ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
SERIA V: DYDAKTYKA MATEMATYKI 29 (2006)

Z WARSZTATU NAUCZYCIELA-BADACZA

Bronislaw Czarnocha
HCC, City University of New York[1]
Vrunda Prabhu
BCC, City University of New York

Teaching Experiment/NYCity Model

Introduction

This presentation outlines the basic features of Mathematics Teaching-Research methodology TR/NYCity model, which underlines the work of Polish teams of the concurrently running International Comenius 2.1 Project: Krygowska Project of Professional Development of Teacher-Researchers; Transforming Mathematics Education with the help of Teaching-Research Methodology (www. pdtr.eu).

TR/NYCity model has been created in the colleges of the Bronx to respond to the challenges of urban education for black and latino minorities of New York City. The aim of the paper is to bring forth salient features of the methodology for the benefit of the participants in the project, and for everyone else interested in the methodology. We start with the definition of the Teaching Experiment/NYCity model and explore several of its consequences in Section 1. We lead the discussion to the example of such a Teaching Experiment based on Achilles and Tortoise paradox of Zeno, which was the part of the larger study Introducing Indivisible into Calculus Instruction supported by the grant from NSF/ROLE (National Science Foundation, Research on Learning in Education)#0126141 . The discussion of Achilles and Tortoise Teaching Experiment in Section 2 allows to introduce the theoretical framework

[1]Spending 2006/2007 sabbatical in Krakow working with the Comenius 2.1 project.

of the Zone of Proximal Development (ZPD) of Vygotsky as way of understanding episodes of the investigation. The discussion of the Section 3 turns to the tool box of a Teacher-Researcher emphasizing its connection to standard didactic tools of the teacher, and with the coordination of class situation to general theories of learning. The theory of the schema as an example of a theory of learning is briefly discussed in Section 4 and the example of its coordination with classroom practice in Section 5 ends the presentation.

As a background to our discussion of the Teaching Experiment, we quote extensively from the (Steffe and Thompson, 2000) who provide more intuitive view on this methodology:

> The methodology of the teaching experiment will unavoidably continue to evolve among researcher who use it. It certainly did not emerge as a standardized methodology nor has it been standardized since. Rather, the teaching experiment is a conceptual tool, which researchers use in the organization of their activities. It is primarily an exploratory tool derived from Piaget's clinical interviews and aimed at exploring students' mathematics. Because it involves experimentation with the ways and means of influencing student mathematical knowledge, the teaching experiment is more than a clinical interview. Whereas clinical interview is aimed at understanding students' current knowledge, the teaching experiment is directed towards understanding the progress students make for an extended period of time. It is a dynamic way of operating ...

1 Definition of the Teaching Experiment and its consequences

Definition: Teaching Experiment of the Teaching-Research methodology NYCity model is the classroom investigation of teaching and/or learning process, conducted simultaneously with teaching and aimed at improvement of learning in the same classroom, and beyond.

The Teaching Experiment is conducted through the cyclical methodology of the design of instructional intervention, its classroom implementation, collection of the relevant data, their analysis and re-design of the intervention on that basis. The requirement of improvement of learning imposes certain constraints on the nature of research questions that could be asked. The classroom investigation must not only investigate the state of affairs relevant to some mathematical situation, but also it must investigate possible routes of the improvement of that state of affairs; consequently Teaching-Research questions have two corresponding components. The definition of the Teaching Experiment is sufficiently general to allow both a teacher and researcher to conduct a cycle of investigations. In the case of a teacher, the design of the intervention is grounded in the practice and professional intuitive knowledge of the teacher, in the case of a researcher it is based on a theory of learning or previous research results. The aim of the TR NYCity model is to promote the development of new profile of a professional equally at home in the classroom with the daily work of the teacher as well as with that of a researcher for whom the same classroom is understood as her or his experimental field, the significance of which may be of general educational character.

Some elements of the new profile are already in place. Cobb and Steffe, show in the paper The Constructivist Researcher as Teacher and Model Builder (Cobb and Steffe,1983), how the activities of the researcher performing the teaching experiment in the classroom turn out to be identical with certain activities of a classroom teacher. (Czarnocha, 1999) discussing the discovery technique of classroom teaching experiment in the paper The Constructivist Teacher as a Researcher, shows why and how the work of a constructivist teacher unites in the classroom with the activities of the investigator of students' cognitive structures bearing on the particular instructional situation. The classroom becomes the forum where the methodology of an educational researcher can integrate, in some special instances, with the classroom instructional methodology of the teacher. At the same time, it is in the classroom context where the difference in goals and attitudes between the researcher and the teacher-researcher are clarified. According to (Cobb and Steffe,1983) the goal and the interest of the researcher during the teaching experiment in the classroom is "in hypothesizing what the child <u>might</u> learn and finding [as a teacher] ways and means of fostering that learning". In contrast, the interest and goal of the teacher-researcher is "to find ways and means to foster what the student <u>needs</u> to learn to reach a particular moment of discovery or understanding [dictated by the curriculum of the class]. Since, however, such moments occur only within students' autonomous cognitive structures, the [constructivist] teacher has to investigate these structures during a particular instructional sequence [in order to be of help to the students]. In this capacity, he or she acts as a researcher". (Czarnocha,1999)

The Teaching Experiment NYCity model is seen as the careful composition of ideas centered around Action Research with the ideas centered around the concept of the Teaching Experiment of Vygotskian school in Russia, where it "grew out of the need to study changes occurring in mental structures under the influence of instruction" (Hunting,1983).

