ANNALES OF THE POLISH MATHEMATICAL SOCIETY
5TH SERIES: DIDACTICA MATHEMATICAE 31 (2008)

DOCTORAL THESES

Ewa Swoboda
Institute of Mathematics
University of Rzeszów

Space, geometrical regularities and shapes in children's learning and teaching[1]

1 Introduction

The message of my book is to reveal that geometrical activities for 4-7 year-old children stimulate their mental development. They help children not only to understand some specific geometrical concepts, but also to develop their mathematical thinking in general.

In recent years one may observe the increase of interest in creating a mathematical way of thinking among young children. Some research results confirm that many children have a well-developed, spontaneous and intuitive mathematical competence before their school education. Research in this field puts a great emphasis on early numeracy and competence in counting, however a very limited number of research is devoted to an early geometry. My previous research results gave me the base for the assumption that it is possible to introduce very little children to a geometrical way of thinking. In this approach the starting point is not the work with geometrical figures, but the work with geometrical regularities. In various aspects of children's activities the role of regularities in intellectual development was investigated: in skills of using language, in children's literature, art, music. Many curriculums and teaching

[1]This text is the modified version of the paper presented on the 22th of May 2008 in Charles University in Prague, Department of Mathematics and Mathematical Education, during defense of habilitation. Proceedings were conducted by prof. Milan Hejný from Charles University. Reviews were made by doc. RNDr. Nadá Stelikóvá Ph.D, from Charles University, doc. RNDr. Alena Hospesová Ph.D, from University of South Bohemia České Budějovice and prof. dr hab. Zbigniew Semadeni from University of Warsaw (Poland).

programmes have suggested the view on possibilities of learning geometry via regularities, but there was a lack of research in this domain.

2 Theoretical background

Some theories stress the fact that geometrical knowledge and understanding are created in a specific way. In those theories the priority is given to perception, although geometrical "seeing" is not identified with the literal meaning of that word. Geometrical world cannot be perceived literally. It is hidden in the real world, and it is emerging from the surroundings through the special intellectual activity which can be called "the geometrical insight" (Hejný, M. 1993, Vopěnka, P. 1989). At the beginning, there is no geometrical world nor geometrical object in a child's mind. Only objects from the real world exist. But we focus our attention on those objects in various way and sometimes we perceive "something". Vopěnka (1989, p. 19) describes such a situation in the following way: *To see "this", means to focus attention on "this", to distinguish "this" from the whole rest.* This, what can absorb the whole attention on itself, we call "phenomenon". Perceiving "something" creates the first understanding. For example, a child can focus his or her attention on the shape of an object or on a specific position of one object in relation to another. *Phenomena* open the geometrical world to a child. In spite of the fact that our attention is attracted by these phenomena, this first understanding is passive: stimulus comes from the phenomenon. In this depiction, the role of perception is important – the perception of "something" is the first step to the creation of the child's own geometrical world.

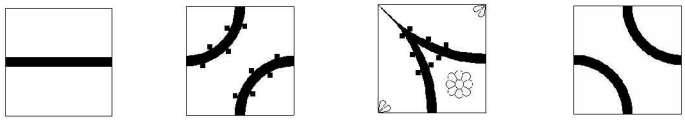
M. Hejný has transformed P. Vopěnka's philosophical depiction. He relied upon his own experimental studies and on conclusions derived from Piaget's, Vygotski's and van Hiele's work. In Hejný's theory, the development of understanding of the geometrical world goes through various levels. On the first level there is the possibility to perceive shapes and some relations, but these are both attributes of real objects. A verbal isolation of these phenomena is also possible by talking about them and calling them. Nevertheless, words such as: *triangle, pyramid, long, high,* or skills of making comparisons like: *longer, broader,* are still words and concepts related to the real, physical world.

3 Methodology

The research was conducted in 2003-2005 and was directed by the following problems:

- What kind of thinking processes guide children's manipulative activities while filling space?
- How is it possible to utilize the research results for opening children's geometrical world?

Tiling was the basic research method. Tiles are squares with 2.5 cm sides, with imprinted one of the four designs below.



