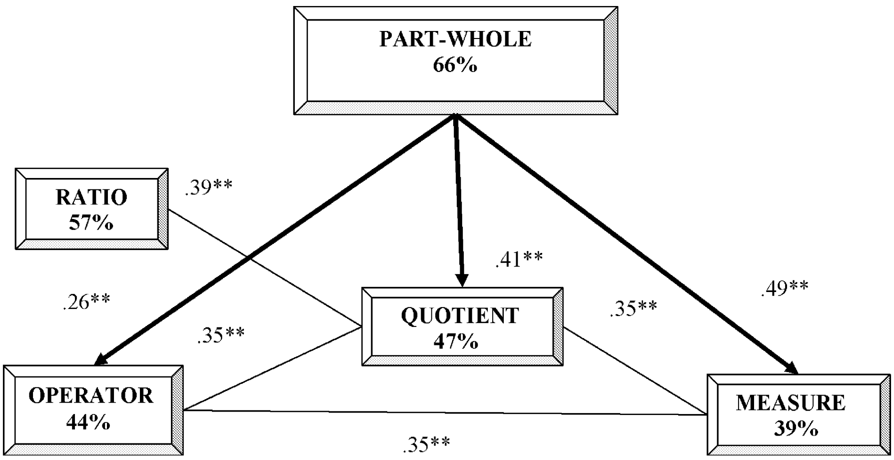

Figure 4. Fraction Schema based upon Significant Correlations with mean scores

The second level of Kiernan's model was used to investigate research questions 3 through 5. The exercises associated with these research questions reflected students' ability to solve problems of equivalence, multiplication and addition of fractions. These exercises are included in Appendix A. Before proceeding to the research questions we need to examine if the exercises used to evaluate the factors were well defined and determine if there were any biases that may have influenced students' responses.

Fractional Equivalence is partially well defined

The seven exercises (equivalence 1) that evaluated equivalence in part I were given by two columns of pairs different or varying geometrical pictures that each contained a total number of objects and a shaded part from this total. The student was asked to determine whether each pair of geometrical objects was equivalent.

An equivalence based application problem was also included in part I (equivalence 2). One example is exercise #15: Ana says $\frac{4}{4}$ is larger than $\frac{7}{7}$ because the pieces are larger. George says $\frac{7}{7}$ is larger than $\frac{4}{4}$ because it contains more pieces. The student is asked to explain who is correct. There may be a certain bias to this question because neither is correct. However, the authors kept the question with original wording to preserve the comparison to the original work of Charalambos and Pitta-Pantazi. The correlation between equivalence 1 and equivalence 2 (0.241) was significant at the 0.05 level. This indicates the adult students understood the connection between geometric and procedural aspects of fractional equivalence.

In part II the equivalence factor was measured by three standard procedural exercises (equivalence 3) involving the equivalence of fractions. For example, $\frac{2}{3} = \frac{?}{12}$.

Equivalence 3 correlated significantly with equivalence 1 (R = 0.29) at the 0.01 level but did not correlate significantly with equivalence 2.

- Correlation between equivalence 1 and equivalence is $R = 0.911^{**}$.
- Correlation between equivalence 2 and equivalence is $R = 0.42^{**}$.

The central component of the equivalence factor is equivalence 1 – the geometric understanding of equivalent fractions.

- Correlation between equivalence 3 and equivalence is $R = 0.0.63^{**}$.
- Correlation between equivalence 2 and equivalence 1 is $R= 0.241^{*}$.
- Correlation between equivalence 3 and equivalence 1 is $R = 0.291^{**}$.

* significant at the 0.05 level,
** significant at the 0.01 level.

Multiplication of Fractions is well defined

Part I contained two exercises with multiplication of fractions (multiplication 1) and part II contained two exercises (multiplication 2) that evaluate multiplication and division of fractions. The correlation between multiplication 1 and multiplication 2 is R = 0.26 which is significant at the 0.05 level and these were combined to form one variable or factor 'multiplication,' which was well defined.

Addition of Fractions is minimally well defined

Students' proficiency with addition of fractions was evaluated by two exercises in part I (addition 1) and two exercises in part II (addition 2) covering addition of proper fractions with both like and unlike denominators. Although addition 1 did not correlate significantly with addition 2 (R = 0.15), they were combined to form the factor 'addition,' and both components correlated significantly with the combined variable or factor addition at the 0.01 level. That addition was only minimally well defined is probably due to the relatively small number of exercises (four) used to evaluated this factor. In addition to more exercises to evaluate the addition factor, there should have been more variety and more challenging exercises for these adult students. For example there were no problems on addition or subtraction with mixed numbers, fractions and whole numbers or addition of fractions that required non-trivial least common denominators.

Procedural Knowledge	Mean	SD
Equivalence	058	0.29
Addition	0.81	0.24
Multiplication	0.57	0.32

Table 4. Mean Score and Standard Deviation for Factors of Procedural Proficiency

Results and analysis: Research question #3

Are Charalambos and Pitta-Pantazi correct when they hypothesize, "the ratio sub-construct is considered the most natural path to promote the concept of equivalence and, subsequently the process of finding equivalent fractions?" (p. 294-295).

