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Construction of the fundamental solution for the equation $\Delta^2 u(X)+ku(X) = 0$

1. The purpose of this paper is an effective construction of the fundamental solution for the equation

$$(1) \qquad \Delta^2 u(X)+ku(X) = 0,$$

$k \neq 0$ being a constant, $X = (x_1, \ldots, x_n)$, in the two, three and n-dimensional case.

In the literature, construction of fundamental solution for the Helmholtz equation $\Delta u+ku = 0$ is known ([1]). It seems, however, that the case of the equation (1) was not dealt with till now. The equation (1) is very important in the elasticity theory.

In the second Section we deal with the solutions of the equation (1) depending only on the distance r of two points, and we solve the (ordinary) differential equation to be satisfied by those solutions. Then in Section 3 we determine effectively the fundamental solution for $k < 0$, say $k = -C^4$, and $n = 2$. In the Section 4 we do the same for $k = C^4$, $k = -C^4$, $n = 3$. In the Section 5 we effectively construct the fundamental solution for $n > 3$ and $k = -C^4$, in the Section 6 a construction is given for $k = C^4$, $n = 2$, and in the Section 7 we deal with the case $n > 3$, $k = C^4$. In the last Section 8, we apply the obtained fundamental solutions to solve the boundary problems of Lauricelli and of Riquier for the equation (1).

2. Now we shall give the ordinary differential equation to be satisfied by each solution of (1), dependent only on the distance of two points. Let

$$\Delta^2 u(X) = \Delta\big(\Delta u(X)\big) = \sum_{i,j=1}^{n} \frac{\partial^4 u(X)}{\partial x_i^2 \partial x_j^2}$$

and let $X(x_1, \ldots, x_n)$, $Y(y_1, \ldots, y_n)$ be two different points in the n-dimensional space, and let $r = \overline{XY} = \big[\sum_{i=1}^{n}(x_i-y_i)^2\big]^{1/2}$.

We shall need the

LEMMA 1. *Let the function $u(X) = U(r)$ be a solution of the equation* (1), *let $U(r)\in C^4$ for $r > 0$, then the function*

$$V(r) = rU(r), \quad r > 0 \tag{2}$$

is the solution of the equation

$$r^4V^{(4)}(r)+(2n-6)r^3V^{(3)}(r)+(n^2-10n+21)r^2V^{(2)}(r)+ \\ +3(-n^2+8n-15)rV'(r)+3(n^2-8n+15)V(r)+kr^4V(r) = 0. \tag{3}$$

Proof. Since

$$\Delta U(r) = U''(r)+(n-1)r^{-1}U(r),$$

$$\Delta^2 U(r) = U^{(4)}(r)+2(n-1)r^{-1}U^{(3)}(r)+(n-1)(n-3)r^{-2}U''(r)+ \\ +(n-1)(3-n)r^{-3}U'(r),$$

each solution of the equation (1) is also a solution of

$$U^{(4)}(r)+2(n-1)r^{-1}U^{(3)}(r)+(n-1)(n-3)r^{-2}U''(r)+ \\ +(n-1)(3-n)r^{-3}U'(r)+kU(r) = 0. \tag{4}$$

In virtue of (2)

$$\begin{aligned} U'(r) &= r^{-1}V'(r)-r^{-2}V(r), \\ U''(r) &= r^{-1}V''(r)-r^{-2}2V'(r)+r^{-3}2V(r), \\ U^{(3)}(r) &= r^{-1}V^{(3)}(r)-r^{-2}3V''(r)+r^{-3}6V'(r)-r^{-4}6V(r), \\ U^{(4)}(r) &= r^{-1}V^{(4)}(r)-r^{-2}4V^{(3)}(r)+r^{-3}12V''(r)-r^{-4}24V'(r)+r^{-5}24V(r), \end{aligned} \tag{5}$$

whence by substitution of U'', $U^{(3)}$, $U^{(4)}$ from (5) into (4) we obtain (3).

In the Sections 3-6 we shall give fundamental solutions in each of the cases $n = 2, 3$ and $n > 3$ and $k = C^4$, $k = -C^4$, separately.

DEFINITION 1. A function $U(r)$ is called a *fundamental solution of* (1'): $\Delta^2 u(x, y)-C^4u(x, y) = 0$, if 1° $U(r) = U(\overline{XY})$ is defined and of class C^4 in the set $\{(X, Y)\colon X \neq Y\}$, 2° $U(r)$, considered as function of X or of Y only, satisfies the equation (1'), 3° for each function $v(X)$ of class $C^{(4)}$ in bounded region D and $C^{(3)}$ in $\bar{D}$, satisfying the equation (1')

$$v(x, y) = -\frac{1}{8C^2}\int\limits_{\partial D}\left(U\frac{d\Delta v}{dn}-\frac{dU}{dn}\Delta v+\Delta U\frac{dv}{dn}-\frac{d\Delta U}{dn}v\right)dS_Y \tag{6}$$

for each $X\in D$, if $n = 2$, and

$$v(X) = \frac{1}{a_n}\iint\limits_{\partial D}\left(U\frac{d\Delta v}{dn}-\frac{dU}{dn}\Delta v+\Delta U\frac{dv}{dn}-\frac{d\Delta U}{dn}v\right)dS_Y \tag{6a}$$

if $n > 2$, a_n being a normalizing constant, depending of n.

3. Let us first consider the case $n = 2$, $k = -C^4$. Each solution dependent only on r satisfies in our case the equation

$$(1') \qquad \Delta^2 u(x, y) - C^4 u(x, y) = 0.$$

The equation (1′) is of form $(\Delta - C^2)(\Delta + C^2)u = 0$.

We shall prove

LEMMA 2. *Every solution* $u(x, y) \in C^{(4)}$ *of the equation*

$$(7) \qquad \Delta u - C^2 u = 0$$

or

$$(8) \qquad \Delta u + C^2 u = 0$$

respectively, is also a solution of (1′).

Proof. It follows from (7) that $\Delta^2 u(x, y) = C^2 \Delta u(x, y) = C^4 u(x, y)$, and from (8) it follows that $\Delta^2 u(x, y) = -C^2 \Delta u(x, y) = -C^2 (-C^2 u(x, y) = C^4 u(x, y))$.

We shall find (fundamental) solutions of (7) and (8) by means of which we shall construct the fundamental solution of (1′). The function $U(r)$ depending on the distance $r = \overline{XY}$, satisfying the equation (8) satisfies also the equation

$$(9) \qquad U''(r) + r^{-1} U'(r) + C^2 U(r) = 0.$$

By the change of the independent variable

$$(10) \qquad R = Cr,$$

we obtain the equation

$$(11) \qquad R^2 U''(R) + R U'(R) + R^2 U(R) = 0,$$

which is obviously the Bessel equation with the parameter $s = 0$ ([3]).

The function

$$(12) \qquad U_1(R) = Y_0(R) = \frac{1}{\pi} \sum_{p=0}^{\infty} \frac{(-1)^p (R/2)^{2p}}{(p!)^2} \left(2\ln\frac{R}{2} - 2f(p+1)\right),$$

where

$$f(k+1) = -\gamma + 1 + \frac{1}{2} + \ldots + \frac{1}{k}, \quad f(1) = -\gamma, \quad \gamma \text{ the Euler const.}$$

is a solution of the equation (11) ([3]).

We conclude from (9), (10), (11), (12) and from Lemma 2 that the function $Y_0(Cr)$ is the solution of the equation (1′).

The function $U(r)$ satisfying the equation (7) satisfies also the equation

$$(13) \qquad U''(r) + r^{-1} U'(r) - C^2 U(r) = 0.$$

Replacing in the equation (9) Ci by C we obtain the equation (13). Consequently the function $\operatorname{Re} Y_0(Cir)$ is a solution of the equation (1').

In all the formulas to follow the symbols $O(1)$, $O(r)$, $O(\varphi(r))$, $\varphi(r)$ being a function of r, will refer to the case $r \to 0$.

Now we are able to prove

THEOREM 1. *In case $n = 2$, $k = -C^4$ the function*

$$U(r) = U_1(r) - U_2(r) \tag{14}$$

where

$$U_1(r) = Y_0(Cr), \qquad U_2(r) = \operatorname{Re} Y_0(Cir)$$

is the fundamental solution of the equation (1').

