ANNALES OF THE POLISH MATHEMATICAL SOCIETY
5TH SERIES: DIDACTICA MATHEMATICAE 31 (2008)

DOCTORAL THESES

Mirosława Sajka
Pedagogical University of Kraków

Mathematical problems designed with the use of functional equations as theoretical tools for examining subject matter knowledge of functions[1]

1 Introduction

Assessing the mathematics teachers' Subject Matter Knowledge (SMK) is currently important problem of research. Mathematics teachers should have profound and thorough SMK, understood as skills and knowledge of mathematics, its methods and history, which are indispensable for teaching. There is a need to find appropriate tools for continuous assessment of that competence of mathematics teachers. However, literature review reveals that functional equations have not been used yet as diagnostic tools. I claim that specially designed tasks related to functional equations can very effectively reveal teachers' SMK of function. As such, this dissertation is a new contribution to the research on possibilities of using the mathematical discipline of Functional

[1]The paper is a modified version of the author's lecture presented 2nd July 2008 at the Pedagogical University of Krakow during the doctoral defense. Prof. Maciej Klakla (Pedagogical University of Krakow, Poland) was a supervisor of the PhD thesis; reviewed by: Prof. Anna Sierpinska (Concordia University, Montreal, Canada); Prof. Stanisław Midura (University of Rzeszow, Poland) and Prof. Bogdan Jan Nowecki (Pedagogical University of Krakow, Poland).

Equations (39BXX AMS Subject Classification) in the field of Didactics of Mathematics.

2 Stages of the research

The purpose of this research was to establish a new group of theoretical tools, designed with the use of functional equations, for diagnosis mathematics teachers' SMK of functions and to verify the usefulness of these tools in the course of empirical research.

The research problem was explored from theoretical and empirical perspectives, enumerated bellow chronologically (A-D).

A. Initial theoretical perspective

In this stage of the research the three areas of problems from the existing literature were analyzed:

- teachers' training and assessing their SMK of chosen topics,
- understanding the notion of function and issues connected with the teaching and learning process,
- advantages of functional equations for the research in didactics of mathematics.

The analysis of the existing literature allowed me to categorize fields where functional equations can be used in didactics of mathematics (see p. 3) and to formulate a pilot research hypothesis (see p. 2. B and p. 4).

B. Pilot research

The pilot study was performed to precisely identify the research problem and verify the pilot hypothesis being:

(PH): Functional equations can serve as theoretical tools to examine chosen aspects of understanding the concept of function by high school students.

The participants of the research were typically skilled high school students. They solved one problem based on the Cauchy's functional equation either in writing or undertook the problem resolution under the researcher individual observation. The analysis of the written answers and discussions led to very interesting results (see: Sajka, 2003).

The pilot research showed that there is a need to perform the investigation among mathematics teachers and led to formulation the main research hypothesis.

C. Modification of Even's framework

Even (1990) formulated theoretical background, widely cited by researchers, describing teachers' SMK of a chosen mathematical concept. Theoretical investigation however led me to the conclusion that the background needs variety of changes and resulted in the formulation of a modification of Even's framework. The modification was afterwards used as the theoretical background of the main empirical phase of the research.

D. Main phase of the research

I investigated the new diagnostic tools by complex verification of the main research hypothesis being:

(RH): Problems related to functional equations can serve as new, multi-sided and effective theoretical tools to examine the quality of pre-service mathematics teachers' SMK of function (SMKf).

The diagnostic theoretical tools could be "*multi-sided*" if they simultaneously reveal several elements of SMKf, even more so if they are able to disclose both positive and negative aspects of the SMKf elements. The "*effectiveness*" translates to providing valid, coherent, authentic and essential diagnosis, consistent with results obtained on the basis of other tools. The *„quality"* of the SMKf is determined by positive and negative aspects of the elements of SMKf, as defined in my theoretical framework (p. 2. C and p. 4).