Action Research is a methodology formulated by (Levin, 1946), where it described a systematic utilization of sociological theory for the planning, implementation and assessment of action aimed at eradication of xenophobia, that is racism, in the neighborhoods of Chicago of the thirties. It was characterized by the cyclic nature of investigation of the situation, planning of the action, its implementation followed by assessment. Several cycles of that nature were usually connected to reach the particular goal by a series of Action Research steps.

Naturally cyclic nature of the teachers' work measured yearly or semestrally, lends itself particularly easily to the utilization of Action Research methodology for improvement of instruction or resolution of problems arising in teaching.

Every instructional cycle during which we teach a particular student cohort can be a part of the cycle of the Teaching Experiment and the analysis of its results can serve as the springboard to the improvement of the intervention the next time we teach the same class. While Action Research relies on the individual investigation of teaching with the purpose to improve it in the classroom, the Teaching Experiment of Vygotsky views the same classroom as one of the sites of a large scale investigation of learning processes designed using a general theory of learning. Its purpose is to investigate „changes in mental structure" of learners, that is changes in their understanding of relevant issues under the impact of a theory-based design of instruction.

Careful integration of both methoodologies allows for the Teaching Experiment to adopt the cyclical nature of Action Research as well as its aim of classroom improvement. As a result we have created a flexible tool of classroom investigations, conducted simultaneously with teaching and aimed at improvement of learning in the same classroom, and beyond.

<u>Cycles and Ethics of the Teaching Experiment.</u>

The Teaching Experiment (TE)is governed by its cycle of design, instruction, data collection and analysis, re-design. The length of such cycle, as well as the question how many TE cycles should there be within one intsructional cycle of a year or a semester is governed by the following considerations:

(How People Learn, 1999) informs about the degree to which the Teaching Research methodology can be very useful in speeding up the progress of reform in mathematics education by shortening the time lag in the flow of information between research and practice. Consequently, one of the principles impacting the decision about the length of TE cycle, is

1. minimalization of the time lag between research and practice;

   One of the few texts, which appeared recently in Poland concerning TR methodology emphasized that in any classroom investigative undertaking, the primary responsibility of the teacher is to the students of the classroom and to their optimal mathematical development (Pawlowski, 2001). One of the strong elements of such a commitment is the ethical principle which asserts that

2. those students who were the subject of the Teaching Experiment leading to the increase of knowledge about their learning, should be the first beneficiares of the new understanding.

The ethics of the classroom teaching experiment formulates strong conditions upon the length of the Teaching-Research Experiment, which require classroom ingenuity of the Teacher-Researcher to satisfy. The Teaching Experiments need to be so situated within the regular cycles of work so that the students who were the subject of investigation are also its immediate beneficiaries. In simple cases, one can state as a guide a minimum of two cycles per unit of the classroom instruction, a semester or a year. Two cycles assure the refinement of instruction based on the qualitative or quantitative analysis, and hence its improvement through incorporation of results of investigation.However one can bypass this simple condition in special circumstances, which still allow for the satisfaction of the ethical principle. For example, if a teacher teaches the same cohort of students the next unit cycle in the school then he/she has the opprtunity to introduce the results of research after one unit and still fullfill the requirement. Or as a teacher, TR apprentice in one of the TR teams of the Socrates Project describes her way of dealng with the problem of the control group through the parallel class taught by the same teacher. The ethical problem teachers encounter here is that in reality it is impossible to have two parallel classes and to implement new instruction one believes in, in only one of them and not both. In other words she doesn't want the control class, whose students are used as object of comparative assessment, not to receive the benefit of the Teaching Experiment conducted in the parallel experimental class. She had divided the teaching experiment into parts within the year and having received the confirmation or rejection of her hypothesis in one class in a given part of the curriuculum, she immediately was introducing the improving technique to the other class, but only for that part. This way she was able to assess the effectiveness of the innovative instruction in its components, and at the same time satisfy the ethical principle. Similar principle was used in the example of Achilles and Tortoise of the next section, where the described sequence of assignments constitutes 3 cycles of the teaching experiment conducted within a semester.

Background for the Teaching Experiment.

For the Teacher-Researchers, investigators of the NSF/ROLE grant Introducing Indivisible into Calculus Instruction, the central question was how to utilize the depth of the famous paradox for the instruction of the limit of sequences in Calculus. The limits of sequences were introduced in the early part of the semester with the help of the geometric definition which asserts that *the point* $(0, L)$ *is the limit of the converging sequence* (n, a_n) *of points, if for every pair of parallel equidistant lines from* $(0, L)$ *there is at most finite number of terms outside of these lines.*

The course was led with the help of the guided inquiry (or discovery) method, which relies on the formulation of challenging open tasks leaving students enough cognitive space to make substantial discoveries and insights by themselves. To large degree the art of facilitation of student discoveries relies on the teacher's familiarity with cognitive steps and cognitive distance needed to reach them.