Picture 1.

The child's task was to create from these tiles as beautiful floor as possible. In this study data was collected in kindergartens and schools in the South-East part of Poland. Children worked in groups, during their regular activities. Some working sessions were videotaped. More than a thousand children participated in this research, and results of their work were analysed.

The analyses of the gathered material were based on:

- *Atomic analysis of chosen children's work and chosen videotapes*: Starting the analysis I did not put any hypotheses a priori. I did not have any ideas on possible directions of the children's work. At this stage I tried to deeply familiarize oneself with the material gathered and to recognize as many phenomena as possible which could be analyzed in detail.
- *Comparative analysis*: Directed by the previous stage I verified the preliminary observations by analysing a larger number of the children's work. At this stage I have resigned from some outlined directions of analysis, therefore some phenomena have appeared in new light.
- *Identification and characterization of dominant phenomena*: It was necessary to define and describe the distinct phenomena, which I did at this stage. Afterwards each child's work was coded in relation to the observed phenomena. The examples of work, where the phenomena were observed intensively in "a pure form", were described in detail.
- *Classification*: The phenomena were classified according to the aim of the research. The preliminary classification was connected with geometrical concepts: space, geometrical regularities, shapes. Within these issues new categories emerged, as a result of the children's ways of working.

- *Statistic elaboration of the research results*: For obtaining a holistic view of the development of the "geometrical regularities" script a quantity depiction was necessary. This depiction was supported by a quantitative analysis. It was particularly necessary for analyzing the connection between the ability of creating geometrical regularities and the success in learning in school.

4 Research results

The analysis of children's work has allowed me to describe some phenomena related to the forming intuitions of:

- geometrical space,
- geometrical regularities,
- shapes (geometrical figures).

While analysing examples from various categories, the thinking processes connected with the notion of opening the geometrical word were traced.

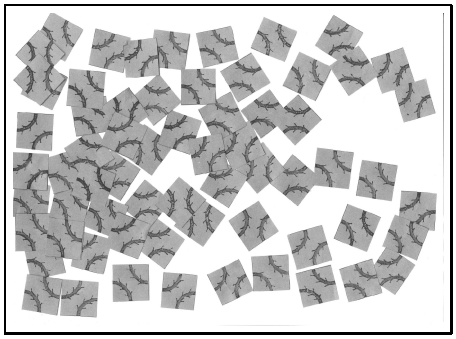
4.1 Geometrical space

Two dimensional geometrical space is unlimited. In the research described, the space was reduced to the limited area of the sheet of paper where a child was obligated to carry out the task. But the way children utilized the paper gave answers to the research question: How did the different age groups of children fill the space represented by a sheet of paper?

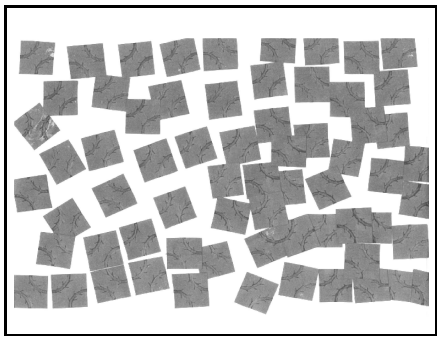
- Are the ways of filling the space capable of being ordered in certain categories?
- Can regular filling stimulate the process of going beyond closed areas towards infinity?

The analyses of children's work have led to distinguishing two different attitudes concerning the use of a sheet of paper as a representative of geometrical space. The first attitude – manifesting the limitation of a sheet; the second one – going into unlimited space. In the frame of the first attitude I have distinguished three categories, namely: "a full sheet of paper", "a frame", and "a picture". As the opposite attitude I have diagnosed the situations where a child demonstrates the possibilities of exceeding the limited area. I have named them "all directions", "stripes", "rug". Examples below illustrate those categories:

- category "a full sheet of paper"