The participants of research were graduate students of mathematics in Pedagogical University of Kraków (Poland) who have completed three-year course preparing them to teach in primary and secondary schools and attended training courses for pre-service mathematics teachers preparing to teach in high schools.

3 Typology of using functional equations in didactics of mathematics

Existing literature on the subject suggest that there are ways of using functional equations in didactics of mathematics. I propose hereby a typology grouping tools in different application areas as shown in Figure 1. It should be noted, that groups are not independent, but influencing each other.

Functional equations are widely used in a teaching practice without sufficient analysis of their effectiveness. They are typically used in high school competitions as a diagnostic tool to reveal specific mathematical talents. My investigations on the frequency of occurrence of such problems in international competitions, the type of problems and possible reasons for including them in mathematical contents are described in (Sajka 1999, Sajka 2006d).

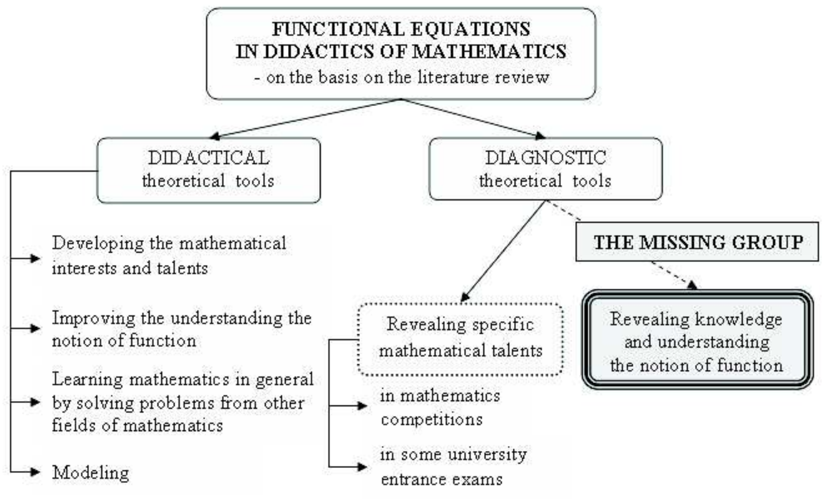

Figure 1. Typology of the use of functional equations in didactics of mathematics.

As a result of typology research, I found that functional equations have not been used yet as diagnostic tools revealing knowledge and understanding the notion of function. My research is concentrated on the group of applications which were identified as missing in the literature available on the subject.

4 Results of the pilot research

The interviews conducted with study participants were analyzed in the terms of *procept theory* formulated by Gray and Tall (1993, 1994), allowing me to describe their *procepts* of function revealed during the dialogue based on the problem being verified (see p. 2. B). An innovative method of the analysis of such a dialogue is presented in (Sajka, 2003).

The task allowed disclosure of both positive and negative aspects of understanding the concept of function and it provided a lot of information about *how* it is understood by a given student. The problem being verified then turned out to be a multi-sided diagnostic tool used to reveal understanding the notion of function and functional symbolism.

The pilot research hypothesis was therefore strengthened in the area relating to the possibility of functional equations serving as theoretical tools to examine chosen aspects of understanding the concept of function.

What is important is the fact that the research done with high schools students was multi-sided only when the researcher could observe and discuss the problem with a student during the individual observation. Very rarely students could deal with the task by themselves, in writing. The reason could be that for the majority of them function is not an abstract and fully shaped mathematical object. The problem being verified then turned out to be a strong diagnostic tool, required higher level of understanding the notion of function. For that reason the diagnostic research based on functional equations for high school students would not be very effective in a writing form.

Difficulties with understanding the notion of function by study participants were based on a limited *procept* of function and a misinterpretation of the functional symbolism. Three kinds of sources of the difficulties were identified:

- the intrinsic ambiguities in the mathematical notation,
- the restricted context in which some symbols occur in teaching and a limited choice of mathematical tasks at school,
- students' idiosyncratic interpretation of mathematical tasks.

