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Remark on a difference equation

Introduction. In the present note we are concerned with the difference equation

(1) $$g(x+1)-g(x) = \varphi(x),$$

where $g(x)$ is the unknown function. W. Krull ([3]) and, independently, the author of the present paper ([4]) have proved the following theorem:

THEOREM 1. *If the function $\varphi(x)$ is concave in an interval (a, ∞), $a \geqslant -\infty$, and fulfils the condition*

(2) $$\lim_{n\to\infty}[\varphi(n+1)-\varphi(n)] = 0,$$

then for every $y_0 \epsilon (-\infty, \infty)$ there exists exactly one convex function $g(x)$ satisfying equation (1) in (a, ∞) and fulfilling the initial condition

(3) $$g(x_0) = y_0, \quad x_0 \epsilon (a, \infty).$$

This function is given by the formula:

(4) $$g(x) = y_0+(x-x_0)\varphi(x_0)- \\ -\sum_{n=0}^{\infty}\{\varphi(x+n)-\varphi(x_0+n)-(x-x_0)[\varphi(x_0+n+1)-\varphi(x_0+n)]\}.$$

In fact, the above theorem has been proved in [3], [4] for $a = 0$ and $x_0 = 1$, but the same argument applies also in the general case (cf. also [5]).

Recently we have proved a theorem ([8], theorem 11.1; cf. also [6], [7]) of which the following is a particular case:

THEOREM 2. *If the function $\varphi(x)$ is defined almost everywhere (a. e.) in (a, ∞), monotonic* [1] *and fulfils the condition*

(5) $$\lim_{x\to\infty}\varphi(x) = 0,$$

(1) When we are dealing with functions defined a.e., monotonic means monotonic on the set where the function in question is defined. Similarly the limit of such a function will be understood as the limit in the set where the function is defined ([9], p. 112). A monotonic function will always be understood as weakly monotonic, and the same remark applies to convex (concave) functions.

then there exists a one-parameter family (with the additive constant y_0) of monotonic functions $g(x)$ which are defined a. e. in (a, ∞) and satisfy equation (1) *a. e. in (a, ∞). These functions are given by the formula*

$$g(x) = y_0 - \sum_{n=0}^{\infty} \{\varphi(x+n) - \varphi(x_0+n)\}, \tag{6}$$

where x_0 is a point from (a, ∞). The family (6) *is unique in the sense that if a monotonic function $\tilde{g}(x)$ is defined and satisfies equation* (1) *a. e. in (a, ∞), then $\tilde{g}(x)$ is equal a. e. in (a, ∞) to one of the functions* (6).

In the present note we are going to prove that Theorems 1 and 2 are equivalent, by which we mean that each can be directly derived from the other. The implication Theorem 2 $\to$ Theorem 1 seems to be of interest, for if one needs for some purpose both theorems, then it would be reasonable to prove Theorem 2 (and rather a more general theorem, as in [8]) and then deduce Theorem 1 from it, instead of presenting the longer independent proof. The converse implication seems to be less interesting, for the direct argument gives a stronger result than Theorem 2 above.

Throughout the paper we shall make frequent use of the properties of convex and concave functions. These properties are proved, for example, in [1] (p. 41-55) and [2].

§ 1. We begin by proving some lemmas.

LEMMA 1. *Let $\sum_{n=0}^{\infty} f_n(x)$ be a convergent series of convex (concave) functions in an interval (a, b), and suppose that the series $\sum_{n=0}^{\infty} f_n'(x)$ converges absolutely a.e. in (a, b). Then*

$$\Big(\sum_{n=0}^{\infty} f_n(x)\Big)' = \sum_{n=0}^{\infty} f_n'(x) \qquad \text{a.e. in } (a, b). \tag{7}$$

