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Remarks on Φ-operators in linear topological spaces

Let X, Y be linear topological spaces. Let A be a linear operator defined on a subset of X with values in Y. The operator A is called *closed* if $x_n \to x$ and $Ax_n \to y$ implies that $Ax = y$. Obviously any continuous operator is closed, but there are closed operators which are not continuous. A closed operator is called *normally resolvable*, cf. [2], if the set E_A of values of A is closed. Let

$$Z_A = \{x \in X\colon\ Ax = 0\}, \qquad \alpha_A = \dim Z_A, \qquad \beta_A = \dim Y/E_A$$

(α_A and β_A may be equal to ∞). The pair of numbers (α_A, β_A) is called the *d-characteristic* of operator A. We say that operator A has a *finite* d-characteristic if α_A and β_A are both finite. By the *index* of operator A of a finite d-characteristic we mean the number

$$\varkappa_A = \beta_A - \alpha_A .$$

A normally resolvable operator with a finite d-characteristic is called a *Φ-operator*, cf. [2]. The definition of Φ-operators in Banach spaces and their basic properties are given by I. C. Gochberg and M. G. Krein in [2].

The note contains a review of the results of Gochberg and Krein which can be generalized to the case of locally bounded spaces, and some examples which show that these results are not true for locally convex (but not locally bounded) spaces.

Let X be a linear topological space. A set Z is called *bounded* if for every sequence of numbers $t_n \to 0$ and every sequence z_n of elements of Z the sequence $t_n z_n$ tends to 0. X is called *locally bounded* if it contains a bounded neighbourhood of zero. The norm is called *p-homogeneous* if $\|tx\| = |t|^p \|x\|$ $(0 < p < 1)$. If X is locally bounded, then there is a p-homogeneous norm in X determining the topology, cf. [1], [3]. Basing ourselves on this fact and on the theory of locally bounded algebras given by W. Żelazko ([5]), we can transfer main results of Gochberg and Krein to the case of locally bounded spaces.

It is easy to see that the proof of Theorem 2.1 of [2] has an algebraic character and we can formulate it here as follows:

THEOREM 1. *Let X, Y be linear spaces. Let A, B be operators with finite d-characteristics, let A map Y into Y and B map X into Y and let A be determined on the whole of Y. Then AB is also an operator with a finite d-characteristic and*

$$\varkappa_{AB} = \varkappa_A + \varkappa_B.$$

If we add the assumption that X and Y are topological linear spaces, we can replace the assumption that A is defined in the whole space by the assumption that A and B are Φ-operators and A is defined in a dense set in Y. This follows from the fact that the following lemma, formulated in [2] for Banach spaces, is also true in general linear topological spaces.

LEMMA. *Let X be a linear topological space and let X be equal to a direct sum $X = R \oplus F$, where F is finite dimensional. Let D be a linear set dense in X. Then the set $D_1 = D \cap R$ is dense in R and we can represent X as a direct sum $X = R \oplus F'$, where $F' \subset D$.*

Let X and Y be locally bounded spaces. Let $\| \; \|_X$, $\| \; \|_Y$ be p-homogeneous norms. Such norms exist, cf. [1], [3], and without loss of generality we can assume that p is identical for both spaces. Let A be a linear continuous operator transforming X into Y. We define the norm of the operator

$$\|A\| = \sup_{\|x\|_X \leqslant 1} \|Ax\|_Y.$$

Obviously it is p-homogeneous and

$$\|Ax\| \leqslant \|A\| \cdot \|x\|.$$

In the same way as Theorem 2.2 of [2] we obtain

THEOREM 2. *Let X, Y be complete locally bounded spaces. Let A be a Φ-operator. Then there is a constant ϱ such that for every B such that $\|B\| < \varrho$ the operator $A+B$ has also a finite d-characteristic and*

$$\varkappa_{A+B} = \varkappa_A.$$

Let X, Y be linear topological spaces. A linear operator T mapping X into Y is called *compact* if there is a neighbourhood $U \subset X$ such that TU is compact. J. H. Williamson ([4]) has shown that if T is a compact operator transforming X into itself, then the operator $I+T$, where I denotes identity, is a Φ-operator and $\varkappa_{I+T} = 0$.