All of them should be taken into account by mathematics teachers preparing the process of teaching the function concept. Awareness of sources of the difficulties together with overcoming the difficulties by the teachers themselves will help them in designing successful teaching activities.

The results of pilot research helped to define the main research hypothesis and to transfer the research on mathematics teachers' level. The research group was limited to prospective mathematics teachers.

5 My modification of Even's theoretical framework

The general theoretical framework defining teachers' SMK of a chosen concept was given by Even (1990). In order to implement the framework as a theoretical background of empirical research a variety of changes, extensions and specifications had to be implemented. Modifications of the Even's background were undertaken by several researchers (e. g. Lloyd and Wilson, 1998; Lucus, 2006) together with me (Sajka 2005a; 2005b; 2006a; 2006e).This shows that working out that kind of framework would be helpful for an analysis of the empirical data and that its formulating still makes a challenge problem in didactics of mathematics.

I have isolated therefore six elements of teachers' SMK of a mathematical concept. I described exemplification of this with the use of the concept of function (SMKf elements in Tab.1). My modification takes into account fundamental results of marked researchers (see references in Tab. 1) and distinguishes six general elements of teachers' SMK related to the concept of function. They are given in Tab.1 and their structure is listed below.

	SMKf	References
1	**The essence of function**	Bergeron&Herscovics, 1982; Even, 1990; Freudenthal, 1983; Dyrszlag, 1978; Sfard, 1991; Semadeni, 2002a, 2002b.
2	**Different representations and languages related to function**	Even, 1990; Klakla, 2003b, Semadeni, 2002a, 2002b; Sierpinska, 1992.
3	**Basic set of function's meanings**	Even, 1990; Dyrszlag, 1978.
4	**Analyzing function's meanings**	Even, 1990; Dyrszlag, 1978; Konior, 2002.
5	**The strength of function**	Even, 1990; Freudenthal, 1983; Krygowska, 1977; Skemp, 1971.
6	**Mathematical culture**	Even, 1990; Krygowska, 1977; Krygowska, 1986; Klakla, 2002a; 2002b; 2003a; Konior, 1993.

Table 1. Modified elements of SMK related to the concept of function.

SMKf-1. On the essence of function

(a) to have a general knowledge of the origin and historical development of the concept of function

(b) to be familiar with modern definitions of the concept

(c) to know *essential features* and understand the notion of function (as a fully shaped object and not only as a process, on the defining levels and levels of abstraction and formalization)

(d) to possess the *deep idea* of the concept of function

SMKf-2. On the different representations and languages related to function

(a) to know different function representations and its *surface forms*

(b) to understand connections between representations; to interchange representations flexibly

(c) to choose appropriate representation depending on the context and needs, remembering that representations are secondary to the concept of function

(d) to know different "languages" used for particular categories of functions (e. g. sequences, geometrical transformations, linear functions – matrixes, and other)

(e) to use different "languages", remembering connections between them and the advantages/ disadvantages of these connections

SMKf-3. On the basic set of objects meant as functions

(a) to understand thoroughly the meaning of function determined in the curriculum

(b) to have in stock a set of objects meant as functions much wider as determined in the curriculum

SMKf-4. On analyzing objects meant as functions

(a) to use different ways of approaching functions while examining them in a given representation; to make the right choice in a given context

(b) to examine a function thoroughly

(c) to construct examples, which are or are not function objects, or, in the former case, fulfil given requirements

SMKf-5. On the strength of function in mathematics

(a) to understand and be able to use operations of composing and inverting functions which create new functions and new possibilities for the development of an analysis

(b) to side-step a problem by mapping into an image set and to solve an easier problem instead e.g. by transport of a structure

(c) to understand generalizations of the function of one variable

SMKf-6. On mathematical culture

(a) to know the elements of a mathematical method

(b) to master basic approaches and behaviours unique to mathematics i.e. mathematical mental activities (e. g.: generalizing, specifying, defining, deducing, reducing; creative activities as transferring methods and knowing the role of examples and counter-examples)

(c) to master approaches and mental activities, which can be developed through mathematics and used in every day life situations outside the mathematical context (e.g. *discipline and critical thinking*)

(d) to possess the ability of self-observation of teachers' mental activity

The aspects of mathematical culture enumerated above are considered here only in the context of solving the problems related to functions.