Proof. From (12) it follows that

$$Y_0(Cir) = \frac{2}{\pi} J_0(Cir) \ln \frac{Cir}{2} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (Cir/2)^{2k}}{(k!)^2} f(k+1),$$

where

$$J_0(Cir) = \sum_{k=0}^{\infty} \frac{(-1)^k (Cir/2)^{2k}}{(k!)^2} = 1 + \sum_{k=1}^{\infty} \frac{(Cr/2)^{2k}}{(k!)^2}. \tag{15}$$

From (15) we get

$$Y_0(Cir) = \frac{2}{\pi}\left(1 + \sum_{k=1}^{\infty} \frac{(Cr/2)^{2k}}{(k!)^2}\right)\left(\ln \frac{Cr}{2} + i\frac{\pi}{2}\right) - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(Cr/2)^{2k}}{(k!)^2} f(k+1) \tag{16}$$

and

$$\operatorname{Re} Y_0(Cir) = \frac{2}{\pi} \ln \frac{Cr}{2} + L_1(r), \tag{17}$$

where

$$L_1(r) = \frac{2}{\pi} \ln \frac{Cr}{2} \sum_{k=1}^{\infty} \frac{(Cr/2)^{2k}}{(k!)^2} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(Cr/2)^{2k}}{(k!)^2} f(k+1) = O(1).$$

From (12) we obtain

$$Y_0(Cr) = \frac{2}{\pi} \ln \frac{Cr}{2} + L_2(r), \tag{18}$$

where

$$L_2(r) = \frac{2}{\pi} \ln \frac{Cr}{2} \sum_{k=1}^{\infty} \frac{(-1)^k (Cr/2)^{2k}}{(k!)^2} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (Cr/2)^{2k}}{(k!)^2} f(k+1) = O(1).$$

In virtue of (14), (17) and (18)

$$U(r) = L_2(r) - L_1(r) = O(1) \tag{19}$$

and

$$U'(r) = L_2'(r) - L_1'(r) = O(1). \tag{20}$$

By (7), (8), (14), (17) and (18)

$$\Delta U = \Delta U_1 - \Delta U_2 = -C^2 U_1 - C^2 U_2 = -C^2\left(\frac{4}{\pi}\ln\frac{Cr}{2} + L_3(r)\right), \tag{21}$$

where $L_3(r) = O(1)$.

From (21) it follows that

$$\frac{d}{dr}\Delta U(r) = -\frac{4C^2}{\pi r} + L_4(r), \tag{22}$$

where $L_4(r) = L_3'(r) = O(r)$.

Let $v(x, y)$ by a function of class $C^{(4)}$ in D satisfying the equation (1′) and let K_R be a circle with the centre X and with radius R. Applying the fundamental formula ([3])

$$\iint_D (U\Delta^2 v - v\Delta^2 U)\,dx\,dy = -\int_{\partial D}\left(U\frac{d\Delta v}{dn} + \Delta U\frac{dv}{dn} - v\frac{d\Delta U}{dn} - \Delta v\frac{dU}{dn}\right)ds_Y$$

to the functions $U(r)$ and $v(x, y)$ in the set $D - K_R$, we get

$$\begin{aligned}\iint_{D-K_R} \big(U(r)\Delta^2 v(Y) - v(Y)\Delta^2 U(r)\big)\,d\xi\,d\eta \\ = -\int_{\partial(D-K_R)}\left(U(r)\frac{d}{dn}\Delta v(Y) - v(Y)\frac{d}{dn}\Delta U(r)\right)ds_Y - \\ -\int_{\partial(D-K_R)}\left(\Delta U(r)\frac{d}{dn}v(Y) - \Delta v(Y)\frac{d}{dn}U(r)\right)ds_Y.\end{aligned} \tag{23}$$

Since $U(r)$ and $v(Y)$ satisfy the equation (1′), the left-hand side of the formula (23) is equal to zero. Therefore it follows from (23) that

$$\begin{aligned}\int_{\partial D}\left(U\frac{d\Delta v}{dn} - v\frac{d\Delta U}{dn} + \Delta U\frac{dv}{dn} - \Delta v\frac{dU}{dn}\right)ds_Y + \\ + \int_{\partial K_R} U\frac{d\Delta v}{dn}\,ds_Y - \int_{\partial K_R}\frac{dU}{dn}\Delta v\,ds_Y + \int_{\partial K_R}\Delta U\frac{dv}{dn}\,ds_Y - \int_{\partial K_R}\frac{d\Delta U}{dn}v\,ds_Y = 0.\end{aligned} \tag{24}$$

By application of the mean value theorem to the curvilinear integrals in the formula (24) we get in view of (19), (20), (21) and (22)

$$
(25)\quad
\begin{aligned}
J_1 &= \int\limits_{\partial K_R} U(r)\frac{d\Delta v(Y)}{dr}\,ds_Y = 2\pi R U(R)\frac{d\Delta v(Q)}{dr} = \\
&= \big(L_2(R)-L_1(R)\big)2\pi R\frac{d\Delta v(Q)}{dr} \to 0 \text{ when } R\to 0,\ Q\in\partial K_R,\\
J_2 &= \int\limits_{\partial K_R} \Delta v(Y)\frac{dU(r)}{dr}\,ds_Y = 2\pi R\Delta v(Q_1)\frac{dU(R)}{dr} = \\
&= 2\pi R\Delta v(Q_1)\big(L_2'(R)-L_1'(R)\big) \to 0 \text{ when } R\to 0,\ Q_1\in\partial K_R,\\
J_3 &= \int\limits_{\partial K_R} \Delta U(r)\frac{dv(Y)}{dr}\,ds_Y = 2\pi R\left(-\frac{4C^2}{\pi}\ln\frac{CR}{2}+L_3(R)\right)\frac{dv(Q_2)}{dr}\to 0\\
&\qquad\qquad\qquad \text{when } R\to 0,\ Q_2\in\partial K_R,\\
J_4 &= \int\limits_{\partial K_R} v(Y)\frac{d\Delta U(r)}{dr}\,ds_Y = 2\pi R v(Q_3)\frac{d\Delta U(R)}{dr} = \\
&= 2\pi R\left(-\frac{4C^2}{\pi R}+L_4(R)\right)v(Q_3)\to -8C^2v(x,y) \text{ when } R\to 0,\ Q_3\in\partial K_R.
\end{aligned}
$$

By (24) and (25) we obtain the formula (6). The function $U(r)$ defined by (14) being obviously sufficiently regular, it follows that $U(r)$ is a fundamental solution.

4. Now we shall construct the fundamental solution for the equation (1) in the three dimensional case. In this case the integrals depending only on the distance r satisfy the equation

$$(26)\qquad V^{(4)}(r)+kV(r) = 0,\qquad r>0.$$

We shall consider two cases: (a) $k=-C^4$, (b) $k=C^4$, C constant.

Case (a). The equation (26) is of the form

$$(27)\qquad V^{(4)}(r)-C^4V(r) = 0.$$

The functions $V_1(r)=e^{Cr}$, $V_2(r)=\cos Cr$, are linearly independent solutions of the equation (27) and the functions

$$U_1(r)=\frac{1}{r}V_1(r)=\frac{1}{r}e^{Cr},\qquad U_2(r)=\frac{1}{r}\cos Cr,$$

are analogously related with the equation (1).

THEOREM 2. *If $n=3$, $k=-C^4$, then the function*

$$(28)\qquad U(r) = U_1(r)-U_2(r) = r^{-1}(e^{Cr}-\cos Cr)$$

is the fundamental solution of the equation (1).