In this study, although the equivalence factor is partially well defined the ratio sub-construct is only minimally well defined. Furthermore, the ratio sub-construct did correlate significantly with the part-whole sub-construct (figure 4) and the correlation between them was not significant R = 0.046, p = 0.66. The following correlations, significant at the 0.05 or 0.01 level, were noted.

- Correlation between equivalence and part-whole is R = 0.53**.
- Correlation between equivalence and quotient is R = 0.4**.
- Correlation between equivalence and measure is R = 0.43**.
- Correlation between equivalence and operator is R = 0.23*.

* significant at the 0.05 level,
** significant at the 0.01 level.

Thus, these adult students used their conceptual knowledge of the other four sub-constructs but not the ratio in answering the equivalence exercises clearly contradicting the hypothesis of research question 3. This result is in stark contrast to those obtained by the original authors working with children, their results strongly support this hypothesis, in particular their correlation between the ratio and equivalence factors is R = 0.85 (figure 2, Charalambos and Pitta-Pantazi, 2007, p. 294-295).

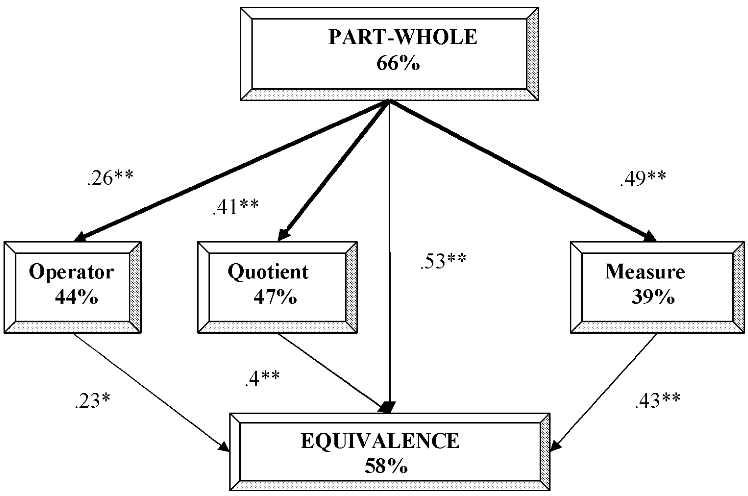

Figure 5. Fraction Schema for Equivalence by Correlation and Mean Score

Figure 5 represents the fraction schema based upon the presumed hierarchical structure that sub-constructs knowledge is necessary for students' equivalence knowledge. Thus, arrows are used to represent correlations significant at 0.01 level between any sub-construct and equivalence.

Results and analysis: Research question #4

In research question 4 we investigate the hypothesis that the operator sub-construct is regarded as helpful for developing understanding of the multiplicative operations of fractions (Charalambos and Pitta-Pantazi, 2007, p. 294-295).

The results of the original study indicate that the operator and quotient sub-constructs are together helpful for developing proficiency with multiplication of fractions.

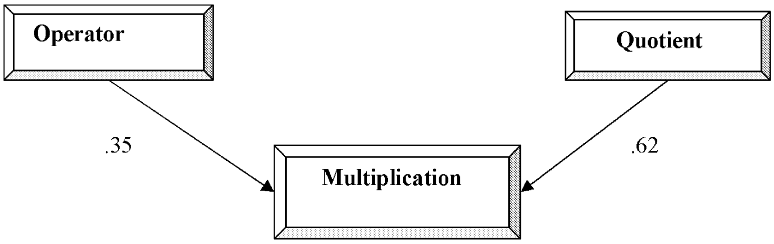

Figure 6a. (Excerpted from figure 2, Charalambos and Pantazi, 2007, p. 306)

In this study:

- The correlation between operator and multiplication is R = 0.25^{*},
- The correlation between quotient and multiplication is R = 0.42^{**},
- The correlation between part-whole and multiplication is R = 0.28^{**}.

Figure 6b is arranged hierarchically according to the model of figure 1, arrows represent correlations significant at the 0.01 or 0.05 level between a sub-construct and a procedural based task, in this case multiplication.

It is of interest to note that both the original study and the present study indicate that the quotient sub-construct is more important than the operator sub-construct in determining or predicting students' proficiency with multiplication of fractions. The original authors speaking on this state, "it cannot go unnoticed that the quotient sub-construct explained much more of this variation compared to the percent explained by the operator sub-construct" (Charalambos and Pitta-Pantazi, 2007, p. 310). In conclusion, the results of both studies imply that the hypothesis of research question 4 should be amended to include that both operator and quotient knowledge are helpful in developing proficiency with fractions.