The question of cognitive distance between different aspects of a problem are well understood with the help of the theoretical framework of ZPD of Vygotskyi (Vygotskyi, 1987). Zone of Proximal Development is the conceptual distance between student's understanding of the concept by himself/herself, and understanding of the same facilitated by the mentor, or an instructor. In the context of classroom instruction, its outer levels are determined by the "spontaneous concepts" of students and "scientific" concepts of the instructor, and the mastery and understanding of the concepts takes place in the zone between the two. The art of the teacher-researcher lies in formulating such questions during the process of classroom facilitation, which allow the student to traverse that distance on his/her own. The sequence of writing assignments below shows 3 TR cycles, through which such effect was reached revealing at the same time the difficulties of the approach as well as the route to victory. The assignment was given during different parts of the semester and 13 students of the class had approximately a week to complete the assignment. The Teaching-Research questions for the Teaching Experiment were:

1. What is the nature, extent and scope of ZPD amongst students of Freshman Calculus, relatively to Achilles and Tortoise?
2. What could be the optimal path of hints for students to traverse their full ZPD?

The discussion below is based on the essay trail of one student.

2 Teaching Experiment

Example: Investigations of the Zone of Proximal Development of students in Freshman Calculus. Achilles and Tortoise (A&T) paradox as a converging sequence.

The first assignment, in the Spring of 2004, served as the diagnostic test for the Fall semester. Assignment 1, Spring 2004
There is a race between Achilles, the fastest of Greek warriors and the Tortoise. Achilles gives Tortoise a head start and is running after him. However, in order to pass the Tortoise, Achilles has to arrive at the point where Tortoise was before, from where however, Tortoise already departed. Hence, having gotten to that point, to pass the Tortoise, although the distance is shorter, Achilles still has to get to where Tortoise is at present. However, by the time Achilles will get there, Tortoise will have departed again. Since this situation can continue with no end, Achilles can never overtake the Tortoise.

Write a 1 page essay discussing the Achilles and Tortoise paradox. Make sure you include the following:

1. *State where is the contradiction in the story. A contradiction has at least two statements which imply opposite conclusions.*
2. *Which of the two statements would you agree with?*

The set of student responses we got contained both views with quite a few who agreed with Zeno, and their arguments show quite a few misconceptions that underline their reasoning. The misconceptions could be quite easily eliminated, teacher-researchers thought, with better formulated assignment.

S1: *One is that you are not able to do an infinite number of tasks in the finite time. This relates to Achilles because he has an infinite number of finite steps to catch up, before he can catch up with the Tortoise.*

S2: *It is obvious that if the distances get shorter each time, its only the matter of timer, however long , till the Tortoise is overtaken.*

S3: *I see that the Achilles will never catch up with the Tortoise because even though the Tortoise is slower, he is not stationed at the same location, which makes the story continue with no end.*

S4: *It is almost impossible for Achilles to get to the point where the Tortoise is and then pass him, because according to Zeno, Tortoise is stationary. In addition if Achilles is to pass the Tortoise would have to be stationary.*

It is interesting to note that the ways of understanding the A&T difficulties by students are touching upon fundamental issues. S2 talks about the distance and her intuition tells her that there should be no problem summing up the decraesing sequences of their lengths. On the other, S3 points to the changing positions of the tortoise and on the basis of this intuition suggests the impossibility of passing. Most striking is S1 who similarly as (Grunbaum, 2002) sees the paradox not in the summation of the infinite number of distances or time intervals, but in the need to perform the infinite number of actions in the finite time. This very profound point of view moves the A&T discussion beyond mathematics, indicating that the mathematical understanding of the situation using the concept of the limit, as for the first time done by St.Gregory in 17th century (Cajori, 1915), solves only the mathematical aspect of the paradox, and not its more intrinsic nature. Here, the teacher-researchers were interested in finding the best sequence of topics of the essays, which would allow students to utilize the definition of the limit of a sequence for their understanding of the situation. In the ZPD terminology, in this teaching experiment, its upper level was given in terms of the „scientific concept" of the limit and the investigations concerned the proper scaffolding to raise the lower level of „spontaneous, intuitive concepts" of students. The design of the themes of essays needed to be such that would allow the integration of both spontaneous and scientific concepts into proper understanding of the paradox:

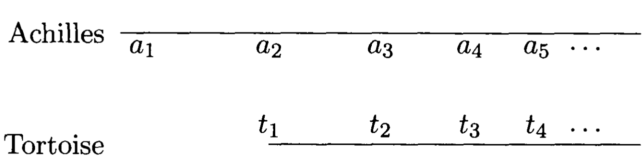
Assignment #2

Achilles and Tortoise

Fall '04

In the fifth century BC the Greek philosopher Zeno of Elea posed 4 problems, now known as Zeno paradoxes that were intended to challenge some of the ideas concerning space and time held in his day. Zeno's second paradox concerns a race between the Greek hero Achilles and a Tortoise that has been given a head start. Zeno argued, as follows below, that Achilles could never pass the Tortoise. Given that Achilles is the fastest Greek hero, Tortoise is one of the slowest animals, the speed difference between is very big, so that the conclusion that Achilles can never pass the Tortoise is contradictory with every day's experience.

Zeno argues as follows: Suppose that Achilles starts at position a_1 *and the Tortoise starts with the position* t_1 *having a head start in terms of the distance. When Achilles reaches the point* $a_2 = t_1$*, the Tortoise is already further away at position* t_2*. When Achilles reaches the point* $a_3 = t_2$*, the Tortoise is at position* t_3*. This process continues without end, and Tortoise is always ahead of Achilles. The conclusion is that Achilles can never overtake the Tortoise.*



Write a one page long essay proposing the resolution of the Achilles and Tor-

toise paradox.
A paradox is resolved if the contradiction between the conclusion as stated and the every day's experience has been eliminated.
In your essay,

1. *state the paradox as you understand it. Make sure your interpretation of the paradox agrees with all facts in the story,*
2. *propose the resolution to the paradox based on your careful rethinking of the process in which the paradox arises.*

Note the introduction of algebraic symbols in the assignment attempting to symbolize the approach to the paradox. We are investigating here levels of algebraic comprehension of the text and student ability to repond to it.