Proof. We shall prove that function $U(r)$, defined by formula (28), satisfies the identity (6a) with $a_3=8C^2$. Let $v(x,y,z)$ be a function of class $C^{(4)}$ in D satisfying the equation (1) and let K_R be a sphere with the centre $X(x, y, z)$ and radius R. Applying the fundamental formula ([2])

$$\iiint_D \big(U(r)\Delta^2 v-(Y)-v(Y)\Delta^2 U(r)\big)dx\,dy\,dz+\iint_{\partial D}\Big(U(r)\frac{d}{dn}\Delta v(Y)-$$
$$-\Delta v(Y)\frac{d}{dn}U(r)+\Delta U(r)\frac{d}{dn}v(Y)-v(Y)\frac{d}{dn}\Delta U(r)\Big)dS_Y=0$$

to the function $U(r)$ and $v(x, y, z)$ in the domain $D-K_R$ we obtain

$$(29)\qquad \iiint_{D-K_R}\big(U(r)\Delta^2 v(Y)-v(Y)\Delta^2 U(r)\big)d\xi\,d\eta\,d_\zeta$$
$$=-\iint_{\partial(D-K_R)}\Big(U(r)\frac{d}{dn_Y}\Delta v(Y)-\Delta v(Y)\frac{d}{dn}U(r)\Big)dS_Y-$$
$$-\iint_{\partial(D-K_R)}\Big(\Delta U(r)\frac{d}{dn_Y}v(Y)-v(Y)\frac{d}{dn}\Delta U(r)\Big)dS_Y.$$

Since the functions $U(r)$ and $v(Y)$ satisfy the equation (1) it follows that the left-hand side of (27) is equal to zero. Hence by (29)

$$(30)\qquad \iint_{\partial D}\Big(U\frac{d}{dn}\Delta v-\Delta v\frac{dU}{dn}+\Delta U\frac{dv}{dn}-v\frac{d}{dn}\Delta U\Big)dS_Y+$$
$$+\iint_{\partial K_R}U\frac{d}{dr}\Delta v\,dS_Y-\iint_{\partial K_R}\Delta v\frac{dU}{dr}dS_Y+\iint_{\partial K_R}\Delta U\frac{dv}{dr}dS_Y-$$
$$-\iint_{\partial K_R}v\frac{d}{dr}\Delta U\,dS_Y=0.$$

Applying the mean value theorem for the surface integrals in formula (30) we obtain in view of (7), (8) and (28)

$$J_1=\iint_{\partial K_R}U(r)\frac{d}{dr}\Delta v(Y)dS_Y=4\pi R(e^{Cr}-\cos Cr)\frac{d}{dr}\big(\Delta v(Q)\big)\to 0,$$
$$\text{when } R\to 0,\ Q\in\partial K_R,$$

$$J_2=\iint_{\partial K_R}\Delta v(Y)\frac{dU(r)}{dr}dS_Y=4\pi R^2\Delta v(Q_1)(-R^{-2}e^{CR}+R^{-1}Ce^{CR}+$$
$$+R^{-2}\cos CR+CR^{-1}\sin CR)\to 0,\quad \text{when}\quad R\to 0,\ Q_1\in\partial K_R,$$

$$J_3=\iint_{\partial K_R}\Delta U(r)\frac{dv(Y)}{dr}dS_Y=-4\pi RC^2(e^{CR}+\cos CR)\frac{dv(Q_2)}{dr}\to 0,$$
$$\text{when } R\to 0,\ Q_2\in\partial K_R,$$

$$J_4=\iint_{\partial K_R}v(Y)\frac{d}{dr}\Delta U(r)dS_Y=4\pi R^2 v(Q_3)(C^2R^{-2}e^{CR}-C^3R^{-1}e^{CR}+$$
$$+C^2R^{-2}\cos CR+C^3R^{-1}\sin CR)\to 8C^2\pi v(x, y, z),\ \text{when } R\to 0,\ Q_3\in\partial K_R.$$

By the above formulas and (30) we get (6a). The function $U(r)$ is obviously of class $C^{(4)}$, whence by Definition 1, it is a fundamental solution in our case.

Case (b). The equation (26) takes the form

$$V^{(4)}(r)+C^4V(r) = 0. \tag{31}$$

The functions $V_1(r) = e^{Ar}\sin Ar$, $V_2(r) = e^{-Ar}\sin Ar$, where $A = C\sqrt{2}/2$, are two linearly independent solutions of (31) as well as the functions $U_1(r) = r^{-1}e^{Ar}\sin Ar$, $U_2(r) = r^{-1}e^{-Ar}\sin Ar$ are linearly independent solutions of (1).

Now we shall prove

THEOREM 3. *If $n = 3$, $k = C^4$, then the function*

$$U(r) = U_1(r)-U_2(r) = r^{-1}\sin Ar(e^{Ar}-e^{-Ar}) \tag{32}$$

is a fundamental solution of the equation (1).

Proof. We shall prove that the function $U(r)$ given by formula (32) satisfies (6a) with $a_3 = -8\pi C^2$. Using analogous arguments as in the proof of the Theorem 2 we may obtain the formula (30). Applying the mean value theorem to the surface integral in the formula (30), using (7), (8) and (32), we obtain formulas analogous to (25)

$$J_1 = 4\pi R^2\left(\frac{d}{dr}\Delta v(Q)\right)R^{-1}\sin AR(e^{AR}-e^{-AR}) \to 0, \quad Q\in\partial K_R,\ R\to 0,$$

$$J_2 = 4\pi R^2\big(\Delta v(Q_1)\big)\big((e^{AR}-e^{-AR})R^{-1}(A\cos AR-R^{-1}\sin AR)+$$
$$+AR^{-1}\sin AR(e^{AR}-e^{-AR})\big) \to 0, \quad Q_1\in\partial K_R,\ R\to 0,$$

$$J_3 = 4\pi R^2\left(\frac{d}{dr}v(Q_2)\right)2R^{-1}(A^2\cos AR)(e^{AR}+e^{-AR}) \to 0, \quad Q_2\in\partial K_R,$$

$$J_4 = 4\pi R^2v(Q_3)\frac{2A^2}{R}\big((e^{AR}+e^{-AR})(-R^{-1}\cos AR-A\sin AR)+$$
$$+A(e^{AR}-e^{-AR})\cos AR\big) \to -8\pi C^2v(x,y,z), \quad R\to 0,\ Q_3\in\partial K_R$$

From these formulas and from formula (30) follows (6a). The function $U(r)$ is evidently sufficiently regular, whence it is a fundamental solution of (1).

5. Now let us consider the case $n>3$, $k = -C^4$. Our equation is of form

$$\Delta^2u(X)-C^4u(X) = 0, \qquad X = (x_1,\dots,x_n). \tag{33}$$

By Lemma 2 the solutions $U(r)$ of the equation (33) satisfying respectively (7) and (8) are solutions of the differential equation

$$U''(r)+(n-1)r^{-1}U'(r)+C^2U(r) = 0 \tag{34}$$

and

(35) $$U''(r)+(n-1)r^{-1}U'(r)-C^2U(r) = 0,$$

respectively. Conversely, solutions of (34) and (35) are solutions of (33). Upon setting $R = Cr$ in equation (33) we obtain the equation

(34a) $$U''(R)+\frac{1-2\dfrac{2-n}{2}}{R}\,U'(R)+U(R) = 0,$$

which is a special case of the equation ([4])

(36) $$U''(z)+\frac{1-2a}{z}U'(z)+(\beta\gamma z^{\gamma-1})^2+\frac{a^2-s^2\gamma^2}{z^2}U(z) = 0,$$

where

$$a = \frac{2-n}{2},\qquad z = R,\qquad \gamma=\beta=1,\qquad s = -a = \frac{n-2}{2}.$$

The solutions of (36) are ([4])

(37) $$U(z) = z^aZ_s(\beta z^\gamma),$$

where in the case when s is a positive integer

(38) $$Z_s(z) = Y_s(z) = \frac{2}{\pi}J_s(z)\ln\frac{z}{2}-\frac{1}{\pi}\sum_{k=0}^{s-1}\frac{(s-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-s}+$$
$$+\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{s-2k}}{k!(s+k)!}\big(-f(k+1)-f(k+s+1)\big),$$

where

$$J_s(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{s+2k}}{k!(s+k)!}$$

and, in the case of non-integer s,

(39) $$Z_s(z) = J_{-s}(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k-s}}{\Gamma(k+1)\Gamma(k-s+1)}.$$

Consequently, the function

$$U(r) = (Cr)^{-(n-2)/2}Z_s(Cr)$$

is a solution of the equation (34).

We shall single out three cases:

$$\left.\begin{array}{l}1^\circ\ s = (n-2)/2 = 2p+1\\ 2^\circ\ s = (n-2)/2 = 2p\end{array}\right\}\quad p \text{ a positive integer},$$

3° $s = (n-2)/2$ for n even.