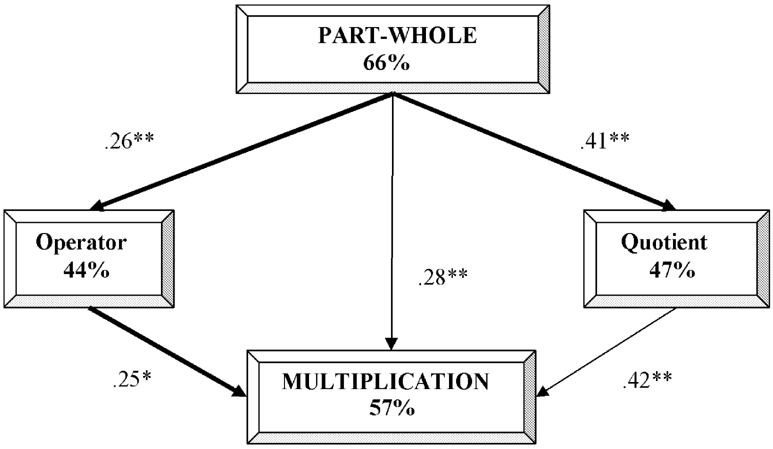

Figure 6b. Fraction Schema for Multiplication by Correlation with mean scores

Results and analysis: Research question #5

In research question 5, we investigate the hypothesis that the measure sub-construct is necessary for developing proficiency with additive operations on fractions. The results of the original study contradict this hypothesis and their results indicate that the correlation between the part-whole sub-construct and addition of fractions R = 0.38 was significant while that between measure and addition was not. (Figure 2, Charalambos and Pitta-Pantazi, 2007, p. 306)

In this study:

- The correlation between part-whole and addition is R = 0.26*,
- The correlation between measure and addition is R = 0.23*.

Arrows represent correlations between sub-constructs and addition significant at the 0.05 level.

The addition factor like the ratio factor was only minimally well defined, and like the ratio factor it experienced relatively low correlation with the rest of the fraction schema. The correlations of figure 7 support the Kiernan model and the resulting hypothesis of research question 5 that the measure sub-construct is important for developing proficiency with additive operations. Figure 7 also supports the previous study with children that the part-whole sub-construct was important in determining students' proficiency with addition of fractions. Students' proficiency with addition of fractions can be used to assist their conceptual understanding of the measure concept as well as the part-whole concept.

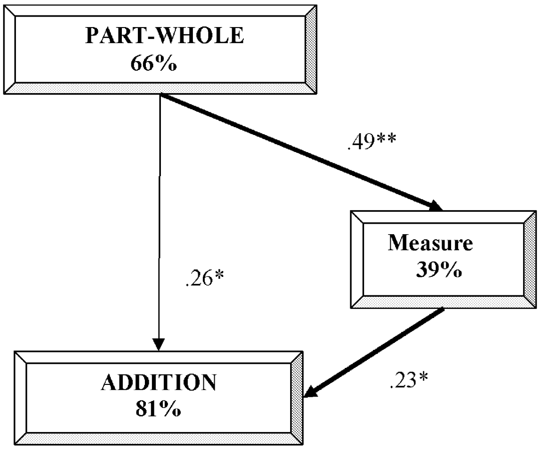

Figure 7. Fraction Schema for Addition by Correlation with Percents Correct
Correlations significant at the 0.05 level

Results and analysis: Research Question #6

In research question #6, we examined the nature of the adult fraction schema, specifically, are there significant correlations between the subordinate sub-constructs?

The ratio sub-construct is significantly correlated at the 0.01 level only with the quotient sub-construct (figure 4). This indicates that the ratio sub-construct was not an integral part of the fraction schema for these students. Although the operator sub-constructs does correlate significantly at the 0.01 level with the part-whole sub-construct (R = 0.26) this correlation is rather low and figure 4 demonstrates the operator sub-construct has more connection to the quotient and measure sub-constructs.

Another surprising result is that the quotient sub-construct, not the part-whole sub-construct, is a central component. It correlates significantly at the 0.01 level with all the other components in this schema while the part-whole does not.