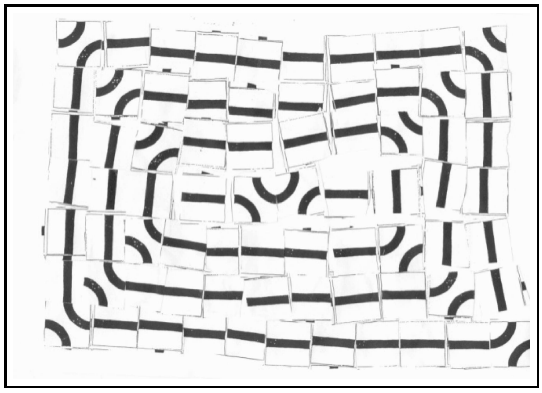
Picture 2. *Boy Filip, 4 years old. Stages of work: 1. he starts from the left side; while filling the space, he tries to connect lines, 2. rigorous filling only starts to be the domination strategy, 3. tiredness, 4. the last work phase consists of the chaotic filling.*



Picture 3. *Girl, Sandra, 4 years old. Stages of work: 1. at the beginning she keeps the rule: rhythmical filling of the paper according to the upper line of edge; during the work she ignores connections of lines, 2. later on, she keeps the same orientation of design and the continuation of lines locally.*

In both works it is possible to see that one of the rules which children try to realize is the rule of filling the paper. However, during the work, Sandra's attention is directed at some geometrical phenomena like creation of shapes or interdependence of lines.

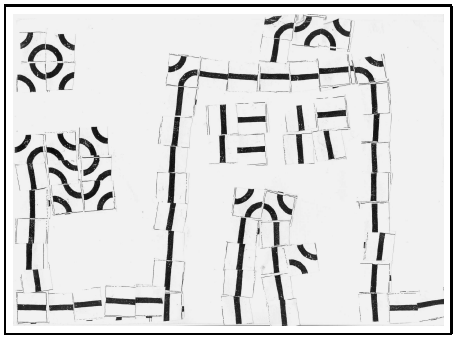
- Category "frame"



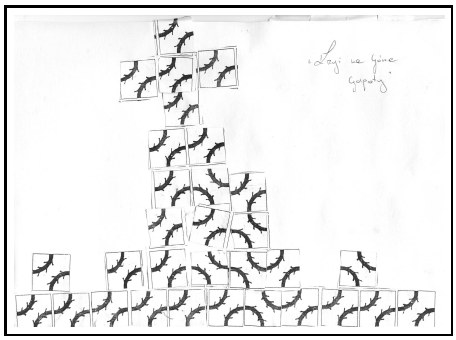
Picture 4. *Boy Wojtek, 6 years old. Phenomena which guide his work: 1. Continuity of the straight lines: vertical and horizontal ones, repetition of the shape indicated by the edges of paper. 2. filling the interior of the closed area. 3.* **D***iscovering: arches, placed askewly create a new rhythmical motif which enriches the whole work.*

This category is related to those pieces of work where a child clearly focuses his or her attention on the interior of a paper but not on the space of a paper. A child presupposed that the place of work is limited. In spite of such restriction, it is visible that working on some new geometrical phenomena attracted the child's attention.

- Category "pictures"



Picture 5. *Boy, Kamil, 6 years old. The content of this picture is clear: a house with windows and with a door, smoke from a chimney, a tree and a sun above.*

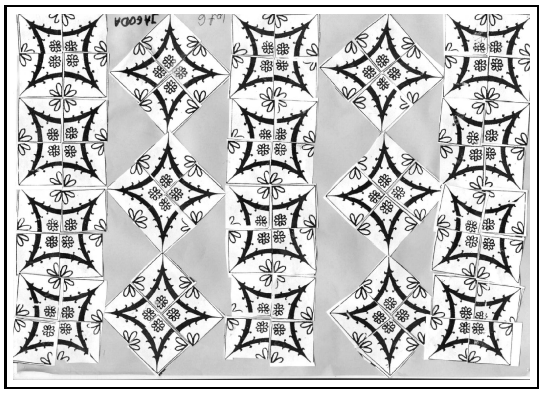


Picture 6. *Boy, Kubuś, 5 years old. "The cross at Golgotha" is placed at a pedestal with regular ornamental design. At the next level, tiles are settled symmetrically, but line continuation is kept everywhere where it was possible.*

Here we can see pieces of work where a child used lines printed on tiles as the lines drawn by crayons, or where a child used tiles like blocks, ignoring the printed lines. In the first case, a child tried to represent "something" : a house, a flower, a computer. In the second case a child built any thematic object. In both cases, the sheet of paper was used as a background for a figure. Also in this category many of pieces were subordinated to geometrical phenomena: continuality of line, symmetry.