It implies (as in [2]) that in Theorem 2 we can replace the norm $\| \; \|$ by the norm

$$\|B\|_C = \inf\{\|B+T\| : T \text{ is a compact operator}\}.$$

As a particular case we obtain

$$\varkappa_{A+T} = \varkappa_A$$

for each compact operator T.

Ler T be a fixed operator transforming X into itself. We consider the operator $T - \lambda I$, where λ is a complex number. Let

$$\Lambda = \{\lambda\colon\ T - \lambda I \text{ is a } \Phi\text{-operator}\}.$$

Theorem 2 implies that if X is a locally bounded space, then the set Λ is open and the index $\varkappa_{T-\lambda I}$ is constant on each component. With this fact also the following theorem is connected:

THEOREM 3. *Let X be a locally bounded space. Let T be an arbitrary operator. If for all complex λ the operator $T - \lambda I$ is a Φ-operator, then X is a finite dimensional space.*

The proof is analogous to that of the corresponding theorem in [2]. It is only necessary to use some results about locally bounded algebras, cf. [4]. However, without the assumption of local boundedness these considerations are not valid (Example 1 below).

Similarly to Theorem 3.3 of [2] we have

THEOREM 4. *Let X be a locally bounded space. Let $\Lambda = \{\lambda\colon\ T - \lambda I$ is a Φ-operator$\}$. The functions $\alpha_{T-\lambda I}$, $\beta_{T-\lambda I}$ are constant on each component of Λ with the exception of an isolated set of points at which these functions are upper semicontinuous.*

Example 2 shows that Theorem 4 is not true in the case of B_0-spaces.

Example 1. Let $C(D)$ be the space of all continuous complex-valued functions $x(z)$ defined on an open set D of the complex plane, with the compact-open topology, i.e. the topology given by uniform convergence on compact sets. In this topology $C(D)$ is a B_0-space, i.e. a complete metrisable locally convex space. Let $(Tx)(z) = zx(z)$. Obviously the operator $T - \lambda I$ is invertible for $\lambda \notin D$. For $\lambda \in D$, the operator $T - \lambda I$ is not a Φ-operator. Therefore Λ is equal to the complement of D and is closed.

Example 2. Let $X = H(C_0)$ be the space of all analytic functions defined in the unit circle C_0 with the compact-open topology. X is a B_0-space. Let $Tx(z) = zx(z)$; then $T - \lambda I$ is a Φ-operator. Thus Theorem 3 cannot be extended to the case of locally convex spaces. Moreover, in this example the index is not continuous with respect to λ. Indeed

$$\varkappa_{T-\lambda I} = \begin{cases} 0 & \text{for} \quad |\lambda| \geqslant 1, \\ 1 & \text{for} \quad |\lambda| < 1. \end{cases}$$

The following question arises: Is Theorem 3 true for locally convex spaces if we assume that $T-\lambda I$ is a Φ-operator and also that $\varkappa_{T-\lambda I} = 0$? The following example shows that the answer is negative.

Example 3. Let C_0^∞ be the space of infinitely many differentiable functions $x(t)$ defined for $0 \leqslant t \leqslant 1$ and such that $x^{(n)}(0) = 0$ $(n = 0, 1, 2, \ldots)$ with the topology determined by the pseudonorms

$$\|x\|_n = \sup_{0 \leqslant t \leqslant 1} |x^{(n)}(t)|.$$

Let $Tx = \dfrac{dx}{dt}$. Then $\varkappa_{T-\lambda I} = 0$ and C_0^∞ is infinite-dimensional.

References

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