The description of the paradox is more detailed and procedural, although attached drawing indicates interpretation of the procedure resulting in the shortening of each successive distance. Below is one of the best students response:

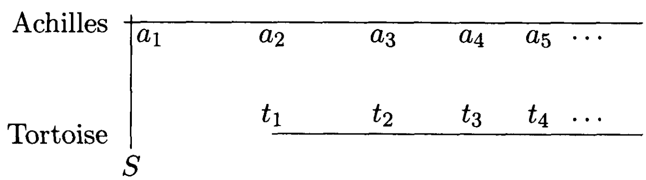
Achilles	a_1	——	a_2	——	a_3	——	a_4	——	a_5	——	a_6
		5 miles		*5 miles*		*5 miles*		*5 miles*		*5 miles*	
Tortoise			t_1	——	t_2	——	t_3	——	t_4	——	t_5

D.S.: *Let us imagine that the race is 25 miles long with a check point at every 5 miles, so that* $a_1 \to a_2 = 5$ *miles., $h_1 \to h_2 = 5$ miles, and so on. Let's also imagine Achilles runs 10 mph and the Tortoise runs 5 mph, since Achilles is faster. Knowing this, we can determine that it will take Achilles 30 minutes to run between each point ...*
Therefore, 30 minutes into the race, Achilles would be at point a_2 and the Tortoise would be between h_1 and point h_2. An hour into the race, Achilles would be at point a_3 and the Tortoise would be at point h_2. Note $a_3 = h_2$. An hour and a half into the race, Achilles would have passed the Tortoise and be at point a_4, while the Tortoise is between the point h_2 and the point h_3.
Though Zeno's conclusion that the quicker will never pass the slower (if given a head start) may have been valid during the fifth century B.C., it does not stand true to today's experiences.

This and other essays informed us about the main difficulty students encounter and that is the absence of the converging sequence in their interpretation. It was not clear whether the issue is of algebraic nature, that is whether absence of algebraic proficiency is responsible for that absence of understanding, or is it the question of conceptual understanding. On the other hand , it is clear that the student took equally spaced „check points" as the basis of analysis rather than points whose distance to the preceeding diminishes. That might indicate the problem solving issue of not understanding that these distances indeed should decrease. To distinquish between the possibilites we have assigned next version of the assignment with concrete numbers assigned to the pertinent concepts. The essays also revealed absence of clear understanding of the phrase „head start", and the assignment is refined with respect to this issue as well.

Assignment #3

You have read about the story of Achilles and the Tortoise in Essay 1. Consider the following. Both, Achilles and the Tortoise start the race at the same time, but Tortoise starts 100 m ahead of Achilles. Achilles starts at the point S, while Tortoise starts 100 m ahead of S. Let the speed of Achilles be 10 m/sec and that of the Tortoise be 1 m/sec.



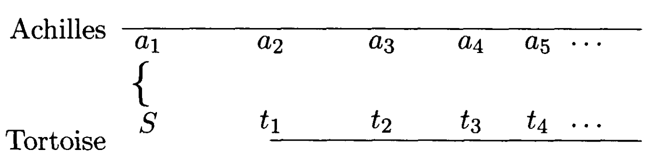
1. *Fill in the dotted lines in the following table:*

Position of Achilles	*Position of Tortoise*	*Distance of Achilles from S*	*Distance of Tortoise from S*	*Time in seconds*
a_1	t_1	....	....	....
a_2	t_2	....	....	....
a_3	t_3	....	....	....
		....	....	....

2. *Are all of Zeno's conditions satisfied in your calculations above? Explain.*

Same student's response:

Let's imagine that Achilles and the Tortoise start the race at the same time, but the Tortoise starts 100 m ahead of Achilles. Achilles starts at the point S (a_1), while the Tortoise starts 100 m ahead of S (t_1). Let's also imagine Achilles runs 10 m/sec and the Tortoise runs 1 m/sec. Knowing this, we can determine the exact position of Achilles and the Tortoise over a period of time.



Position of Achilles	*Position of Tortoise*	*Distance of Achilles from S*	*Distance of Tortoise from S*	*Time in seconds*
a_1	t_1	$0\,m$	$100\,m$	0
a_2	t_2	$100\,m$	$110\,m$	10
a_3	t_3	$200\,m$	$120\,m$	20
a_4	t_4	$300\,m$	$130\,m$	30

At a_1, we know Achilles is 0 m from S because this is his starting point. We know the Tortoise is 100 m from S at t_1, because this is his starting point. [Note: only 0 seconds have elapsed in the first row because this is the beginning of the race]. Since we know $a_2 = t_1$ (refer to first diagram), we can conclude that Achilles is 100 m from S at a_2 because the Tortoise began at t_1, which was 100 m ahead of S. If Achilles is 100 m from S at a_2, 10 seconds must have passed because he runs 10 m/sec [Solve by: 100 m / 10 m = 10 sec]. Since we know the Tortoise runs 1 m/sec, he must be 110 m from S at t_2 [100 m + (10 sec · 1 m) = 110 m].