Ad 1°. In this case in view of (38)

$$Y_s(Cr) = -\frac{(2p)!}{\pi}\left(\frac{Cr}{2}\right)^{-2p-1} + V_1(r), \tag{40}$$

where

$$V_1(r) = \frac{2}{\pi} J_s(Cr)\ln\frac{Cr}{2} - \frac{1}{\pi}\sum_{k=1}^{s-1}\frac{(s-k-1)!}{k!}\left(\frac{Cr}{2}\right)^{2k-s} -$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k (Cr/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big) = O(r^{-2p+1}).$$

In virtue of (37), (38), (40) and Lemma 2 the function

$$U_1(r) = (Cr)^{-2p-1}\left(-\frac{(2p)!}{\pi}\left(\frac{Cr}{2}\right)^{-r-1} + V_1(r)\right) \tag{41}$$

is a solution of the equation (34). Replacing C by Ci in (34) we obtain (35), whence $\operatorname{Im}(Cir)^{-2p-1} Y_s(Cir)$ as well as $U_2(r) = (Cr)^{-2p-1}\operatorname{Im} Y_s(Cir)$ are solutions of (35). From (40) it follows that

$$Y_s(Cir) = -\frac{1}{\pi}(2p)!\left(\frac{Cir}{2}\right)^{-2p-1} + V_2(Cir),$$

where

$$V_2(Cir) = \frac{2}{\pi}\left(\ln\frac{Cir}{2}\right)\cdot J_s(Cir) - \frac{1}{\pi}\sum_{k=1}^{2p}\frac{(2p-k)!}{k!}\left(\frac{Cir}{2}\right)^{2k-2p-1} -$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k (Cir/2)^{2p+2k+1}}{k!(2p+2k+1)!}\big(f(k+1)+f(k+s+1)\big).$$

Thus

$$\operatorname{Im} Y_s(Cir) = (-1)^p\left(\frac{Cr}{2}\right)^{-2p-1}\frac{(2p)!}{\pi} + \operatorname{Im} V_2(Cir),$$

and therefore

$$U_2(r) = (Cr)^{-2p-1}\left((-1)^p\left(\frac{Cr}{2}\right)^{-2p-1}\frac{(2p)!}{\pi} + V_3(r)\right), \tag{42}$$

where

$$V_3(r) = \operatorname{Im} V_2(Cir) = O(r^{-2p+1}).$$

THEOREM 4. *If* $n > 3$, $s = (n-2)/2 = 2p+1$, p *is a positive integer, then the function*

$$U(r) = C_1 U_1(r) + C_2 U_2(r),$$

where $C_1 = 1$, $C_2 = (-1)^p$; $U_1(r)$ *and* $U_2(r)$ *are defined by the formulas* (41) *and* (42), *or*

$$U(r) = (Cr)^{-2p-1}\big(C_1V_1(r)+C_2V_3(r)\big) \tag{43}$$

is a fundamental solution of the equation (33).

Proof. In view of (43),

$$U(r) = O(r^{4-n}), \tag{44}$$

$$\begin{aligned} U'(r) &= C(-2p-1)(Cr)^{-2p-2}\big(C_1V_1+C_2V_3(r)\big)+ \\ &\quad + (Cr)^{-2p-1}\big(C_1V_1'(r)+C_2V_3'(r)\big) = O(r^{3-n}), \end{aligned} \tag{45}$$

because $V_1'(r) = O(r^{-2p}) = V_3'(r)$. Using (7) and (8) we get

$$\begin{aligned} \Delta U(r) &= -C^2\big(C_1U_1(r)-C_2U_2(r)\big) \\ &= 2C^2(Cr)^{-4p-2}2^{2p+1}\frac{(2p)!}{\pi} - C^2(Cr)^{-2p-1}\big(C_1V_1(r)-C_2V_3(r)\big) \\ &= O(r^{2-n}), \end{aligned} \tag{46}$$

$$\begin{aligned} \frac{d}{dr}\Delta U(r) &= C^3(-4p-2)(Cr)^{-4p-3}2^{2p+2}\frac{(2p)!}{\pi}+ \\ &+C^3(2p+1)(Cr)^{-2p-2}\big(C_1V_1(r)-C_2V_3(r)\big)-C^2(Cr)^{-2p-1}\big(C_1V_1'(r)-C_2V_3'(r)\big) \\ &= \beta_n r^{1-n}+O(r^{3-n}), \end{aligned} \tag{47}$$

where

$$\beta_n = -\frac{1}{\pi}C^{4-n}2^{(n+2)/2}\left(\frac{n-2}{2}\right)!.$$

Noting that the function (43) satisfies the conditions 1° and 2° of the Definition 1, it is sufficient to show the condition 3°. Applying similar arguments as in the proof of Theorem 2 we obtain the formula (30). The surface integrals are $(n-1)$-fold integrals. Applying the mean value theorem to the surface integrals, by (44), (45), (46), and (47), we get

$$\begin{aligned} J_1 &= \iint_{\partial K_R} U(r)\frac{d}{dr}\Delta v(Y)\,dS_Y = \Omega_n R^{n-1}O(R^{4-n}) = O(R^3) \to 0, \\ J_2 &= \iint_{\partial K_R} \Delta v(Y)\frac{dU(r)}{dr}\,dS_Y = \Omega_n R^{n-1}O(R^{3-n}) = O(R^2) \to 0, \\ J_3 &= \iint_{\partial K_R} \Delta U(r)\frac{dv(Y)}{dr}\,dS_Y = \Omega_n R^{n-1}O(R^{2-n}) = O(R) \to 0, \\ J_4 &= \iint_{\partial K_R} v(Y)\frac{d}{dr}\Delta U(r)\,dS_Y = \Omega_n R^{n-1}\big(\beta_n R^{1-n}+O(R^{4-n})\big)v(Q) \to \\ &\qquad \to \alpha_n v(X),\ Q\in\partial K_R, \end{aligned} \tag{48}$$

where $a_n = \Omega_n \beta_n$ and Ω_n denotes the area of the unit sphere in the n-dimensional space.

In virtue of (48) and (38) we obtain (6a). Our assertion follows, since $U(r)$ is obviously of class $C^{(4)}$.

Ad 2°. Let $s = (n-2)/2 = 2p$. In view of (38)

$$Y_s(Cr) = -\frac{(2p-1)!}{\pi}\left(\frac{Cr}{2}\right)^{-2p} + T_1(r), \tag{49}$$

where

$$T_1(r) = \frac{2}{\pi} J_s(Cr)\ln\frac{Cr}{2} - \frac{1}{\pi}\sum_{k=1}^{s-1}\frac{(s-k-1)!}{k!}\left(\frac{Cr}{2}\right)^{2k-s} +$$
$$+\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k (Cr/2)^{s+2k}}{k!(s+k)!}\big(-f(k+1)-f(k+s+1)\big) = O(r^{2-2p}).$$

By Lemma 2 and by (38), (37) and (49) the function

$$U_1(r) = (Cr)^{-2p}\left(-\frac{(2p-1)!}{\pi}\left(\frac{Cr}{2}\right)^{-2p} + T_1(r)\right) \tag{50}$$

is a solution of the equation (34).

Since by (38) and (40)

$$Y_s(Cir) = -\frac{1}{\pi}(2p-1)!\left(\frac{Cir}{2}\right)^{-2p} + T_2(Cir),$$

where

$$T_2(Cir) = \frac{2}{\pi} J_s(Cir)\ln\frac{Cir}{2} - \frac{1}{\pi}\sum_{k=1}^{2p-1}\frac{(2p-k-1)!}{k!}\left(\frac{Cir}{2}\right)^{2k-2p} -$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k (Cir/2)^{2p+2k}}{k!(2p+k)!}\big(f(k+1)+f(k+s+1)\big) = O(r^{2-2p}),$$

we have

$$\operatorname{Re} Y_s(Cir) = -\frac{1}{\pi}(2p-1)!\left(\frac{Cr}{2}\right)^{-2p}(-1)^p + \operatorname{Re} T_2(Cir)$$

and it follows that the function

$$U_2(r) = (Cr)^{-2p}\left(-(-1)^p\frac{(2p-1)!}{\pi}\left(\frac{Cr}{2}\right)^{-2p} + T_3(r)\right), \tag{51}$$

where $T_3(r) = \operatorname{Re} T_2(Cir) = O(r^{-2p+1})$ is a solution of the equation (34).