6 Empirical verification of the new diagnostic tools designed with the use of functional equations

6.1 Organization and methodology of the main research

The main research was carried out in the two parts (Fig. 2):

- *Verificative – Referential*,
- *Verificative – Qualitative*.

As described below, this two research ways are mutually dependent.

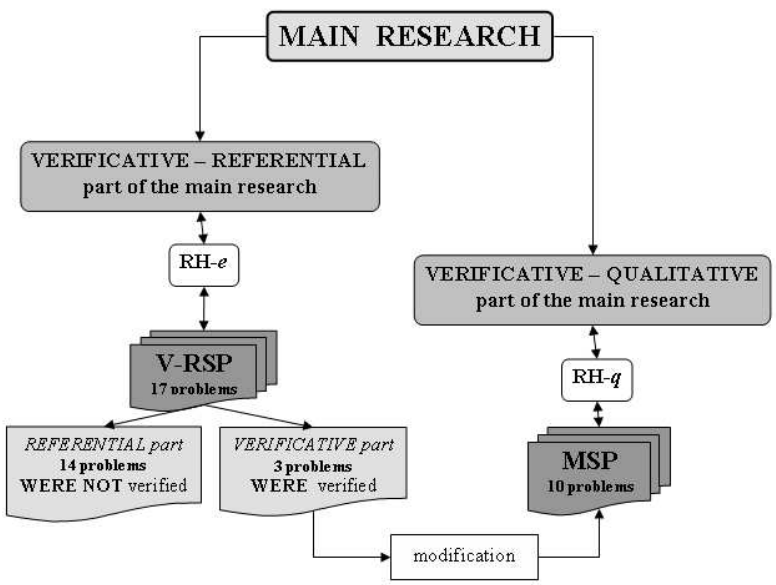

Figure 2. Organization of the main research.

6.1.1 *Verificative – Referential* part of the main research

The aim of the *Verificative – Referential* part was to verify the auxiliary research hypothesis RH-e concerning the *effectiveness* of the tools.

(RH-e): Problems related to functional equations can serve as effective theoretical tools for revealing the quality of prospective mathematics teachers' subject matter knowledge of function.

To satisfy this part of the research, the *Verificative – Referential Set of Problems* (*V-RSP*) was designed and divided in two parts.

- The *Verificative* part involved three tools being investigated.
- The *Referential* part contained fourteen problems including tools already verified by other researchers (Even, 1990; Vinner, 1983) as well as new problems designed specifically for the purpose of this research.

Comparison of the results from each of the two parts provided means to determine validity of the SMK assessment obtained with the use of the new tools and resulting modifications to the problems were implemented to the *Verificative – Qualitative* part of the research.

6.1.2 *Verificative – Qualitative* part of the main research

The *Verificative – Qualitative* part (Fig. 3) of the study was focused on *quality* of the SMK as revealed by the new tools. The diagnostic potential of these tasks and the kind of information they deliver was used to set up the second auxiliary research hypothesis RH-q being:

(RH-q): Problems related to functional equations can reveal the quality of prospective mathematics teachers' subject matter knowledge of function in the multi-sided way.

Main Set of Problems (*MSP*) containing ten tasks designed with the use of functional equations was then created to provide verification of the RH-q.