To find the distance of Achilles from S at a_3, we must find a pattern between a_1 and a_2. <u>Since he was 0 m away from S at a_1 and 100 m away from S at a_2, we notice that it is going in increments of 100.</u> So at a_3 Achilles should be 200 m from S [100 m + 100 = 200 m]. If Achilles is 200 m from S, we know 20 seconds must have passed [200 m /10 m = 20 sec]. That means the Tortoise is 120 m from S at t_3 [100 m + (20 sec · 1 m) = 120 m]. If you were to continue the pattern, <u>30 seconds into the race, Achilles would be 300 m</u> from S at a_4 and the Tortoise would be 130 m from S at t_3. Therefore it is impossible for the Tortoise to win the race. Achilles would always win because for every 10 seconds, he increases his position by 10 m every 10 seconds.

We see that certain elements of understanding have appeared, for example an attempt at the coordination of the actions of Achilles with those of the Tortoise — one of the central mental acts in understanding the nature of the paradox. At the same time we see that student still can not push the analysis beyond uniform motion because of the fundamental error in observing the patterns of the increase of the distance without taking into account sufficient number of cases. This error reveals a good student whose way of reasoning was affected by absence of clarity on the meaning of „patterns". It maybe reinforced by the basic conception of a uniform movement of the race's participants, whose irregularity is imposed somewhat artificially by Zeno condition. How many cases are necessary to consider before one is ready to generalize? — is the question for future investigations, which should be done in the framework of strategy of hints mentioned by a mathematics teacher from NYC, Mark Saul in (Saul, 1995). What is the most important is that with all the technical improvement we still didn't have a single procedural evidence of the converging sequence of decreasing distances present in the thinking of students, our main object of investigation. Hence, we have not yet found the lower level of the ZPD of students to enable the proper integration of both levels.The 4th essay was assigned in the second half of the semester, when more mathematical tools became available to students. It is re-designed accordingly to the analysis above:

Assignment #4
November 3, 2004
Due: November 10, 2004

Suppose Achilles runs ten times as fast as the tortoise and gives him a hundred yards head start. In order to win the race Achilles must first make up for his initial handicap by running a hundred yards; but when he has done this and has reached the point where the tortoise started, the animal has had time to advance ten yards. While Achilles runs these ten yards, the tortoise gets one yard ahead; when Achilles has run this yard, the tortoise is a tenth of a yard ahead; and so on.

Graph the distance traveled by Achilles as a function of time and on the same grid, also the distance traveled by Tortoise as a function of time. Project the sequence of distances onto the vertical axis. Is this an increasing sequence of numbers, which is bounded above (Hint: is the length of the racecourse finite?) By Axiom 1, this sequence must have a smallest number greater than every term of the sequence. How will you prove that this number is the limit of the sequence?

Same student:
For example, let's imagine Achilles runs ten times as fast as the tortoise and gives him a hundred yards start. The projection of the race would look as follows:

TIME	*DISTANCE RAN IN YARDS*	
	ACHILLES	*TORTOISE*
—	0	100
t_1	100	110
t_2	110	111
t_3	111	111.1
t_4	111.1	111.11
$t5$	111.11	111.111

From the graph, we can determine that this is an increasing sequence of numbers because the distance ran at t_{n+1} is always greater than t_n. To find out *if this sequence is bounded above, we must look at Axiom 1. Axiom 1 states, "If M is a set of numbers that is bounded above, then either there is a largest number in M or there is a smallest number that is larger than every number in M." Since the length of the racecourse is infinite, there isn't a largest number. On the other hand, there is a smallest number greater than every term of the sequence, although it may not be as clear to some people as to what it is. In past problems it was easy to determine the smallest number greater than every term of the sequence [its clear that the sequence* $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, ...\}$ *is bounded above at 1]. However, the sequence in this particular problem is { ... , 111.1, 111.11, 111.111, ... }; which can also be written as { ... , 111.*$\underline{1}$*}. In situations like this we can implicitly use infinite sums.*

111 1/10 + 111 1/100 + 111 1/1000 + 111 1/10000 + ... = 111

Observe that as you add more and more terms, the partial sums become closer and closer and closer to 111 1/9. Therefore, we can say 111 1/9 is the smallest number greater than every term of the sequence (and also the limit of the sequence).

*To prove that 111 1/9 is the limit of the sequence, we can apply the geometric definition of convergence of a sequence. This definition states, "The number L is the limit of the infinite sequence a if and only if for any two equally spaced horizontal lines with the point (0, L) between them, there are at most finitely many points outside the two horizontal lines and infinitely many points between the lines." For example, if you were to draw 2 horizontal lines at 111 and 111 2/9 (both lines being 1/9 away from 111 1/9), there will be at most finitely many terms outside of the horizontal lines and infinitely many between the lines (*note: the horizontal lines can be drawn anywhere as long as it contains the limit). Hence, 111 1/9 is the limit of the sequence.*