Proof. We take α, β, $a < \alpha < \beta < b$, such that the series $\sum_{n=0}^{\infty} |f_n'(\alpha)|$ and $\sum_{n=0}^{\infty} |f_n'(\beta)|$ converge. The functions

$$F_n(x) = f_n(x) + x\big(|f_n'(\alpha)| + |f_n'(\beta)|\big)$$

are increasing in $\langle \alpha, \beta\rangle$, since

$$F_n'(x) = f_n'(x) + |f_n'(\alpha)| + |f_n'(\beta)| \geqslant 0 \qquad \text{a.e. in } \langle \alpha, \beta\rangle$$

(the derivative of a convex or concave function is monotonic) and the functions $F_n'(x)$, being convex (concave) just like $f_n(x)$, are absolutely continuous ([2], p. 15). On account of the theorem of Fubini ([9], p. 403)

$$\Big(\sum_{n=0}^{\infty} F_n(x)\Big)' = \sum_{n=0}^{\infty} F_n'(x) \quad \text{a.e. in } \langle a, \beta\rangle,$$

i.e.

$$\Big(\sum_{n=0}^{\infty} f_n(x) + x\sum_{n=0}^{\infty}\big(|f_n'(a)| + |f_n'(\beta)|\big)\Big)'$$
$$= \sum_{n=0}^{\infty} f_n'(x) + \sum_{n=0}^{\infty}\big(|f_n'(a)| + |f_n'(\beta)|\big) \quad \text{a.e. in } \langle a, \beta\rangle.$$

Hence

$$\Big(\sum_{n=0}^{\infty} f_n(x)\Big)' = \sum_{n=0}^{\infty} f_n'(x) \quad \text{a.e. in } \langle a, \beta\rangle.$$

Letting $a \to a$, $\beta \to b$, we obtain (7).

LEMMA 2. *If $\varphi(x)$ is a monotonic function, defined a.e. in an interval (a, ∞) and fulfilling condition* (5), *then the series* (2)

$$(8) \qquad \sum_{n=0}^{\infty}\Big[\varphi(x+n) - \int_n^{n+1} \varphi(x_0+t)\,dt\Big], \qquad x, x_0 \in (a, \infty),$$

converges absolutely a.e. in (a, ∞).

Proof. For an arbitrary set $A \subset (a, \infty)$ and an integer k we denote by $A+(k)$ the set of numbers $x+k$ where x ranges over the set A.

Let B be the set of $x \in (a, \infty)$ at which the function $\varphi(x)$ is not defined and put

$$(9) \qquad E = (a, \infty) - \bigcup_{k=-\infty}^{+\infty} [B+(k)].$$

The set $(a, \infty) - E$ has measure zero. We shall show that series (8) converges absolutely in the set E.

We take an arbitrary $x \in E$. Then $\varphi(x+n)$ is defined for every n. For the argument's sake we assume that $\varphi(x)$ is decreasing and $x \leqslant x_0$. Then

$$\varphi(x_0+n+1-0) \leqslant \int_n^{n+1} \varphi(x_0+t)\,dt \leqslant \varphi(x_0+n+0) \text{ (3)}$$

(2) Here and in the sequel the integral is always taken in the Lebesgue sense)

(3) In this and the following formulae we must take $\varphi(x_0+n+0)$ $\big(= \lim_{\xi\to x_0+n+0} \varphi(\xi)\big)$ instead of $\varphi(x_0+n)$, because the function φ need not be defined at the point x_0+n.

and

$$0 \leqslant \varphi(x+n)-\varphi(x_0+n+0) \leqslant \varphi(x+n)-\int_n^{n+1} \varphi(x_0+t)\,dt$$
$$\leqslant \varphi(x+n)-\varphi(x_0+n+1-0).$$

Thus it is enough to show that the series

(10) $$\sum_{n=0}^{\infty} [\varphi(x+n)-\varphi(x_0+n+1-0)]$$

converges. Let N be an integer such that $x+N > x_0+1$. Consequently

(11) $$\varphi(x+n)-\varphi(x_0+n+1-0) \leqslant \varphi(x+n)-\varphi(x+N+n).$$

Now

(12) $$\sum_{n=0}^{\infty} [\varphi(x+n)-\varphi(x+N+n)]$$
$$= \sum_{k=0}^{N-1} \sum_{n=0}^{\infty} [\varphi(x+k+n)-\varphi(x+k+1+n)].$$

The series

$$\sum_{n=0}^{\infty} [\varphi(x+k+n)-\varphi(x+k+1+n)], \qquad k = 0, 1, \ldots, N-1,$$

converge in view of (5). This shows, according to (12) and (11), that series (10) converges, which was to be proved.