The basic stage of research (Fig. 3) was designed it such a way that all study participants were asked to attempt a written solution of one MSP tasks per day. Just after solving the task they had to complete a survey, with questions about the difficulty, usefulness, clearness of the task they worked on. In the next step selected participants took part in an interview, which was followed by another questionnaire designed in order to check some conclusions.

In a supplementary stage (Fig. 3) the main research problem was looked at two different methodological perspectives. The first was a case study based on an individual observation of a person while solving the problems from MSP. The second, utilized analysis of the written answers of 101 participants who attempted to solve the MSP(2) problem.

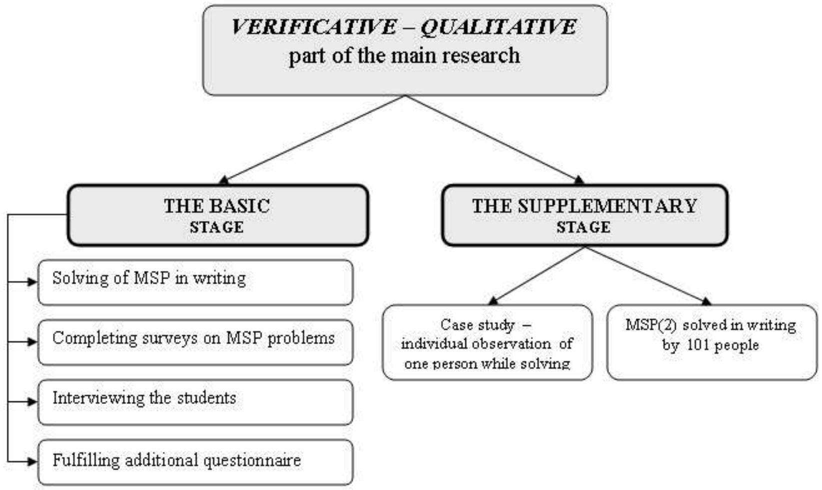

Figure 3. Two stages of the *Verificative – Qualitative* part of the research.

6.1.3 Examples of the tools and their modification

The first three problems MSP(1)-(3) are presented to illustrate the types of diagnostic tools. The following sample demonstrates the nature of modifications applied to the problems. Its primary formulation V-RSP(1), designed specially for the purpose of *Verificative – Referential* part of the main research, stated:

V-RSP(1)

A function $f\colon \mathbf{R} \to \mathbf{R}$ is such that holds the requirement $f(x+1)-f(x+3)=0$ for any arguments.

Is it possible to give an example of a function that fulfils this requirement and, in addition:

(a) is linear (justify your answer)?
(b) is not linear but is continuous in the domain (justify your answer)?
(c) is discontinuous in the domain (justify your answer)?
(d) what can you say about all the functions fulfilling this requirement?

Changes aimed at improving this tool were made in terms of editing and substance. As a result V-RSP(1) was developed which consisted of two sections. The first section was expanded to include an additional question which turned into a sub-item (a) of the problem, and the second section, now complemented by addition of questions (f), (g), (h).

MSP(1)

It is known that there are functions $f\colon \mathbf{R} \to \mathbf{R}$ that hold the requirement

$$f(x+1) - f(x+3) = 0$$

for any $x \in \mathbf{R}$.

(a) Give an example of such a function. Justify your answer.
(b) Is there a function which fulfils this requirement and, in addition, is linear (justify your answer)?
(c) Is there a function which fulfils this requirement and, in addition, is not linear but is continuous in the domain (justify your answer)?
(d) Is there a function which fulfils this requirement and, in addition, is discontinuous in the domain (justify your answer)?
(e) What can you say about all the functions fulfilling this requirement?
Consider the class of all functions $f\colon \mathbf{Z} \to \mathbf{R}$ that hold the requirement

$$f(x+1) - f(x+3) = 0$$

for any whole arguments.
(f) Give an example of a function from this class.
(g) Is there a continuous function in this class (justify your answer)?
(h) What can you say about all the functions from this class?