<u>Discussion</u>

It is clear that the last assignment had a strong impact upon the particular student enabling him to make the full connection between the content of A&T paradox and the definition of the limit of a sequence. Coordination of two situations is excellent, showing all necessary connections. Except for the final interpretation of the definition which misses the universal quantifier. In terms of the inquiry technique, the teacher-researcher was able to facilitate a significant moment of understanding in student mental apparatus relative to the problem, in terms of ZPD, she was able to outline the full scope of ZPD of the student starting at the hint leading to grasping the decreasing aspect of the distances to full understanding. A significant step in the construction of understanding was also the recognition of interaction of distances and time intervals between Achilles and Tortoise. We see in the example above te process of coordination of the classroom experience with the theoretical framework of ZPD of Vygotskyi. The guide of this coordination was the very definition of ZPD as the conceptual distance between what student can do on his/her own and what she/he can grasp with the help of the intsructor or a properly chosen peer group. Having introduced during the course the appropriate definitions of the limit of a converging sequnce as „scientific" concept, teacher-researchers were searching for such a hint of intuitive nature, which would allow the student to construct the understanding reaching the level of the definition. We found out that correct hint, and hence the extent of ZPD in the case of the student whose exerpts we discussed, was the indication of the systematic decrease in the length of Achilles' steps.

3 Tools of Teaching-Research

Teaching tools as instruments of investigations. The previous example highlights the particular nature of the Teaching-Research tools. These are, to large extent, the traditional but highly refined tools of classroom teaching: a questioning attitude in the classroom can become a clinical interview of a student, checking their homework can become highly sophisticated analysis of errors and misconceptions, creating a worksheet can become a diagnostic tool. Classroom roster can become the source of statistical analysis with 2, 3 classrooms giving a sufficient sample to be able to draw some conclusions. An interesting homework problem can become the tool through which we assess student level and nature of understanding, or an exam problem can become the tool of pattern detection for important student errors leading, as in the following section, to coordination with a general theory of learning. In the case of instructional sequence designed to achieve certain mental effect, one of the central competencies is the capability to design the sequences of problems and questions, which are at the appriopriate conceptual distance from each other and from the level of familiarity of the student. If they are too high relatively to the lower lever of related ZPD, then the student can't do anything with it. If they are too low they do not provide an appropriate challenge and their learning potential is wasted. This difficulty, present daily in a class of a reflective teacher, magnifies significantly in the class of the teacher-researcher because the levels needed to be investigated are very subtly separated from each other and requires high level of professional imagination and creativity to get them right. A master of the technique was the Robert. L. Moore, a mathematician from the Chicago School, the creator of the Texan Discovery technique designed for gradutae mathematics curses. Each such course has been a series of 100-200 problems, definitions and axioms whose careful cognitive organization had allowed students to climb the tree of knowledge to large degree independently.

It is the process of such task formation that gives a special twist to the classroom of a Teacher-Researchers – they become very interesting to students as a necessary byproduct of classroom investigations. The preparation of such classrooms stimulate intellectual development of teachers and their actual implementation stimulates the intellectual development of students. One of the most characteristic moments of TR, the moments of „comradship in thinking" take place, when a child is thinking on the problem at hand while the teacher is thinking about the nature of the thoughts of the student about the same problem. It is possible that the moments of this unity of mental action between student and teacher are the proper unit of analysis for the concept of Teachng-Research NYCity model, in a similar fashion in which verbal thought of Vygotsky is the unit of analysis for the dual aspects of thought and language.

<u>Theory and a theoretical framework as the tool of a Teacher-Researcher</u>. It's proper introduction into the work of a Teacher-Researcher requires careful thought, because Teaching-Research as a development and particularization of Action Research approach to the conditions of the mathematics classroom has also, in general, inherited the methodological problems that Action Research encounters in relation to the general educational knowledge base. According to (King and Lonnquist, 1992) „ action research (teaching research, classroom research) has been unable till now to make a clear contact between the particularity of classroom research and the general educational knowledge base of the profession. For (Noffke, 1994), one of the present challenges ahead of Action Research is to clarify precisely, if and how action research can contribute to the traditional knowledge base of the profession. The absence of clear cut, unambiguous answers to this fundamental question has seriously limited the role of teaching-research as a viable research tool.

In Mathematics Education, this limitation can be now eliminated both conceptually and practically by the introduction into professional discourse of the notion of constructive or cognitive theory of learning. Since the time such theories, which apply across wide range of studied phenomena (e.g. process-object duality theories of (Asiala, 1996) or (Sfard, 1994), Anderson's cognitive psychology (Anderson, 1995) or Vygotsky theory (Vygotskyi. 1986)) appeared in the context of mathematics education, the bridge between the particular classroom situation and educational base knowledge can be much easier established. As soon as a teacher in the classroom can coordinate her or his teaching-research methodology with one of the theories proven useful and general in the profession, at that moment he or she leaves the particularity of the classroom situation and can view the same classroom as one of the many instances where the particular theory is known to work. At this moment, the whole conceptual apparatus and methodology of the particular theory can be applied to bear upon the particular classroom teaching-research situation. Thus the general character and trustworthiness of the classroom teaching-research becomes grounded in the generality and trustworthiness of a particular theory of learning with which the classroom situation was coordinated. What's more, the very process of coordination between a theory and practice, one of the modes of reflective abstraction, becomes the central skill required from the teacher-researcher. (Czarnocha, 2001). The next two sections describe one of more useful theories of learning, the theory of the schema, and show how its application to the context of the classroom can help in the formulation of Teaching-Research questions.