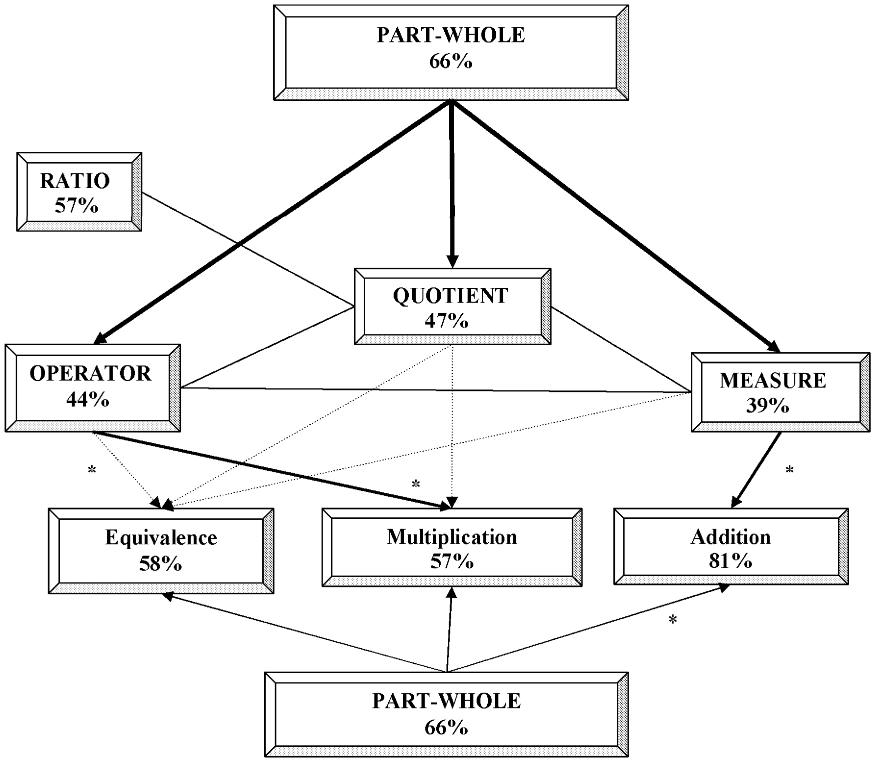

Figure 8. Model of Fraction Schema
Lines and arrows represent correlations significant at the 0.01 level unless indicated with (∗). Those arrows that have a (∗) are significant only at the 0.05 level. Arrows represent correlations between sub-constructs predicted by the model or between sub-constructs and the procedural tasks in the operations of addition and multiplication and fractional equivalence which, following the model of figure 1 are in the bottom level.
* Significant at the 0.05 level, all other lines are arrows are significant at the 0.01 level.

Comparison between the fraction schema of Figure 4 & 8 with the model of Figure 1

The Kiernan model of figure 1 predicts that only the part-whole sub-construct will correlate significantly with the other four sub-constructs. Figure 4 demonstrates that the part-whole sub-construct does not correlate significantly at the 0.01 level with the ratio sub-construct and its correlation with operator, while significant, is rather low. In contrast, the quotient sub-construct does correlate significantly at the 0.01 level with each of the other faction sub-constructs and is thus a central concept in the fraction schema. Figure

8 demonstrates the fundamental role that the part-whole sub-construct plays in determining students' proficiency with fraction operations and equivalence, a result not predicted by the model of figure 1.

7 Discussion

Research question #1: Are there five well defined conceptual entities that comprise the notion of fraction?

The first task in the original research was to "examine whether each of the 50 tasks-variables used in the test was significantly correlated with the factor it was designed to load" (Charalambos and Pitta-Pantazi, 2007, p. 305). The criterion of significant correlations between the individual tasks and the combined factor used in the original study is called minimally well defined in this study. Two stronger criteria put forth and used in this study are called well defined and partially well defined. In the present study all sub-constructs except for 'ratio' were partially well defined or well defined. That the ratio sub-construct was only minimally well defined indicates a sense of randomness to students' responses to the ratio tasks and may explain why the ratio factor did not significantly correlate with most of the other factors in the fraction schema.

In the original article, using t-tests the mean value of all pairs of sub-constructs were found to be significantly different from one another at $p < 0.01$ level of significance. In this study, using t-tests the mean value of all sub-constructs were found to be significantly different from one another except for operator and quotient as well as operator and measure.

Research question #2: Are the other four sub-constructs subordinate to part-whole construct?

The hypothesis that the part-whole sub-construct is foundational and the other four are subordinate was supported by Charalambos and Pitta-Pantazi and they state the data generally support the five sub-construct model. As to the hierarchy of the part-whole sub-construct with the others they ask: "Does the finding justify the traditional instructional approach in using the part-whole notion as an inroad to teaching fractions?" To some extent they agree, although they note that it could be because the part-whole method is the dominant method of instruction and affects student's fraction schema because this type of representation is used more frequently in students' mathematics textbooks.

In looking at the mean scores of the sub-constructs in table 2b the results of this study tend to support the hierarchical structure of the model with students doing better on the part-whole sub-construct than the others. Figure 3b indicates the part-whole sub-construct does correlate significantly with the other sub-constructs except for ratio. Thus, we can conclude that the part-whole sub-construct is foundational conceptual knowledge.

Research question #3: Does an understanding of ratio imply proficiency with fractional equivalence?

Charalambos and Pitta-Pantazi found that there was a significant relationship between ratio and fractional equivalence. In this study, the correlation was close to zero, indicating these students experienced almost no connection between the ratio sub-construct and equivalence of fractions. As noted earlier, the correlation between the ratio and part-whole sub-constructs was also very low thus the ratio sub-construct which was not an integral part of these adult students' schema in determining students' proficiency with fractional equivalence.

Although figure 5 indicates the sub-constructs of quotient and measure correlated significantly with the fractional equivalence factor, the mean score for this factor was higher than that of either of these two sub-constructs. This suggests that student's knowledge of fractional equivalence was used to assist them to develop conceptual quotient and measure knowledge.

Research question #4: Does an understanding of the concept of operator imply proficiency with fraction multiplication?

Our results agreed with the original authors and we found that the quotient sub-construct is more important than the operator sub-construct in determining or predicting students' proficiency with multiplication of fractions. These results imply that the hypothesis of research question 4 should be amended to include that both the operator and quotient sub-constructs are helpful in understanding the multiplicative aspect of fractions. Figure 6b also indicates that knowledge of the part-whole is useful in proficiency with multiplication.

Note, in figure 6b, the mean score of the multiplication factor is higher than the mean score of either the quotient or operator sub-construct. Thus, student's proficiency with multiplication of fractions can be viewed as assisting their conceptual understanding of the quotient and operator concept.

Research question #5: Does an understanding of the measure concept imply proficiency with the operation of fraction addition?

The results of the original study do not support this hypothesis; however, Charalambos and Pitta-Pantazi note that this may be because students have a great deal of difficulty using a number line.