THEOREM 5. *If $n > 3$, $s = (n-2)/2 = 2p$, and p is a positive integer, then the function*

$$U(r) = C_1 U_1(r) + C_2 U_2(r) = (Cr)^{-2p}\big(C_1 T_1(r) + C_2 T_3(r)\big), \tag{52}$$

$C_1 = -1$, $C_2 = (-1)^p$, *$U_1(r)$ and $U_2(r)$ are defined by formulas* (50) *and* (51), *is a fundamental solution of the equation* (33).

Proof. In virtue of (52)

$$U(r) = O(r^{4-n})$$

and

$$U'(r) = -2pC(Cr)^{-2p-1}\big(C_1 T_1(r) + C_2 T_3(r)\big) + (Cr)^{-2p}(C_1 T_1' + C_2 T_3') = O(r^{3-n}).$$

In view of (7), (8), (50) and (51)

$$\begin{aligned} \Delta U(r) &= C_1 \Delta U_1(r) + C_2 \Delta U_2(r) = -C^2\big(C_1 U_1(r) - C_2 U_2(r)\big) \\ &= C^2 2^{2p}\frac{(2p-1)!}{\pi}(Cr)^{-4p}\big(C_1 - (-1)^p\big) - C^2 (Cr)^{-2p}\big(C_1 T_1(r) - C_2 T_3(r)\big) \end{aligned}$$

and consequently $\Delta U(r) = O(r^{2-n})$ and

$$\begin{aligned} \frac{d}{dr}\Delta U(r) &= -C^3 p 2^{2p+2}\frac{(2p-1)!}{\pi}(Cr)^{-4p-1}\big(C_1 - (-1)^p\big) + \\ &\quad + C^3 2p(Cr)^{-2p-1}\big(C_1 T_1(r) - C_2 T_3(r)\big) - C^2 (Cr)^{-2p}\big(C_1 T_1'(r) - C_2 T_3'(r)\big) \\ &= -C^3 p 2^{2p+2}(Cr)^{-4p-1}\frac{(2p-1)!}{\pi}\big(C_1 - (-1)^p C_2\big) + O(r^{3-n}) . \\ &= \beta_n r^{1-n} + O(r^{3-n}) \end{aligned}$$

where

$$\beta_n = \frac{1}{\pi} C^{4-n} 2^{(n-2)/2}(n-2)(n-4)!. \tag{53}$$

Applying similar arguments as in the proof of Theorem 4 we get formulas analogous to (48) with $a_n = \Omega_n \beta_n$, where β_n is given by formula (53). We may obtain our conclusion similarly as in the proof of Theorem 5.

Ad 3°. In the case 3° we single out two subcases

3°A) $\quad s = \dfrac{n-2}{2} = (2q+1)\dfrac{1}{2} = 2p + \dfrac{1}{2} \quad$ for $\quad q = 2p,$

3°B) $\quad s = \dfrac{n-2}{2} = (2q+1)\dfrac{1}{2} = 2p + \dfrac{3}{2} \quad$ for $\quad q = 2p+1.$

In view of Lemma 2 and formulas (37) and (39) the functions

$$(54)\qquad U_1(r) = u_1(Cr)$$
$$= \frac{1}{\Gamma(1)\Gamma(1-s)}2^{(n-2)/2}(Cr)^{2-n} - \frac{1}{\Gamma(2)\Gamma(2-s)}2^{(n-6)/2}(Cr)^{4-n} + W_1(Cr),$$

where

$$W_1(Cr) = (Cr)^{(2-n)/2}\sum_{k=2}^{\infty}\frac{(-1)^k(Cr/2)^{2k-s}}{\Gamma(1+k)\Gamma(k-s+1)} = O(r^{6-n}),$$

and

$$u_1(Cir) = \frac{2^{(n-2)/2}(Cr)^{2-n}i^{2-n}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{(n-6)/2}(Cr)^{4-n}i^{4-n}}{\Gamma(2)\Gamma(2-s)} + W_1(Cir),$$

where $W_1(Cir) = O(r^{6-n})$, are solutions of (34) as well as the functions $\operatorname{Re} u_1(Cir)$ and $\operatorname{Im} u_1(Cir)$.

Ad 3°A). In this case $i^{-n} = i$ whence

$$u_1(Cir) = -i\left(\frac{2^{(n-2)/2}(Cr)^{2-n}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{(n-6)/2}(Cr)^{4-n}}{\Gamma(2)\Gamma(2-s)}\right) + W_1(Cir).$$

Let $U_2(r) = \operatorname{Im} u_1(Cir)$ and let

$$(55)\qquad U(r) = -\big(U_1(r)+U_2(r)\big) - \big(W_1(Cr)+\operatorname{Im} W_1(Cir)\big).$$

Ad 3°B). In this case $i^{-n} = -i$ whence like in case 3°A) the function defined by (34), the function $u_3(Cir)$ and

$$U_3(r) = \operatorname{Im} u_3(Cir) = \frac{2^{(n-2)/2}(Cr)^{2-n}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{(n-6)/2}(Cr)^{4-n}}{\Gamma(2)\Gamma(2-s)} + \operatorname{Im} W_1(Cir)$$

are solutions of (55). Let

$$(56)\qquad U(r) = U_3(r) - U_1(r) = \frac{2^{(n-4)/2}(Cr)^{4-n}}{\Gamma(2)\Gamma(2-s)} + \operatorname{Im} W_1(Cir) - W_1(Cr).$$

THEOREM 6. *Let $n > 3$, $s = (n-2)/2 = 2p+\frac{1}{2}$ or $s = 2p+\frac{3}{2}$, p being a positive integer. Then the functions defined by* (55) *and* (56) *are respectively solutions of* (33).

Proof. We shall only consider the case 3°A) since in the case 3°B) the proof is analogous. We have

$$U(r) = A_1 r^{4-n} + O(r^{6-n}), \qquad \frac{d}{dr}U(r) = A_2 r^{3-n} + O(r^{5-n}),$$

$$\Delta U(r) = A_3 r^{2-n} + O(r^{4-n}), \qquad \frac{d}{dr}\Delta U(r) = A_4 r^{1-n} + O(r^{3-n}),$$

where A_1, A_2, A_3, A_4 are constants and $A_4 = 2^{(n-4)/2}C^{4-n}[\Gamma(2)\Gamma(2-s)]^{-1}$. We conclude the proof similarly as this of Theorem 4, setting $a_n = \Omega_n A_4$.

6. Now we shall construct fundamental solutions for the equation

$$(1'') \qquad \Delta^2 u(x,y) + C^4 u(x,y) = 0,$$

C a positive constant. Since (1'') may be obtained from (1') when replacing C by $C\sqrt{i}$, the function $U(r) = \operatorname{Im} Y_0(Cr\sqrt{i})$ is a solution of (1''). By (12)

$$Y_0(Cr\sqrt{i}) = \frac{2}{\pi}\left(\ln\frac{Cr\sqrt{i}}{2} + \gamma\right) - \frac{2}{\pi}\cdot\frac{Cr\sqrt{i}}{2}\left(2\ln\frac{Cr\sqrt{i}}{2} - k\right) + P_1(r),$$

where

$$P_1(r) = \frac{1}{\pi}\sum_{k=2}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{2k}}{(k!)^2}\left(2\ln\frac{Cr\sqrt{i}}{2} - 2f(k+1)\right) = O(r^{4-\varepsilon}),$$

ε being an arbitrary positive constant, and $k = f(2)$. Hence

$$Y_0(Cr\sqrt{i}) = \frac{2}{\pi}\left(\ln\frac{Cr}{2} + \gamma + \frac{\pi}{2}\left(\frac{Cr}{2}\right)^2\right) + \\ + \frac{2}{\pi}\left(\frac{\pi}{4} - 2\left(\frac{Cr}{2}\right)^2 \ln\frac{Cr}{2} + k\left(\frac{Cr}{2}\right)^2\right) i + P_1(r)$$

and it follows that

$$(57) \quad U(r) = \operatorname{Im} Y_0(Cr\sqrt{i}) = \frac{2}{\pi}\left(\frac{\pi}{4} - 2\left(\frac{Cr}{2}\right)^2 \ln\frac{Cr}{2} + k\left(\frac{Cr}{2}\right)^2\right) + \operatorname{Im} P_1(r).$$