4 Schema Theory as the tool of Teaching-Research

The presented model of the Teaching Experiment can interact with many different theories of learning and research results. A theoretical framework found useful in the work of a Teacher–Researcher is the Schema Theory that is <u>theory of a structure of the mental network of relationships between different components of a given concept</u>. Of special importance for the work in the classroom whose purpose is to find more effective means of learning is the knowledge of the developmental process of the schema. Understanding that process gives a powerful tool in the hands of a Teacher–Researcher, which allows to develop innovative instruction in agreement with the developmental process for a particular concept. (Shepard, 1993; Clark et al., 1998; Baker and Czarnocha, 2002)).

According to the theories of schema development (Skemp, 1976; Shuell 1993; Piaget & Garcia 1989), the process of discovering and formulating the relationships between different problem contexts where a particular concept manifests itself, is a fundamental step for the full and rich development of the concept in student's mind. In the language of Schema Theory, this process helps the learner to construct the relationships between different and separated individual manifestations of the concept, and by this virtue to enter the higher, second level of schema development.

The importance of schema for the process of thinking derives from several different directions of research, such as the developmental psychology of

(Piaget and Garcia, 1983), cognitive psychology of (Rumelhart and Normans, 1978; Shuell, 1992) and theory of mathematics education of Skemp (1975). All approaches seem to agree on the schema development structure where the first stage is characterized by the learner paying attention to individual cases of the concept (i.e., the Intra-operational level for Piaget), and the second stage where the learner becomes cognizant of the relationship between different manifestations (i.e., the Inter-operational level). The third stage is characterized by several slightly different descriptions either by state of automation, i.e., a smooth execution of all necessary steps, or as the moment of transcendence, i.e., the instance of grasping the full structure of the emerging concept (or the Trans-operational in Piagetian terms (Garcia, 1983). This stage is followed by the process of thematization of concepts by appropriate use. For Shuell (1992) however, it is the stage where "the knowledge structures and schemata formed during the preceding stage become better integrated and function more automatically. In most situations performance will be automatic, unconscious and effortless" (p. 543). Depending on the particular interest one can use either model; if our interest is in understanding and facilitation of the full process of comprehetion we can guide ourselves by the Triad of Piaget and Garcia, if on the other hand our aim is to facilitate the technical mastery, we should guide our selves by the Shuell model grounded in the automation effects of Artificial Intelligence research.

Mathematics teachers start recognizing the structure of the schema in their every day's work. Lampert (2001), a teacher-researcher concerned about the nature and development of children's thinking within her problem-centered math curriculum, notes that "in contrast to the familiar topic-by-topic approach, I worked on constructing lessons for my students to investigate a number of different but related topics, and to investigate them repeatedly in different problem contexts" (pp. 259-261). Important questions regarding such an approach to instruction are: what are its implications for the organization of teacher knowledge for addressing children's intellectual needs and how can this approach be made most effective? Lampert's (2001) objective "to investigate a number of different but related topics" allows to direct students' attention not only to the particular computational skills needed in each instance but also toward the relatedness of topics to each other, toward the network of relationships between them. The reflection needed to discover and formulate these relations allows students to develop a broad conceptual point of view about the mathematical situations in question. This broader view is the beginning of the formation of a new mathematical schema that would enable their thinking about the particular concept. A similar reason underlies Lampert's decision to investigate a particular topic in different problem contexts.

The central competency of a teacher-researcher here is the capacity to coordinate the structure of a syllabus or a structure of a particular classroom event with the structure of the relevant schema. Example 2 below shows the utility of such a coordination for understanding student mastery of negation of simple quantified statements. The need to study mastery of simple logical functors arose in the same calculus investigation of understanding the definition of the limit of sequences. One of the evidence for such understanding is the ability to explain when a sequence doesn't have a particular number as its limit; such explanation requires the negation of the definition what has been a serious problem amongst students of calculus, and more generally amongst students of community colleges.

5 Example. Schema-Theory development of a TR question

Development of a Teaching-Research question. The discussion below illustrates the process of the development of a Teaching-Research question from the results of a final exam for students in an Introduction to Mathematics course, one of two first level colleges courses, at the Hostos CC of CUNY. Introduction to Mathematics has a large chunk of time devoted to the basic introduction to logic, and the exam questions discussed here are dealing with logical negation of simple quantified statements.

We are showing here how the recognition of an interesting pattern in student errors leads to the coordination with a learning theory, which in turn provides suggestions for the modification of instruction, in agreement with the principles of Research into Practice standard of Beyond Crossroads.

Negation data	**26 – correct**
N = 61 (two sections)	**18 – many errors**
	12 – one error
	5 – non standard solutions

Negate the sentences:

- **All calculus problems are difficult**

  Some calculus problems are not difficult.

  Most calculus problems are not difficult.

  All calculus problems are not difficult.

  Some calculus problems are difficult.

- **<u>All students in the class have dark hair.</u>**

  Some students do not have dark hair.

  Most students in the class does not have dark hair.

  No students in the class have dark hair.

  All students in the class do not have dark hair.

  Some students in the class have dark hair.

- **<u>Some differentiation problems are easy to calculate</u>.**

  All differentiation problems are not easy to calculate.

  All differentiation problems are easy to calculate.

  Some differentiations are not easy to calculate.

  Not differentiation problems are easy to calculate.

  Not all differentiation problems are easy to calculate.

- **<u>There is x in the domain of the function where the derivative</u>. <u>is undefined.</u>**

  For every x in the domain of the function, the derivative is defined (not undefined).