The results of this study give more support to Kiernan's model than the original study. The adult students had a higher mean score on the measure sub-construct than the children. Furthermore, in this study, the correlation between measure and part-whole was significant at the 0.01 level while the correlation between measure and addition was significant at the 0.05 level.

These results support the model more than those of the original study; however, a look at figure 7 reveals two important considerations. First, both the part-whole and measure sub-constructs correlated significantly at the 0.05 level with students' proficiency in addition, and second, the mean score on these addition exercises was much higher than that of the measure sub-construct or the part-whole sub-construct. This indicates students' proficiency with addition can be viewed as assisting development of their conceptual measure knowledge.

Research question #6: What is the nature of the adult fraction schema? In particular, are there significant correlations between the sub-constructs and in particular between the four subordinate sub-constructs?

It is of interest to note that the connection of the three sub-constructs in figure 4 (quotient, measure and operator) with the part-whole sub-construct can be interpreted as those concepts being variations of the partitioning process. Fractions are introduced using partitioning as foundational conceptual knowledge (part-whole) and number line placement of fractions reinforces this partitioning concept (measure). The operator concept is used briefly to introduce multiplication of fractions through the action of taking a fraction of a quantity and can be explained as partitioning the quantity according to the denominator. It is reasonable to conclude the partitioning process that defines the part-whole concept had been assimilated into and is an integral part of students' fraction schema.

Figure 4 highlights three important results not predicted by the Kiernan model shown in figure 1 or in the results of the original study displayed in figure 3a. First, the ratio sub-construct only correlates with the quotient sub-construct and not with the part-whole sub-construct or the equivalence factor as predicted by figure 1 and 3a, neither does it correlate with any other sub-construct or factor and thus it is barely integrated into the fraction schema of figures 4 and 8. Second, the quotient sub-construct is a central component of this fraction schema because it correlates significantly at the 0.01 level with each of the other sub-constructs. Third, the operator sub-construct, although significantly related to the part-whole sub-construct, does not have a correlation value with the part-whole sub-construct even close to that of the original study for children.

The result that the ratio sub-construct was only minimally well defined indicates students did not understand the relationship between the exercises used to evaluate this concept. They did not connect the task that involved the duality of fraction notation to represent either part-part or part-whole with tasks that involved proportional reasoning. Furthermore, they could not relate concrete and abstract proportional reasoning tasks. While it is predictable that a minimally well defined factor would not show significant correlations to the rest of the fraction schema, the contrast between these results and the significant correlations of ratio with part-whole for children is dramatic. One possible explanation is that the ratio and proportions section is covered in the syllabus well after fractions and these concepts are treated as relatively separate entities. Another explanation is that the other sub-constructs all share a common conceptual interpretation based upon partitioning a quantity into equal parts while the ratio sub-construct is more typically interpreted as a relationship between two partial quantities.

Figure 4 demonstrates a significant correlation between the part-whole and operator sub-constructs; however, this correlation is noticeably less than the correlations between the part-whole sub-construct and the other sub-constructs (not including ratio). This result could be due to the more abstract nature of the operator sub-construct.

Figure 4 indicates that the quotient sub-construct is a central component of the fraction schema because it is the only subcontract that correlates significantly with all the others. Therefore, the quotient sub-construct is central to the fraction schema.

One explanation for the centrality of the quotient sub-construct is that most of the tasks used to evaluate the quotient sub-constructs involve problem solving which has a central role in the fraction model of figure 1. Application problems involve the coordination of knowledge from several constructs which may explain why the quotient sub-construct correlates significantly with the other sub-constructs.

8 Conclusion

The primary objective of this study was to replicate the work of Charalambos and Pitta-Pantazi with adult students in order to investigate any cognitive differences between the fraction schema of children and adult students in remedial mathematics classes.

The Kiernan fraction schema contains five distinct sub-constructs arranged in a hierarchy with the part-whole serving as a foundation for learning the other subordinate sub-constructs. The results of the original study support the existence of the five distinct sub-constructs in the fraction schema and they also support the hierarchical structure of the fraction schema. Charalambos and Pitta-Pantazi state that "this finding provides support to the claim that the part-whole interpretation of fractions should be considered a necessary, but not sufficient condition for developing an understanding of the remaining notions of fractions." (p. 310)

The results of this study were in agreement with the original; a comparison between the mean scores of the children and adult students on each task (table 2a and 2b) demonstrated remarkable similarity. The original authors' conclusion, based upon table 2.4 (table 2a) that "the differences in students' performance on the five interpretations of fractions mirrors the imbalance in emphasis placed on them during instruction" (Charalambos, Pitta-Pantazi, 2007, p. 309) equally applies to the adult students as the part-whole sub-construct receives the majority of emphasis in elementary and remedial mathematics textbooks in the United States. This implies that mathematics instructors and authors of mathematics textbooks would better serve their adult students if they would take a more balanced approach and utilize all five representations of fractions. Educators should be aware that the part-whole interpretation only serves as an inroad for learning fractions; one can not rely completely on this interpretation.