THEOREM 7. *Let $n = 2$, then the fundamental solution of (1'') is given by formula (57).*

Proof. The function $U(r)$ defined by (57) satisfies obviously

$$U(r) = O(1),$$

$$U'(r) = \frac{2}{\pi}\left(-C^2 r\ln\frac{Cr}{2} - \frac{C^2 r}{2} + \frac{C^2 kr}{2}\right) + \big(\operatorname{Im} P_1(r)\big)' = B_1 \ln\frac{Cr}{2} + O(r),$$

$$U''(r) = \frac{2}{\pi}\left(-C^2\ln\frac{Cr}{2} + \frac{C^2 r}{2} - \frac{3}{2}C^2\right) + \big(\operatorname{Im} P_1(r)\big)'',$$

$$U'''(r) = -\frac{2C^2}{\pi r} + \big(\operatorname{Im} P_1(r)\big)''',$$

$$\begin{aligned} \Delta U(r) &= U''(r) + r^{-1}U'(r) \\ &= \frac{2}{\pi}\left(-2C^2\ln\frac{Cr}{2} - 2C^2 + C^2 k\right) + \big(\operatorname{Im} P_1(r)\big)'' + r^{-1}\big(\operatorname{Im} P_1(r)\big)' \\ &= B_2 \ln\frac{Cr}{2} + O(r), \end{aligned}$$

$$\begin{aligned} \frac{d}{dr}\Delta U(r) &= U'''(r) + r^{-1}U''(r) - r^{-2}U'(r) \\ &= -\frac{4C^2}{\pi r} + \big(\operatorname{Im} P_1(r)\big)''' + \frac{1}{r}\big(\operatorname{Im} P_1(r)\big)'' - r^{-2}\big(\operatorname{Im} P_1(r)\big)' \\ &= B_3 r^{-1} + O(r), \end{aligned}$$

where B_1, B_2 are constants and $B_3 = -4\pi^{-1}C^2$. The remaining part of the proof is similar to this of Theorem 1.

7. In this Section the construction of the fundamental solution of

$$(1''')\qquad \Delta^2 u(X)+C^4 u(X) = 0, \quad X(x_1, \ldots, x_n), \quad C \text{ const}$$

will be given for $n > 3$.

Equation (1''') is obtained from (33) when replacing C by $C\sqrt{i}$.

We shall distinguish eight cases

1°A) $s = 2p+1 = 8q+1$ for $p = 4q$,

1°B) $s = 2p+1 = 8q+3$ for $p = 4q+1$,

1°C) $s = 2p+1 = 8q+5$ for $p = 4q+2$,

1°D) $s = 2p+1 = 8q+7$ for $p = 4q+3$,

2°A) $s = 2p = 4q$ for $p = 2q$,

2°B) $s = 2p = 4q+2$ for $p = 2q+1$,

3°A) $s = q+\frac{1}{2} = 2p+\frac{1}{2}$ for $q = 2p$,

3°B) $s = q+\frac{1}{2} = 2p+\frac{3}{2}$ for $q = 2p+1$.

Ad 1°A). By (38)

$$(58)\qquad Y_s(Cr\sqrt{i}) = -\frac{1}{\pi}(8q)!\left(\frac{Cr\sqrt{i}}{2}\right)^{-8q-1} - \frac{1}{\pi}(8q-1)!\left(\frac{Cr\sqrt{i}}{2}\right)^{-8q} - \\ -\frac{1}{\pi}\sum_{k=2}^{8q}\frac{(8q-k)!}{k!}\left(\frac{Cr\sqrt{i}}{2}\right)^{2k-8q-1} + \frac{2}{\pi}J_s(Cr\sqrt{i})\ln\frac{Cr\sqrt{i}}{2} - \\ -\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big).$$

In virtue of (37) and (58) the function

$$F_1(r) = (Cr\sqrt{i})^{-8q-1}Y_s(Cr\sqrt{i}) \\ = -\frac{1}{\pi}(8q)!2^{8q+1}(Cr\sqrt{i})^{-16q-2} - \frac{1}{\pi}(8q-1)!2^{8q-1}(Cr\sqrt{i})^{-16q} + W_1(r) \\ = \frac{1}{\pi}(8q)!2^{8q+1}(Cr)^{-16q-2}i - \frac{1}{\pi}(8q-1)!2^{8q-1}(Cr)^{-16q} + W_1(r),$$

where

$$W_1(r) = (Cr\sqrt{i})^{-8q-1}\left(-\frac{1}{\pi}\sum_{k=2}^{8q}\frac{(8q-k)!}{k!}\left(\frac{Cr\sqrt{i}}{2}\right)^{2k-8q-1}\right) + \\ + \frac{2}{\pi}J_s(Cr\sqrt{i})\ln\frac{Cr\sqrt{i}}{2} - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big) \\ = O(r^{6-n})$$

and the function

$$(59)\qquad U_1(r) = -\operatorname{Re} F_1(r) = \frac{1}{\pi}(8q-1)!\,2^{8q-1}(Cr)^{-16q} + \operatorname{Re} W_1(r)$$

are solutions of (1‴).

Ad 1°B). The function

$$F_2(r) = (Cr\sqrt{i})^{-8q-3} Y_s(Cr\sqrt{i})$$

$$= -\frac{1}{\pi}(8q+2)!\,2^{8q+3}(Cr)^{-16q-6} i + \frac{1}{\pi}(8q+1)!\,2^{8q+1}(Cr)^{-16q-4} + W_2(r),$$

where

$$W_2(r)$$

$$= (Cr\sqrt{i})^{-8q-3}\left(\sum_{k=0}^{8q+2} \frac{(8q+2-k)!}{k!}\left(\frac{Cr\sqrt{i}}{2}\right)^{2k-8q-3} + \frac{2}{\pi} J_s(Cr\sqrt{i}) \ln\frac{Cr\sqrt{i}}{2} - \right.$$

$$\left. - \sum_{k=0}^{\infty} \frac{1}{\pi}\cdot\frac{(-1)^k (Cr\sqrt{i}/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(s+k+1)\big)\right) = O(r^{6-n})$$

and the function

$$(60)\qquad U_2(r) = \operatorname{Re} F_2(r) = \frac{1}{\pi}(8q+1)!\,2^{8q+1}(Cr)^{-16q-4} + \operatorname{Re} W_2(r)$$

are solutions of (1‴).

Ad 1°C). The function

$$F_3(r) = (Cr\sqrt{i})^{-8q-5} Y_s(Cr\sqrt{i})$$

$$= \frac{1}{\pi}(8q+4)!\,2^{8q+5}(Cr)^{-16q-10} i - \frac{1}{\pi}(8q+3)!\,2^{8q+3}(Cr)^{-16q-8} + W_3(r),$$

where

$$W_3(r)$$

$$= (Cr\sqrt{i})^{-8q-5}\left(\sum_{k=2}^{8q+4} \frac{(8q+4-k)!}{k!}\left(\frac{Cr\sqrt{i}}{2}\right)^{2k-8q-5}\right) + \frac{2}{\pi} J_s(Cr\sqrt{i}) \ln\frac{Cr\sqrt{i}}{2} -$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k (Cr\sqrt{i}/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big) = O(r^{6-n}),$$

and the function

$$(61)\qquad U_3(r) = -\operatorname{Re} F_3(r) = \frac{1}{\pi}(8q+3)!\,2^{8q+3}(Cr)^{-16q-8} + \operatorname{Re} W_3(r)$$

are solutions of (1‴).

Ad 1°D). The function

$$F_4(r) = (Cr\sqrt{i})^{-8q-7}Y_s(Cr\sqrt{i})$$

$$= -\frac{1}{\pi}(8q+6)!2^{8q+7}(Cr)^{-16q-14}i+\frac{1}{\pi}(8q+5)!2^{8q+5}(Cr)^{-16q-12}+W_4(r),$$

where

$$W_4(r)$$

$$= (Cr\sqrt{i})^{-8q-7}\left(\sum_{k=2}^{8q+6}\frac{(8q+6-k)!}{k!}\left(\frac{Cr\sqrt{i}}{2}\right)^{2k-8q-7}\right)+\frac{2}{\pi}J_s(Cr\sqrt{i})\ln\frac{Cr\sqrt{i}}{2}-$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big) = O(r^{6-n})$$

and the function

$$U_4(r) = \operatorname{Re}F_4(r) = \frac{1}{\pi}(8q+5)!2^{8q+5}(Cr)^{-16q-12}+\operatorname{Re}W_4(r) \tag{62}$$

are solutions of (1‴).