  There is x in the domain of the function where the derivative is not undefined.

  There is not x in the domain where the derivative is undefined.

<u>Observation</u>: Student errors (in *italic*) are interesting in that many consist in completing one out of two operations, which together form the negation of the quantified statements: the change of the quantifier and the negation of the property. For example, in the first case, instead of negating the sentence *All problems in Calculus are difficult* by the sentence *Some problems of calculus are not difficult*, the students in this group would say *All problems in Calculus are not difficult* or *Some problems in Calculus are difficult.*

In other words, students don't see the change of the quantifier and the negation of the property as one, conceptual or structural, whole whose components are coordinated with each other. Generally, without the support of the relevant research, an instructor has to treat these difficulties as misconceptions, hoping that with repetition of instruction or another technique of explanation, the student will get it. However, if one coordinates these results with the learning theory, one can see the errors more as a necessary stage in the development of the concept, and consequently one can prepare the activities, which, while taking that stage into account, facilitate student's development to the next one. In other words, the didactic action of the instructor using the TR methodology acquires higher precision in facilitating student's grasp of the issue.

In order to coordinate these observations with a learning theory, we need to find a common element of the structure of a theory with the structure of the results above. Separateness of student observations might be such an element. Students who make discussed errors see the two mental actions of negation as separate, not connected with each other. That understanding of components as separate and isolated from each other seems to coordinate well with Intra level, described by the ability to deal with individual problems in isolation from others. If the Schema Theory is correct here, then after one designs and implements the instruction, which focuses student intellectual attention on the relationships between two separate for them mental actions, then students should be able to construct such a relationship, finding themselves on the stage Inter in their own understanding of the negation. Consequently, the number of such type of errors should diminish.

Will it? – is the new Teaching-Research question.

(1) *Will the mastery and understanding of the negation of simple quantified statement increase if instruction is designed on the developmental Schema Theory?*

In view of these considerations, it's natural to pose the second TR question:

(2) *Can other basic logical competencies be understood in a similar manner by coordinating their development with the Schema-Theory development?*

Whereas the first question is concerned with the improvement of learning, the second question refers to the generality of the approach, in agreement with the general discussion above concerned with the structure of Teaching-Research questions.

Warsaw, Poland, December 2006

References

Anderson, J. R.: 1995, *Cognitive Psychology and its Implications*, 4th edition, W. H. Freeman and Company.

Cajori, F.: 1915, The History of Zeno's Arguments On Motion, AMM, Vol XXII.

Clark, J., Cordero, F. Cottrill, J., Czarnocha, B., St. John, D., Tolias, G., Vidakovic, D.: 1998, Constructing a Schema – The Case of Chain Rule, *Journal of Mathematical Behavior,* **16(4)**.

Cobb, P. Steffe, L. P.: 1983, The Constructivist Researcher as Teacher and Model Builder. *Journal for Research in Mathematics Education.* **14(2)**, 83-94.

Czarnocha, B.: 1999, El Maestro Constructivista como Investigador. In *Educacion Matematica* **11(2)**, 52-63, Mexico.

Czarnocha, B.: 2002, Teacher Researcher for the 21^{st} Century. Short Oral Report, *Proceedings of the 24^{th} Annual Meeting of NA-PME*, Athens, Georgia, October 25-28.

Donovan, M. S. et al, (Eds).: 1999, How People Learn: Bridging Research and Practice Committee on Learning Research and Educational Practice, National Research Council, The National Academies Press.

Grindbaum, A. (...) in Salmon, C. W. (ed).: *Zeno Paradoxes*, Hackett Publishing Company, Inc.

Hunting, R.: 1983, Emerging Methodologies for Understanding Internal Processes Governing Children's Mathematical Behavior. *The Australian Journal of Education* **27(1)**.

King, J. A. Lonnquist, M. Peg: 1992, A Review of Writing on Action Research (1944-Present) *Eric document #ED3555664.*

Noffke, S.: 1994, Action Research: Towards the new generation. *Educational Action Research* **2(1)**, 9-21.

Pawlowski, M.: 2001, Badać jak uczyć, *Nauczyciele i Matematyka* **38**, 22-25.

Rumelhart, D. E., Norman, D. A.: 1978, Acretion, Tuning and Restructuring: Three modes of Learning, in Cotton, J.W. Klotzky, R.L. (eds) *Semantic Factors in Cognition.*

Piaget, J., Garcia, R.: 1989 *Psychogenesis and The History of Science*, Columbia University Press.

Saul, M.: 1995, The nourishment is palatable: A teacher's view of Research in Mathematics Education *Journal of Mathematical Behavior* **14**.

Skemp, R.: 1976, *Psychology of Mathematics Education*, Erlbaum Associates, Mahwah, New Jersey.

Shepard, R. S.: 1993, Writing for conceptual development in mathematics. *Journal of Mathematical Behavior* **12**, 287-293.

Shuell, T. J.: 1990, The phases of meaningful learning, *Review of Educational Research, Winter, R.G.* **60(4)**.

Steffe, L. P. Thompson, P. W..: 2000, Teaching Experiment Methodology in Kelly, A.E., Lesh, R.A. (eds) *Handbook of Research Design in*

Mathematics and Science Education. Lawrence Erlbaum Associates, Publishers. Mahwah, New Jersey.
V y g o t s k y, L.: 1986, *Thought and Language* The MIT Press, Cambridge, Mass.