For adult students, we were able to find partial support for the Kiernan fraction schema because all the subordinate sub-constructs except ratio (figure 4, table 3) correlated significantly with the part-whole sub-construct. Thus, the adults engaged in mathematical reasoning about ratios and answering questions about the abstract operator concept with approximately the same level of proficiency as the children; however, the adult students were not able to assimilate this information into their fraction schema nearly as efficiently as the children.

Charalambos and Pitta-Pantazi conclude: "though the findings of the study are suggestive of the preponderance of the part-whole interpretation of fractions, they also underline the need that instruction emphasizes the other sub-constructs of fractions" (p. 310).

The results of the present study indicate not only the importance of an instructional emphasis on all four subordinate sub-constructs but raises the question of whether these students ever did understand the connections between the concrete part-whole sub-construct and the more abstract sub-constructs of ratio and operator. If reasoning skills and the ability for abstraction arise as separate from their foundational conceptual knowledge then these students would likely struggle with learning fractions and with multiple representations in mathematics. In reviewing the effect of the sub-construct knowledge on procedural proficiency, the original authors state, "as a whole, these finding suggest that a profound understanding of the different interpretations of fractions can uplift student's performance on tasks related to the operations of fractions and to fraction equivalence" (Charalambos and Pitta-Pantazi, 2007, p. 311).

The results of this study concerning the effect the sub-constructs demonstrated on fraction operations and equivalence are contained in figure 8. These results indicate that the part-whole sub-construct was foundational because students tended to do better in this sub-construct than the other factors. Furthermore the part-whole sub-construct was a central component of the fraction schema, it related to all sub-constructs (except ratio) and it correlated significantly at the 0.01 level with the factors of equivalence and multiplication and at the 0.05 level with the addition factor. Thus, students' part-whole knowledge appears to be assimilated into almost every aspect of their schema and it is reasonable to conclude they used this knowledge to solve tasks in every factor and sub-construct except ratio. Figure 8 also reveals that the subordinate sub-constructs together with the part-whole sub-construct demonstrated significant correlations with the fraction operations and equivalence. This result follows and expands upon those of the original study in which operator and quotient both significantly correlated with multiplication and part-whole was the only sub-construct to correlate significantly with addition (figure 2, Charalambos and Pitta-Pantazi, 2007, p. 306). Thus, the results of this study affirm and extend the results of the earlier study. An understanding of all the different interpretations of fraction including part-whole is beneficial when performing tasks on operations of fractions and equivalence.

It is important however to note that the mean score for the multiplication, addition and fractional equivalence factors were higher than the mean score for the subordinate sub-constructs (figure 8). Thus, the basic premise contained in the second level of the fraction schema which implies that students use their conceptual knowledge of the four subordinate sub-constructs to gain proficiency with fraction operations and equivalence (figure 1) may perhaps be backwards. For example, it is unlikely that students use their knowledge of a subordinate sub-construct such as operator (with mean score of 44%) to assist them in learning multiplication (with a mean score of 57%). In contrast, it is easy to believe students use their foundational part-whole knowledge (with a mean score of 66%) to learn multiplication. Instead, these results point to students using both their foundational conceptual knowledge of the part-whole sub-construct and their more procedural knowledge for operating on fractions and fractional equivalence to understand the three subordinate concepts of operator, quotient and measure. This result appears to be an example of students using previously learned conceptual knowledge and acquired procedural knowledge to further their conceptual development. That is an iterative cycle of conceptual knowledge advancing procedural knowledge which in turn advances further conceptual knowledge (Rittle-Johnson et. al., 2001).

The second objective of this present study is to determine whether there are relatively strong relationships between the four sub-constructs in the fraction schema as evidence of flexible thinking.

In an effort to investigate students' capacity for flexible thinking, correlations between the subordinate sub-constructs were studied (table 3) and the resulting schema is depicted in figure 4. Two important comments on these results are first, those sub-constructs that can be conceptually related to partitioning a whole into equal parts: part-whole, measure, quotient and to a lesser extent operator all correlate significantly with one another (figure 4) at the 0.01 level. Second, the quotient sub-construct is the only sub-construct that correlates significantly with each of the other sub-constructs. Thus, the quotient sub-construct is a central sub-construct since it was the only sub-construct that correlated significantly at the 0.01 level with each of the other sub-constructs.

One plausible explanation is that the partitioning nature of tasks used to evaluate quotient were easy to relate to other sub-constructs. It is also possible that the quotient exercises involved problem solving which itself has a central role in the fraction schema. In reflecting upon the relationship between the observed fraction schema in children, Charalambos and Pitta-Pantazi state "the part-whole interpretation of a fraction has a significant role in developing understanding of the multifaceted construct of fraction, since this construct is significantly related to all four subordinate constructs" (p. 309). The fraction schema obtained in this study (figure 4) suggests that adult remedial mathematics curriculum should introduce the fraction sub-constructs while stressing their relationship to partitioning. Instructors may have more success reaching these adult learners if they present problems as partitioning a quantity instead of introducing operator notation. For example, instead of taking two fifths of a quantity an instructor could present the problem as a portioning the quantity into five pieces and taking two of them.

The adult students had much more difficulty assimilating the ratio sub-construct and the operator sub-construct into their schema than the children (figure 3a, figure 3b, figure 4). Thus, the adult students exhibited difficulty with flexible thinking when it involved reasoning or abstraction of the concrete notion of partitioning. The authors conjecture is that this lack of flexible thinking may be cognitive evidence of the difficulties remedial arithmetic students have in completing their remedial courses.