MSP(2)

(a) Draw the graph of the function $h\colon \mathbf{R} \to \mathbf{R}$ knowing that $h(x) = |x|$ for $x \in [-1, 2)$ and for any $x \in \mathbf{R}$ the following requirement $h(x) = h(x-3)$ is fulfilled.
(b) What properties does the function h have? Justify your answer.

MSP(3)

Give an example of at least three functions f such that for any real numbers x, y in the domain of f the following equation holds: $f(x+y) = f(x) \cdot f(y)$. Justify your answer.

6.2 Results of the main research

6.2.1 Results of the *Verificative* – *Referential* part of the research

Outcomes from the *Verificative* and *Referential* parts of *V-RSP* were compared on the basis of marks given for correctness of participants' answers. They have been analyzed according to the following two criterions: a) comparison of general marks, and b) comparison of conformity of the marks.

Presented below are general conclusions based both on the general analysis of the results and the detailed qualitative analysis of chosen representative answers.

1. The outcomes of *Verificative* part stayed in full agreement with the results of *Referential* part of V-RSP. None of the diagnosis obtained on the basis of the *Verificative* part of V-RSP were in contradiction to the outcomes of the *Referential* part. Some of them could not to be confirmed because of the different area of diagnostic potential of the problems from the *Referential* part. Moreover surprisingly similar outcomes were typical, especially of high marks answers.
2. The average easiness of the problems from *Verificative* part was 0.4 whereas of the *Referential* part was 0.6.
3. The tools being verified proved to be very strong tools to detect participants with poor SMKf. All participants acquiring very weak outcomes from the *Verificative* part of V-RSP (e. g. four people obtained 0%) belonged to the group that showed the weakest general outcomes (below 40%). They lacked many aspects of SMKf and revealed significant difficulties in understanding the notion of function and interpreting functional symbolism.
4. The tools being verified proved to be very strong tools to detect participants with strong SMKf. All participants having high outcomes from the *Verificative* part of V-RSP achieved also very high results from the *Referential* part (89% – 94%).
5. The tools being verified proved to be tools of a significant *differentiating power* which means that they were faultlessly solved mainly by participants achieving higher general marks.
6. The tools being verified proved to be multi-sided tools, providing many credible and essential both positive and negative information on participants' SMKf. They could simultaneously reveal many aspects of SMKf. Moreover, even answers of students having very weak outcomes provided not only negative but also positive aspects of their SMKf. For instance, answer of a student who acquired the percentage outcome of 0% from the *Verificative* part of V-RSP disclosed her idiosyncratic interpretation of the symbol $f(x+1)$ as a function with the formula $f(x) = x + 1$. Applying a standard diagnostic process, this probably would not have been disclosed. *Verificative* part of V-RSP was able to disclose some

crucial and hidden till that moment students' misinterpretations and misunderstanding the notion of function.

7. Analysis of the answers with high percentage conformity of the marks from the two parts proved that the *Verificative* part provided many significant pieces of information about the participants' SMKf. Some of them were not able to be revealed by the *Referential* part. For example analysis of the answer of the best percentage conformity proved that if we take into account only the diagnosis obtained on the basis of problems constructed by Vinner (1983), the conclusions concerning this person's image of function would be false whereas the tasks being verified provided accurately diagnosis. The evident effectiveness of diagnostic tools (true RH-e) confirmed directly the main research hypothesis (RH). The elements of SMKf revealed by the problems being verified were authentic and essential as well.
8. In general, *Referential* part of V-RSP exhibited weaker diagnostic potential of SMKf than the *Verificative* one. *Verificative* part turned out to be more advantageous comparing the *Referential* one. That conclusion was confirmed by all the results, moreover, it was also well visible in the case of people whose outcome in *Verificative* part of V-RSP deviate mostly from the percentage results obtained in *Referential* part. Their answers to the *Verificative* part revealed many-sided difficulties in interpretation of functional symbolism. These not yet overcome difficulties might remain hidden even by solving other problems, e.g. from *Referential* part of V-RSP.