THEOREM 8. *Let* $n>3$. *If* $s=(n-2)/2 = 8q+1$, $8q+3$, $8q+5$, $8q+7$ *then the functions* (59), (60), (61) *and* (62) *are respectively fundamental solutions of* (1‴).

Proof. We shall only consider the case $s=8q+1$, the remaining ones are worked out similarly. By (59)

$$U_1(r) = C_1r^{4-n}+O(r^{6-n}),\qquad U_1'(r) = C_2r^{3-n}+O(r^{5-n}),$$

$$\Delta U_1(r) = C_3r^{2-n}+O(r^{4-n}),\qquad \frac{d}{dr}\Delta U_1(r) = C_4r^{1-n}+O(r^{3-n}),$$

where C_1, C_2, C_3 are constants and $C_4 = 2^{(n+2)/2}C^{4-n}\big((n-2)/2\big)!$

The proof runs down like this of Theorem 4.

The next theorem will deal with the case 2°A) and 2°B).

If $s=2p=4q$ then the function

$$F_5(r) = (Cr\sqrt{i})^{-4q}Y_s(Cr\sqrt{i})$$

$$= -\frac{1}{\pi}(4q-1)!2^{4q}(Cr)^{-8q}-\frac{1}{\pi}(4q-2)!2^{4q-2}(Cr)^{-8q+2}i+W_5(r),$$

and the function

$$U_5(r) = -\operatorname{Im}F_5(r) = \frac{1}{\pi}(4q-2)!2^{4q-2}(Cr)^{-8q+2}-\operatorname{Im}W_5(r), \tag{63}$$

where

$$W_5(r)$$

$$= (Cr\sqrt{i})^{-4q}\left(\sum_{k=2}^{s-1}\frac{(s-k-1)!}{k!}\left(\frac{Cr\sqrt{i}}{2}\right)^{2k-s}+\frac{2}{\pi}J_s(Cr\sqrt{i})\ln\frac{Cr\sqrt{i}}{2}-\right.$$

$$\left.-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big)\right) = O(r^{6-n})$$

are solutions of (1''').

If $s = 2p = 4q+2$ the function

$$F_6(r) = (Cr\sqrt{i})^{-4q-2}Y_s(Cr\sqrt{i})$$

$$= \frac{1}{\pi}(4q+1)!2^{4q+2}(Cr)^{-8q-4}+\frac{1}{\pi}(4q)!2^{4q}(Cr)^{-8q-2}i+W_6(r),$$

where

$$W_6(r)$$

$$= (Cr\sqrt{i})^{-4q-2}\left(\frac{1}{\pi}\sum_{k=2}^{s-1}\frac{(s-k-1)!}{k!}\left(\frac{Cr\sqrt{i}}{2}\right)^{2k-s}\right)+\frac{2}{\pi}J_s(Cr\sqrt{i})\ln\frac{Cr\sqrt{i}}{2}-$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{s+2k}}{k!(s+k)!}\big(f(k+1)+f(k+s+1)\big) = O(r^{6-n})$$

and the function

$$U_6(r) = \operatorname{Im}F_6(r) = \frac{1}{\pi}(4q)!2^{4q}(Cr)^{-8q-2}+\operatorname{Im}W_6(r) \tag{64}$$

are solutions of (1''').

THEOREM 9. *Let $n > 3$. If $s = (n-2)/2 = 4q$ or $s = (n-2)/2 = 4q+2$ then the functions* (63) *and* (64) *are respectively fundamental solutions of* (1''').

Proof. We shall only consider the case 2°A), the second one being analogous. By (63)

$$U_5(r) = D_1r^{4-n}+O(r^{6-n}),\qquad U_5'(r) = D_2r^{3-n}+O(r^{5-n}),$$

$$\Delta U_5(r) = D_3r^{2-n}+O(r^{4-n}),\qquad \frac{d}{dr}\Delta U_5(r) = D_4r^{1-n}+O(r^{3-n}),$$

where D_1, D_2, D_3 are constants,

$$D_4 = \big((n-6)/2\big)!2^{(n-4)/2}C^{4-n}(4-n)(2-n)$$

and the proof is analogous to this of Theorem 4.

Let now $s = q+\frac{1}{2} = 2p+\frac{1}{2}$. The function

$$z^{-s}Y_s(z) = \frac{z^{-2s}2^s}{\Gamma(1)\Gamma(1-s)} + \frac{z^{-2s+2}2^{s-2}}{\Gamma(2)\Gamma(2-s)} + z^{-s}\sum_{k=2}^{\infty}\frac{(-1)^k z^{2k-s}}{\Gamma(k+1)\Gamma(k-s+1)}$$

is a solution of (36), whence the function

$$F_6(r) = (Cr\sqrt{i})^{-s}Y_s(Cr\sqrt{i})$$
$$= \frac{2^s}{\Gamma(1)\Gamma(1-s)}(Cr\sqrt{i})^{-2q-1} - \frac{2^{s-2}}{\Gamma(2)\Gamma(2-s)}(Cr\sqrt{i})^{1-2q} + W_6(r),$$

where

$$W_6(r) = (Cr\sqrt{i})^{-q-1/2}\sum_{k=2}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{2k-q-1/2}}{\Gamma(k+1)\Gamma(k-s+1)} = O(r^{6-n})$$

and the function

$$U_6(r) = (-1)^p\mathrm{Re}F_6(r)$$
$$= -\frac{2^{s-1/2}(Cr)^{-4p-1}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{s-5/2}(Cr)^{1-4p}}{\Gamma(2)\Gamma(2-s)} + (-1)^p\mathrm{Re}W_6(r)$$

satisfy (1‴). Also the function

$$F_7(r) = (Cri\sqrt{i})^{-s}Y_s(Cri\sqrt{i})$$
$$= \frac{2^s(Cr)^{-4p-1}}{\Gamma(1)\Gamma(1-s)}i^{(3/2)(-4p-1)} + \frac{2^{s-2}(Cr)^{-4p+1}}{\Gamma(2)\Gamma(2-s)}i^{(3/2)(1-4p)} + W_7(r),$$

where

$$W_7(r) = (Cri\sqrt{i})^{-s}\sum_{k=2}^{\infty}\frac{(-1)^k(Cri\sqrt{i}/2)^{2k-s}}{\Gamma(k+1)\Gamma(k-s+1)} = O(r^{6-n})$$

and the function

$$U_7(r) = (-1)^p\mathrm{Im}F_7(r) = \frac{2^{s-1/2}(Cr)^{-4p-1}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{s-5/2}(Cr)^{1-4p}}{\Gamma(2)\Gamma(2-s)} + (-1)^p\mathrm{Im}W_7(r)$$

are solutions of (1‴). If follows that the function

$$U(r) = U_6(r)+U_7(r) = \frac{2^{s-3/2}(Cr)^{1-4p}}{\Gamma(2)\Gamma(2-s)} + (-1)^p\big(\mathrm{Re}W_6(r)+\mathrm{Im}W_7(r)\big) \tag{65}$$

is a solution of (1‴).