Reflections on Flexible Thinking and the Mathematics Curricula

We now look at some specific errors that adult students frequently make when working with fraction that can be interpreted as due to lack of flexible thinking and consider how the curriculum might be adapted to address them.

The ratio sub-construct is typically used to represent a comparative process between quantities and the operator sub-construct represents a process of taking a fraction of a total. The lack of flexible thinking in understanding the subtle relationship between the fraction as a numerical quantity corresponding to the number of parts taken out of a partitioned whole and as a comparative process in the ratio sub-construct is evidenced by a common addition error. When a student adds: $\frac{1}{3} + \frac{2}{5}$ and obtains $\frac{3}{8}$ they are incorrectly employing addition of ratios as a natural extension of the addition of whole numbers. The fact that the ratio concept is separate from the rest of their fraction schema and the operation of addition of ratios is a natural extension of whole number addition suggests changes in the curriculum. Perhaps the ratio concept needs to be more extensively taught including operations on it while highlighting the subtle relationships between the ratio and part-whole sub-constructs.

Difficulties these students experience understanding rates in fraction notation can be seen in proportions. For example, given a rate of 25 mg per 500 ml and asked to find the number of mg per 10,000 ml students will rarely simplify the given rate to 1 mg per 40 ml before setting up the proportion. The student compartmentalizes fraction equivalence and rates used in proportions which are taught much later in the curriculum. Clearly, the relationship between the two should be stressed.

The fact that the operator sub-construct demonstrated a rather low correlation value with the part-whole construct is perhaps indicative of the underestimation of the role of fraction multiplication in the curriculum. Given that addition is generally problematic for students, it precedes and receives more time than multiplication. Consequently, serious consideration should be given to introducing multiplication and the related operator concept earlier in the fraction curriculum while stressing its relationship to partitioning. In contrast to the task of taking a fraction of a total, adult students experience much less difficulty taking a percent of a total, which has real life meaning for them. The relationship between these concepts is absent in the curriculum; it should definitely be included.

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Appendix A

This appendix is excerpted from the appendix of the original authors (Charalambos and Pitta-Pantazi, 2007, p. 312-314). These exercises were used with their permission. Do not copy or use this test without permission from the original authors.

A.1 PART-WHOLE

1. The following figure (A-1) represents $\frac{2}{3}$ of a set of marbles, draw the whole set of marbles.
2. Which of the following diagrams in figure (A-2) corresponds to the fraction $\frac{2}{3}$?

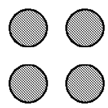
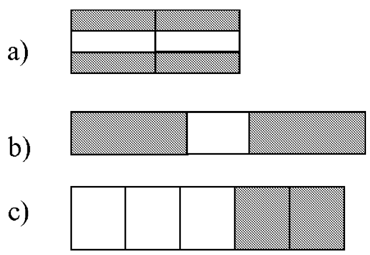


Figure A1.



d) Take a set of objects, divide it into three equal parts and take two of them.

Figure A2.

A.2 RATIO

1. John and Mary are preparing orange juice for their party. Presented below are the recipes they are used. What recipe will make the juice the most orangey?

   John' s recipe: Two cups of concentrate juice-five cups of water.

   Mary' s recipe: Four cups of concentrate juice-eight cups of water.

2. Seven girls equally split up three pizzas while three boys equally split up one pizza. Who gets more pizza, the boys or the girls? Please see figure (A-3) below.

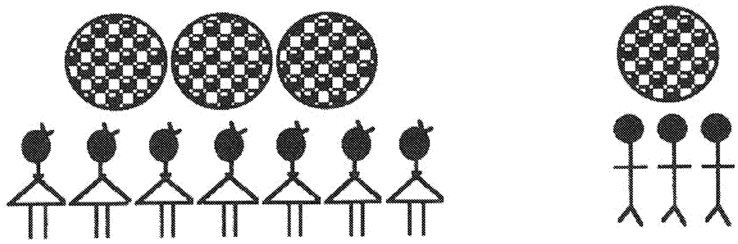


Figure A3.

A.3 OPERATOR

1. Without carrying out any operation, decide whether the following statement is correct: If we divide a number by four and then multiply the result by 3 we are going to get the same result we would get if we multiplied this number by $\frac{3}{4}$.
2. The following diagram (A-4) represents a machine that outputs $\frac{2}{3}$ of the input quantity. Which is the output quantity if the input quantity is equal to 12?

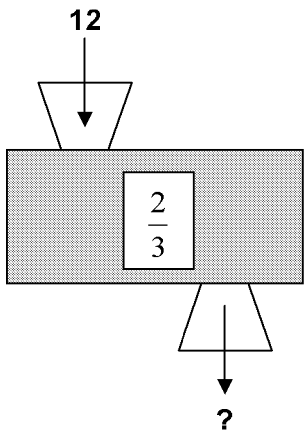


Figure A4.

A.4 QUOTIENT

1. Decide whether the following statement is correct or not: The fraction $\frac{2}{3}$ is equal to the quotient of the division 2 divided by 3.
2. Three pizzas are evenly divided among four children. How much pizza will each child get?
3. Three pizzas were evenly shared among some friends. If each of them got $\frac{3}{5}$ of the pizza, how many friends were there altogether?