Therefore my research proved the truthfulness of RH-e. The new diagnostic tools were very effective in providing authentic information about SMKf. They exhibit high differentiating power and reveal simultaneously many aspects of SMKf.

6.2.2 Results of the basic stage of *Verificative – Qualitative* part of the research

This part of research was preceded by formulation of assumptions on diagnostic possibilities of investigated tools. Then *Main Set of Problems* (MSP) was given to a group of ten students. The obtained solutions disclosed both positive and negative aspects of SMKf elements (see: fig. 4).

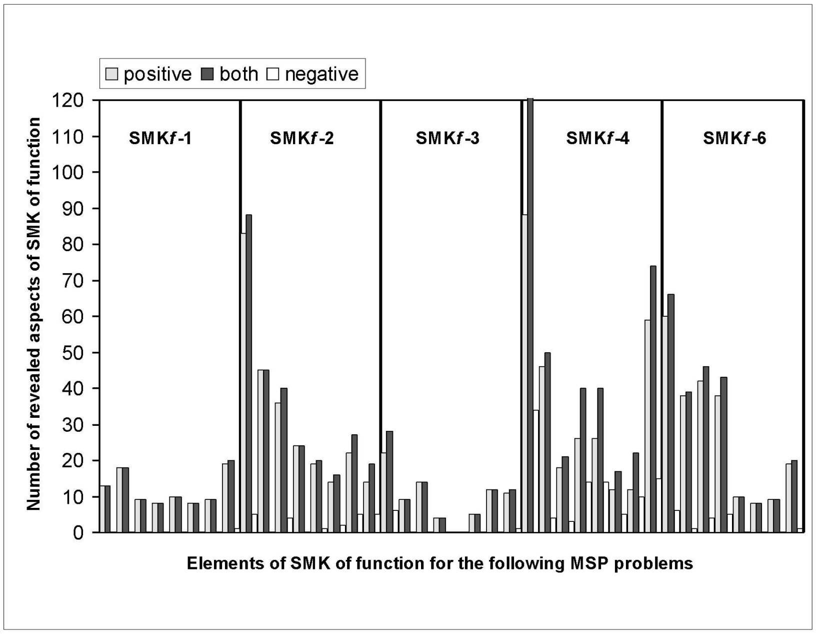

Figure 4. Number of aspects of SMKf elements gained from MSP solutions.

The empirical results were very promising in comparison to the assumed *a priory* ones. Tab. 2 presents the diagnostic possibilities of MSP components based on this part of research.

MSP	**SMK*f*-i**					
	1	**2**	**3**	**4**	**5**	**6**
MSP 1	X	X	X	X		X
MSP 2	X	X	X	X		X
MSP 3	X	X	X	X		X
MSP 4	X	X	X	X		X
MSP 5	X	X		X		X
MSP 6	X	X	X	X		X
MSP 7	X	X	X	X		X
MSP 9	X	X	X	X		X
MSP 10	X	X	X	X		X

Table 2. Diagnostic possibilities *a posteriori* of the tasks from MSP.

Majority of the problems from MSP allowed us to reveal all aspects of SMKf elements with exception of SMKf-5. Disclosing the SMKf-5 (strength of function) was not attempted in this research because of its nature. A complex problem would be required in order to reveal this element, with the level of difficulty too high to be used as a tool for diagnosis SMKf in written form.

The symptoms of SMK$f-6$ (mathematical culture) were revealed most frequently and in many different ways. Detailed analyses of examples was presented in dissertation (Sajka, 2008) .

In the last part of the basic stage of research, students involved in solving the MSP were asked to fill out a questionnaire and provide a personal opinion about the problems included in MSP. They found tasks reasonably difficult, but at the same time interesting ones. In general, students found themselves deficient in this area of knowledge and lacking experience. They appreciated however, the repetition of function problems leading to a better understanding of the subject being explored.