Let $s = q+\frac{1}{2} = 2p+\frac{3}{2}$, the function

$$F_8(r) = (Cr\sqrt{i})^{-s}Y_s(Cr\sqrt{i})$$
$$= \frac{2^s(Cr)^{-4p-1}(\sqrt{i})^{-4p-1}}{\Gamma(1)\Gamma(1-s)} - \frac{2^{s-2}(Cr)^{1-4p}(\sqrt{i})^{1-4p}}{\Gamma(2)\Gamma(2-s)} + W_8(r),$$

where

$$W_8(r) = (Cr\sqrt{i})^{-s}\sum_{k=2}^{\infty}\frac{(-1)^k(Cr\sqrt{i}/2)^{2k-s}}{\Gamma(k+1)\Gamma(k+s+1)} = O(r^{6-n})$$

and the function

$$U_8(r) = (-1)^p\operatorname{Re}F_8(r)$$
$$= \frac{2^{s-1/2}(Cr)^{-4p-3}}{\Gamma(1)\Gamma(1-s)} + \frac{2^{s-5/2}(Cr)^{-4p-1}}{\Gamma(2)\Gamma(2-s)} + (-1)^p\operatorname{Re}W_8(r)$$

are solutions of (1‴). Solutions of (1‴) are also the function

$$F_9(r) = (Cri\sqrt{i})^{-s}Y_s(Cri\sqrt{i})$$
$$= \frac{2^s(Cr)^{-4p-3}}{\Gamma(1)\Gamma(1-s)}i^{(3/2)(-4p-3)} - \frac{2^{s-2}(Cr)^{-4p-1}}{\Gamma(2)\Gamma(2-s)}i^{(3/2)(-4p-1)} + W_9(r),$$

where

$$W_9(r) = (Cri\sqrt{i})^{-s}\sum_{k=2}^{\infty}\frac{(-1)^k(Cri\sqrt{i}/2)^{2k-s}}{\Gamma(k+1)\Gamma(k-s+1)} = O(r^{6-n})$$

and the function

$$U_9(r) = (-1)^p\operatorname{Re}F_9(r)$$
$$= \frac{2^{s-1/2}(Cr)^{-4p-3}}{\Gamma(1)\Gamma(1-s)} - \frac{2^{s-5/2}(Cr)^{-4p-1}}{\Gamma(2)\Gamma(2-s)} + (-1)^p\operatorname{Re}W_9(r).$$

Hence the function

$$(66)\quad U(r) = U_8(r) - U_9(r) = \frac{2^{s-3/2}(Cr)^{-4p-1}}{\Gamma(2)\Gamma(2-s)} + (-1)^p\big(\operatorname{Re}W_8(r) - \operatorname{Re}W_9(r)\big)$$

is a solution of (1‴).

Now

$$U(r) = E_1r^{4-n} + O(r^{6-n}),\qquad U'(r) = E_2r^{3-n} + O(r^{5-n}),$$

$$\Delta U(r) = E_3r^{2-n} + O(r^{4-n}),\qquad \frac{d}{dr}\Delta U(r) = E_4r^{1-n} + O(r^{3-n}),$$

E_1, E_2, E_3 being constants, and $E_4 = \dfrac{2^{n/2}C^{2-n}(2-n)}{\Gamma(1)\Gamma\big((4-n)/2\big)}$.

By the already familiar procedure we may prove the

THEOREM 10. *Let $n > 3$. If $s = 2^{-1}(n-2) = 2p+1$ or $s = (n-2)/2 = 2p+\frac{3}{2}$, then the functions* (65) *and* (66) *are respectively fundamental solutions of* (1‴).

8. In this Section we shall apply the fundamental solutions $U(r)$ of (1) to the boundary problems of Lauricelli ([2]) and of Riquier ([2]) for the equation (1).

Let $H(X, Y)$ be a function of X, Y defined in $\bar{D}\times\bar{D}$ and satisfying the conditions

1. $H(X, Y)\in C^{(4)}$ for $(X, Y)\in D\times D$,
2. $H(X, Y)\in C^{(3)}$ for $(X, Y)\in \bar{D}\times\bar{D}$,
3. $H(X, Y) = U(r)$ for $Y\in\partial D$,
4. $\dfrac{dH}{dn_Y} = \dfrac{dU}{dn_Y}$ for $Y\in\partial D$,
5. As a function of Y, $H(X, Y)$ satisfies the equation

$$\Delta^2 H(X, Y)+kH(X, Y) = 0, \qquad Y\in D.$$

Definition 2. The function

$$G(X, Y) = U(r)-H(X, Y) \tag{67}$$

is called the *Green function of type* (L) for the equation (1) in the (bounded) domain D.

We shall now give a formula solving the Lauricelli problem for the equation (1). Taking in the fundamental formula

$$\iiint\limits_{D} (u\Delta^2 v - v\Delta^2 u)\,dx_1\ldots dx_n + \iint\limits_{\partial D}\left(\Delta u\frac{dv}{dn} - v\frac{d\Delta u}{dn} + u\frac{d\Delta v}{dn} - \Delta v\frac{du}{dn}\right)dS_Y = 0 \tag{68}$$

a solution of class $C^{(4)}$ of (1) in D as $u(Y)$, and as $v(Y)$ the function $\dfrac{1}{a_n}H(X, Y)$ we obtain from (68) and by identity $u\Delta^2 H - H\Delta^2 u = 0$

$$0 = \frac{1}{a_n}\iint\limits_{\partial D}\left(\Delta u\frac{dH}{dn} - H\frac{d\Delta U}{dn} + U\frac{d\Delta u}{dn} - \Delta u\frac{dU}{dn}\right)dS_Y. \tag{69}$$

Upon setting $v(Y)$ as $u(Y)$ into (6a) we get

$$u(X) = \frac{1}{a_n}\iint\limits_{\partial D}\left(\Delta U\frac{du}{dn} - u\frac{d\Delta U}{dn} + U\frac{d\Delta u}{dn} - \Delta u\frac{dU}{dn}\right)dS_Y. \tag{70}$$

If we add formulas (69) and (70) then we have

$$u(X) = \frac{1}{a_n}\iint\limits_{\partial D}\left((\Delta U-\Delta H)\frac{du}{dn} + \left(\frac{d\Delta H}{dn} - \frac{d\Delta U}{dn}\right)u + \left(\frac{dH}{dn} - \frac{dU}{dn}\right)\Delta u + (U-H)\frac{d\Delta u}{dn}\right)dS_Y.$$

By conditions 3 and 4 and by Definition 2 we obtain

$$u(X) = \frac{1}{a_n} \iint_{\partial D} \left(\Delta G \frac{du}{dn} - \frac{d\Delta G}{dn} u \right) dS_Y. \tag{71}$$

THEOREM 11. *Suppose that there exists Green function of type* (L). *Let the functions $f(Y)$ and $h(Y)$ be continuous on ∂D, the boundary of D, then the function*

$$u(X) = \frac{1}{a_n} \iint_{\partial D} \left(h(Y) \Delta G(X, Y) - f(Y) \frac{d\Delta G(X, Y)}{dn_Y} \right) dS_Y$$

solves the boundary problem of Lauricelli:

$$u(X) = f(Y), \quad \frac{du(Y)}{dn_Y} = h(Y) \quad \textit{for} \quad Y \in \partial D.$$

The Green function for the Riquier (R) problem is defined as follows. Let $H_1(X, Y)$ be a function of X, Y defined in $D \times D$ and satisfying conditions:

6. $H_1(X, Y) \in C^{(4)}$ for $(X, Y) \in D \times D$,
7. $H_1(X, Y) \in C^{(3)}$ for $(X, Y) \in \bar{D} \times \bar{D}$,
8. $\Delta H_1 = \Delta U(r)$ for $Y \in \partial D$,
9. $\dfrac{d\Delta H_1}{dn_Y} = \dfrac{d\Delta U}{dn_Y}$ for $Y \in \partial D$,
10. As a function of Y, H_1 satisfies the equation (1).

DEFINITION 3. The function $G_1(X, Y) = U(r) - H_1(X, Y)$ is called the *Green function of type* (R) for the equation (1) in the (bounded) domain D.

Analogously as above

$$u(X) = -\frac{1}{a_n} \iint_{\partial D} \left(G_1(X, Y) \frac{d\Delta u}{dn} - \frac{dG_1(X, Y)}{dn} \Delta u \right) dS_Y$$

which implies

THEOREM 12. *Suppose that there exists Green function of type* (R). *If $f_1(Y)$ and $h_1(Y)$ are continuous functions on ∂D then the formula*

$$u(X) = -\frac{1}{a_n} \iint_{\partial D} \left(h_1(Y) \frac{dG_1(X, Y)}{dn_Y} - \frac{d\Delta G_1(X, Y)}{dn_Y} f_1(Y) \right) dS_Y$$

solves the equation (1) *with the Riquier boundary values*

$$u(Y) = f_1(Y), \quad \Delta u(Y) = h_1(Y) \quad \textit{for} \quad Y \in \partial D.$$

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