- Category "stripes"

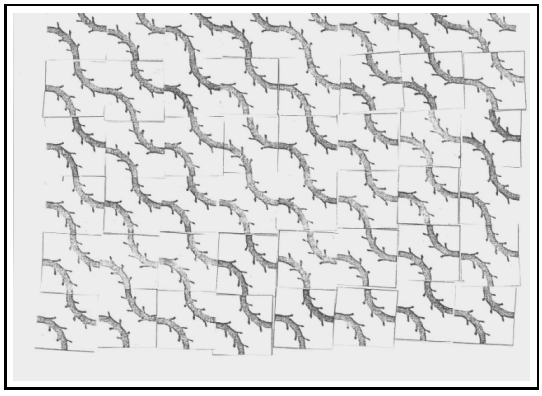
A design is created on a certain width and it is continued as stripes on a sheet of paper. One stripe contains many identical motifs and these motifs can be repeated many times. Below the first stripe, the second one (the third..) is placed. Stripes could be various or could be created in any relations. Sometimes designs appear in an alternate way and then the work has a very clear structure.



Picture 7. *Girl Jagoda, 6 years old. Phenomena: 1. Work is submitted to esthetic feelings. 2. Those feelings cause the creation of a very clear structure of the work. 3. Because of intuitions, the structure is built of various types of symmetries: – translation – axis symmetry – rotation on* 90° *i* 45°*. 4. Tendency of exceeding the area of a sheet of paper.*

- Category "full sheet of paper"

The most important feature of those pieces of work is a very restricted abidance of the created rule and a very consistent filling of the sheet of paper. In the child's work, all the most important properties of the applied regularity were coded. Also in this case, it is possible to see that the awareness and understanding of those properties were established through manipulations.



Picture 8. *Girl Małgosia, 7 years old. Stages of work: 1. She creates the first row rhythmically (manipulation). 2. She connects the second row with the first one. (+ strategy) 3. She connects the third row with the second one. (+ strategy) 4.* **D***iscovering: the oblique stripe. Everything is autonomic.*

One may risk a statement here, that there is a close relation between having intuition of unlimited space and the ability to create regularities. Perceiving the continued regularity causes a child to turn its attention to the fact that the working place can be extended. Such relation is supported by the percentage of children who create 'unlimited' surfaces composed of tiles: the breakdown shows that this rate is constantly increasing in older age groups. Such functioning in the world of 'regularities' is supported by the children's intellectual development. While creating a regular pattern, a child is not focused on the edges of a sheet of paper; by following the regularity he/she can think about its continuity beyond the limited area.

4.2 Geometrical regularities

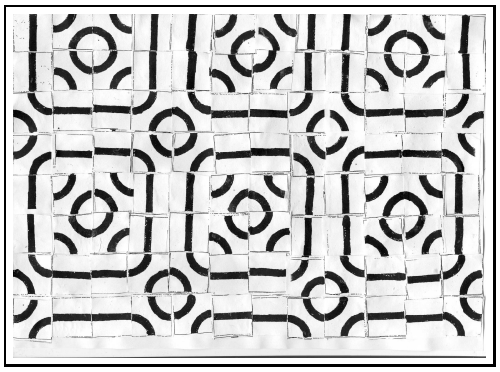
The second research question was as following: How does creating relations on a plane by children aged 4-7 take place? In details, I was looking for answers to the questions:

- Is it possible to order in certain categories the ways of organizing regularity on a plane?
- Among the regularities created by children, are there any that forecast the creation of mathematical concepts of plane transformations such as rotation, translation and mirror symmetry?