6.2.3 Results of the supplementary stage of *Verificative – Qualitative* part of the research

Solving MSP(2) by a group of a hundred and one students

The typology of interpreting the functional equation given in the MSP(2) was presented as part of the CIEAEM 58 Conference (Sajka, 2006b). The written answers delivered various data on participants' SMKf, especially on their ability in interpreting the functional symbolism (SMK$f-2$).

Individual observation

Extending the methodology by including the individual observation delivered very accurate diagnosis and interesting data about the revealed elements of student's SMKf. The analysis of an excerpt from the case study is described in Sajka (2007). One of the diagnosis of the prospective teacher's SMKf-2 (representations and languages related to function) is provided here to illustrate the importance of the diagnosis obtained during the observation:

Diagnosis: According to the participant, it is x and not $x+1$ and $x+3$ that is the "true" argument of the function f written as $f(x+1) = f(x+3)$. This interpretation was used by the study participant even though at the same time the expressions $x+1$ and $x+3$ were called and used as arguments. Students were then looking for the function fulfilling another requirement: $f(x) = f(x+1) = f(x+3)$. Until the time of solving the MSP(1), this misinterpretation was not only not recognized but also not unrealized by the study participant.

Is it worth mentioning that the students of four-year mathematics studies have been credited a number of advanced courses in higher mathematics, including for example: algebra, calculus, functional analysis, differential equations, topology and many others. The interpretation of the basic symbol $f(x+1)$ in this way was very surprising. The legitimacy of the diagnosis was repeatedly confirmed several times by the study participant while solving other problems from the MSR. The diagnosis describes a particular type of a *false conviction* (Pawlik, 2004), which is difficult to identify.

The presence of the observer while the student was attempting to solve problems related to functional equations created a circumstance, which tuned out helpful in disclosing errors. A standard diagnostic process probably would not disclose the difficulty related to SMKf-2. Thus, it could not to be ruled out that the participant would have started teaching mathematics with incorrect interpretation of functional symbolism.

7 Conclusions

Outcomes from theoretical and empirical research described in the thesis distinguished the following main groups of results:

1. Working out the new group of theoretical tools for revealing chosen aspects of subject matter knowledge of function including a set of ten problems (MSP), designed with the use functional equations.
2. Thorough verification of the diagnostic potential of these tools in the course of empirical research carried out among students of the Pedagogical University of Kraków, all of them soon-to-be teachers, showed that the tasks developed on the basis of functional equations can serve as new, *multi-sided* and *effective* theoretical tools for revealing SMK of function in prospective teachers. One such problem can simultaneously reveal at least a few elements of the SMK of functions as well as its positive and negative aspects.
3. The general theoretical background defining teachers' SMK of a given concept was outlined, exemplified and described with the use of the concept of function.
4. Implementing the theoretical background for analysis of empirical data allowed to define the language of the background, which lead to conclusion that each one of the six classified elements of SMK consists of aspects that can be revealed by positive or negative symptoms.
5. The empirical research also shows a variety of the information about prospective teachers' SMK of functions as revealed by the theoretical tools (see: Tab. 3).

SMKf-2	– understanding, interpreting and ability of using functional symbolism – sketching graphs of functions, given by requirements
SMKf-3	– knowing and understanding the basic set of objects meant as functions
SMKf-4	– knowing and understanding general properties of functions – examining function objects – constructing examples which fulfil given requirements
SMKf-6	– knowing logic and set theory – using methods of proving a theorem and invalidating a hypothesis – understanding a mathematical text – ability of self-observation

Table 3. Disclosed prospective teachers' difficulties with SMKf.

6. Typology of using functional equations in didactics of mathematics was proposed and one missing group was diagnosed.
7. This dissertation outlines an innovative way of using the mathematical discipline of Functional Equations in the field of Didactics of Mathematics and opens new possibilities for their applications.

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