A.5 MEASURE

1. Locate the number one on each of the following number lines in figure (A-5) below.
2. Name one fraction that appears between $\frac{1}{9}$ and $\frac{1}{8}$.

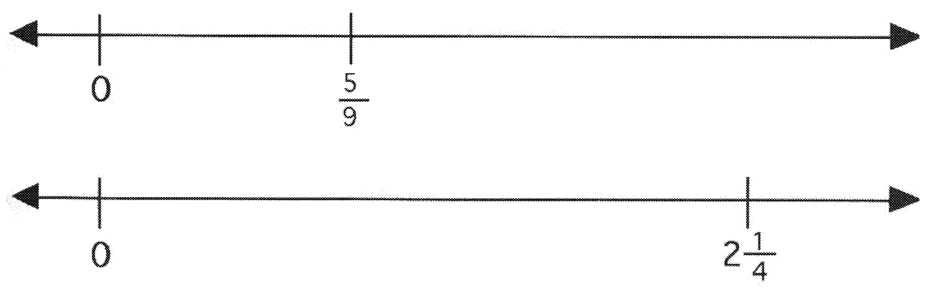


Figure A5.

A.6 OPERATIONS

1. Select the answer that represents the best estimation of the product of $6\frac{3}{4} \times 4\frac{3}{7}$ The product is between:

   a) 18 and 24,

   b) 25 and 28,

   c) 29 and 32,

   d) 33 and 40.
2. Find the sum of the following two fractions: $\frac{5}{8} + \frac{4}{5}$.
3. Find the result of the following: $2 \div \frac{1}{2}$.

A.7 EQUIVALENT FRACTIONS

Fill in the boxes with the missing number.

1. $\frac{2}{3} = \frac{?}{12}$,
2. $\frac{25}{40} = \frac{5}{?}$,
3. $\frac{7}{9} = \frac{42}{?}$,

In figure (A-6) use the diagram on the right (by shading in the appropriate number of figures) to represent an equivalent fraction to the one presented on the left.

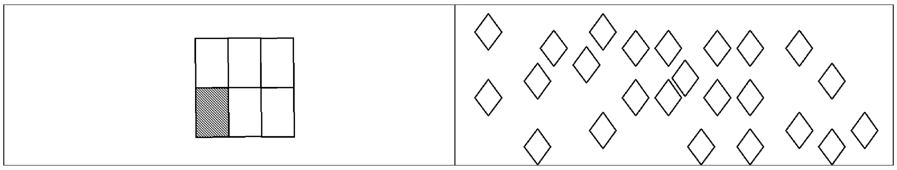


Figure A6.

O uczeniu się ułamków przez dorosłych

Streszczenie

Przedstawiona praca zajmuje się uczeniem się ułamków przez dorosłych, uczestniczących w wyrównawczych zajęciach z matematyki na koledżach w Bronksie, dzielnicy Nowego Jorku. Artykuł relacjonuje badanie rozwoju sieci poznawczej ułamków przy pomocy modelu wypracowanego przez Charalambos i Pitta-Pantazi (2007) i opartego na badaniach Tomasza Kiernana, a także na założeniu, że ta sieć poznawcza składa się z pięciu podstruktur: podstawowej struktury część/całość oraz podstruktur stosunku, ilorazu, operatora i miary. Zastosowanie tych badań w przypadku dorosłych dostarcza ciekawej wiedzy porównawczej na temat sieci poznawczej dzieci i dorosłych.

Udało się pokazać, że model Kiernana opisuje w miarę dobrze właściwości sieci poznawczej dorosłych, ponieważ otrzymano dobre korelacje pomiędzy strukturą część/całość oraz trzema podstrukturami. Ważnym wyjątkiem była podstruktura stosunku, która nie dała dobrej korelacji z pozostałymi. Okazało się, że chociaż dorośli zaangażowali się równie intensywnie w myślenie o stosunku i operatorze jak dzieci, to nie byli oni w stanie zasymilować tej wiedzy w podobnym stopniu efektywności.

To, że dorośli zrozumieli stosunek jako strukturę niezależną od struktury część/całość, daje dużo do myślenia. Stosunek liczb jest najczęściej rozumiany jako ich porównanie, a nie jako liczba, podczas kiedy struktura część/całość oznacza branie ułamka z całości. Brak elastycznego myślenia i rozumienia delikatnego związku pomiędzy ułamkiem jako wielkością liczbową odpowiadającą liczbie części wziętych z podzielonej całości oraz stosunkiem jako porównaniem liczb często ujawnia się w powszechnym błędzie. Kiedy uczniowie dodają $\frac{1}{3}+\frac{2}{5}$ często acz nieświadomie traktują to jako dodawanie stosunków i otrzymują $\frac{3}{8}$. To, że dorośli uczniowie widzą stosunek jako osobną strukturę względem siatki poznawczej ułamków, sugeruje zmiany w programie nauczania. Być może, stosunek powinien być uczony w szerszym wymiarze, włączającym działania na stosunkach i jednocześnie podkreślającym subtelny związek ze strukturą część/całość.