As the analysis showed, the organization of a pattern on a plane can be:

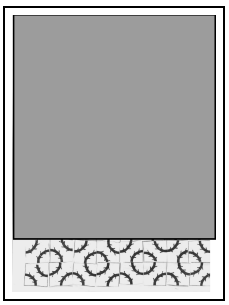
- 1. locally relational,
- 2. generally relational, very consequent throughout the whole area or with only some little distractions.

Works classified as generally-relational are the ones in which a child consequently (or only with little distractions) is guided by one idea. Generally, the arrangement of figures suggested the departure beyond the sheet of paper – by his/her work a child manifested that the presented area only signalizes the regularities.

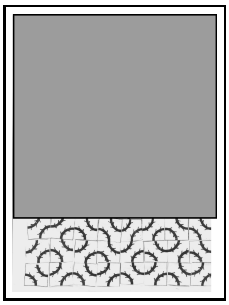


Picture 9. *Girl, 6 years old. The whole sheet of paper is very tightly filled (except the bottom where there was too little space for the whole tile). The basic motif is seen entirely only in the middle line but the infinity and the structure of the whole pattern is very clear.*

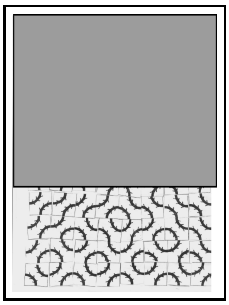
Some children were starting to work with a clear idea on how to conduct their task. But what was the most common was a situation when children were discovering an interesting, specific geometric phenomena during the work . As an example I will describe one work done by 6 years old boy, Kamil. His work was videotaped.



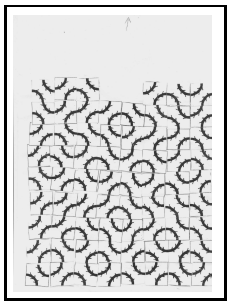
Picture 10. *Stage 1. The begining of work: rhythmical filling.*



Picture 11. *Stage 2. A change (casual?)*

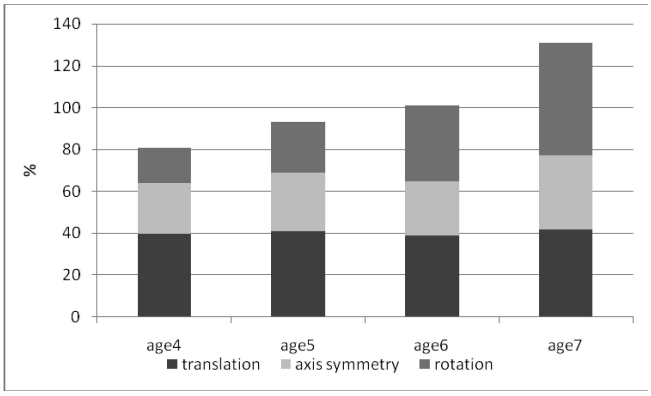


Picture 12. *Stage 3. Emergence of a new idea, a new relation.*



Picture 13. *Stage 4. Clarifying an idea of an axis symmetry.*

Among global relations there were intuitions of all well-known geometrical transformations. The most numerous group of examples represented the geometrical translation. Some 4-year old children were able to make regular patterns and this ability grows according to the age. A breakdown concerning the use of regularities by children with the division between age groups is presented below.



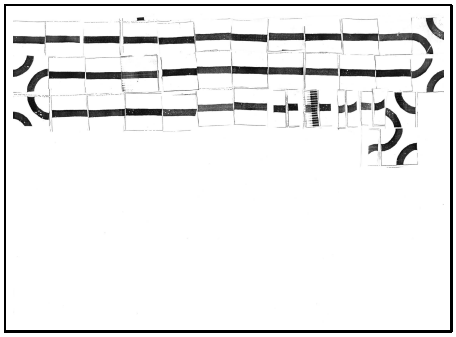
Picture 14. Intuition of geometrical transformation in children's work.

It is impossible to overestimate the fact that children spontaneously create the relations which adhere to the basic geometrical transformations. According to my research, I can assume that such relations as translation, axial symmetry and, to some extent, rotation are intuitively known to children. This knowledge is qualitatively different from the knowledge possessed by mathematicians, mainly because it functions on different rules. In the child's situation it is impossible to talk about a figure and its image but it is possible to create a mutual relation of many congruent figures which co-exist on a plane. To make certain relations legible for a child, "the rich structure" is necessary. In this structure not only two figures remain in a definite relation but a certain fragment of a space is organized according to this relation. The goal for a child: filling up the area of a paper automatically causes a shift of attention from a single figure to the relation of one figure to another. Introducing the understanding of certain geometrical relations is being created by the sense of regularity in a statically organized space. Here, a child arranges and organizes a paper, and on the basis of his reflections during the work , creates some new ideas. The arrangements very often led them beyond the paper's plane.

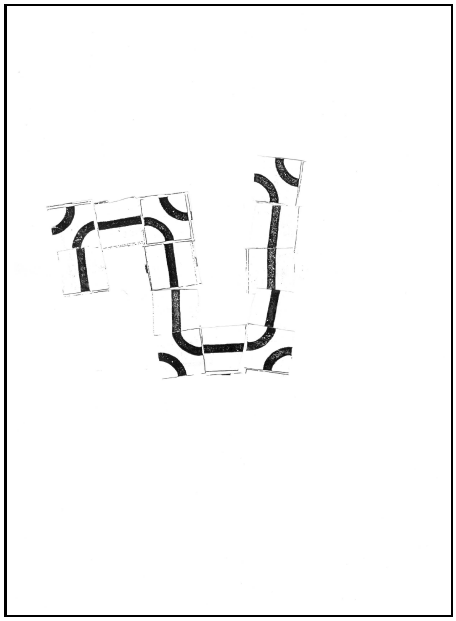
4.3 Shape as a construction functioning in an ornament

The pieces of work prepared in the frame of the diagnostic task "tiles" provide very rich material which enables us to express a new view on the creation of geometrical shapes by a child. They give a base for the recognition that children's creative skills related to building and investigating shapes are much broader than it is contemporarily accepted. Children are able not only to reproduce shapes but also to investigate them. When a child wants to create "something nice", s/he usually starts from joining two, (three, four) tiles

together. The child is guided by the will to continue the line or by his/her imagining of any real object. But these are not all possible approaches. Very often, a "casual play" with tiles leads to discoveries that open new geometrical meanings. Such experiments enrich children's geometrical intuitions significantly. For instance, one of these discoveries was the awareness that a line can be perceived as an object that has a recurrent infinity.



Picture 15. *Boy, Kamil, 4 years old. His work is very consistent. A boy used also small pieces of tiles for making a continuity of the line.*



Picture 16. *Boy, Oskar, 5 years old. He started his work from the middle of the paper. During his work he only tried to continue the line independently to the direction indicated by edges of the paper.*

Making figures from tiles is something different from creating figures through the process of abstracting shapes from real objects. When a child starts to manipulate with tiles, s/he has usually no idea of a shape which s/he wants to obtain.

5 Summary

Geometrical activities for young children cannot be oriented towards the 'ready-made' products like geometrical concepts or skills. An early childhood education could be dedicated to the gathering of experiences, which will be the

base for a conscious mental process of creating such concepts at later stages of mathematical education. In other words – it is a very important, indispensable preparatory period for 'true' mathematics, and at the same time, it is a period which can be devoted to stimulating the child's intellectual development. Vopěnka wrote: *In order to penetrate the geometrical world, we must turn attention to it*. Hejný underlines the fact that the geometrical world emerges from the real one, from the observations and actions in this world. Analyses showed that the process of creating geometrical regularities goes beyond entertainment. It is a means of focusing attention on the geometrical world. During fun activities a child can perceive various geometrical phenomena and subordinate all further actions to them. In this way, relations on a plane acquire a status of a geometrical individuality.

References

S w o b o d a, E.: 2006, *Przestrzeń, regularności geometryczne i kształty w uczeniu się i nauczaniu dzieci*, Wydawnictwo Uniwersytetu Rzeszowskiego. Rzeszów.