§ 2. Now we assume that Theorem 1 holds true and we shall prove Theorem 2.

Let $\varphi(x)$ be a monotonic function, defined a.e. in an interval (a, ∞) and fulfilling condition (5). We may assume that $\varphi(x)$ is decreasing, for otherwise we consider the equation

$$-g(x+1)-[-g(x)] = -\varphi(x)$$

instead of equation (1). Consequently, according to (5), the function $\varphi(x)$ must be non-negative.

We put

$$\Phi(x) \stackrel{\text{df}}{=} \int_{x_1}^{x} \varphi(t)\,dt, \qquad x_1 \in (a, \infty).$$

The function $\Phi(x)$ is defined and concave in (a, ∞). By (5) we get

$$0 \leqslant \Phi(n+1)-\Phi(n) = \int_n^{n+1} \varphi(t)\,dt \leqslant \varphi(n+0) \to 0 \quad \text{as} \quad n \to \infty.$$

On account of Theorem 1 there exists a convex function $G(x)$, satisfying in (a, ∞) the equation

$$G(x+1)-G(x) = \Phi(x) \tag{13}$$

and we have

$$G(x) = Y+(x-x_0)\Phi(x_0) - \\ -\sum_{n=0}^{\infty}\{\Phi(x+n)-\Phi(x_0+n)-(x-x_0)[\Phi(x_0+n+1)-\Phi(x_0+n)]\},$$

Y being a constant.

Let B be the set of $x\epsilon(a, \infty)$ at which the function $G(x)$ is not differentiable. B is a set of measure zero. Define the set E by formula (9). Thus $(a, \infty)-E$ has measure zero and for every $x\epsilon E$ the functions $G(x)$, $G(x+1)$ and (in view of (13)) also $\Phi(x)$ are differentiable. Hence it follows that the functions [4]

$$g(x) = G'(x)-G'(x_0)+y_0, \qquad y_0\epsilon(-\infty, +\infty), \tag{14}$$

are defined and increasing in E, i.e. almost everywhere in (a, ∞). According to (13) functions (14) satisfy equation (1) in E.

On the other hand, let a function $g_1(x)$ be defined and monotonic a.e. in (a, ∞) and satisfy equation (1) a.e. in (a, ∞). Since $\varphi(x) \geqslant 0$, $g_1(x)$ must be increasing (cf. (1)). Let $g_0(x)$ be the function obtained from (14) by putting $y_0 = g_1(x_0)$. Thus

$$g_1(x_0) = g_0(x_0). \tag{15}$$

The function

$$G_1(x) = \int_{x_1}^{x} g_1(t)\,dt \tag{16}$$

is defined and convex in (a, ∞). Integrating the relation

$$g_1(x+1)-g_1(x) = \varphi(x)$$

we get

$$G_1(x+1)-G_1(x)-\int_{x_1}^{x_1+1} g_1(t)\,dt = \Phi(x),$$

whence, writing

$$c = \int_{x_1}^{x_1+1} g_1(t)\,dt$$

(4) The point x_0 in formula (4) may be taken quite arbitrarily from (a, ∞); moreover, according to Theorem 1, taking another x_0 we obtain a function, which may differ from $G(x)$ at most by an additive constant, and thus the set E does not depend on the choice of x_0. Consequently we may assume that $x_0\epsilon E$.

and

(17) $$G_2(x) = G_1(x) - cx,$$

we obtain

$$G_2(x+1) - G_2(x) = \Phi(x).$$

The function $G_2(x)$ is convex in (a, ∞), just like $G_1(x)$, and so, by Theorem 1, it may differ from $G(x)$ only by an additive constant:

(18) $$G_2(x) = G(x) + C.$$

By (16) and (17) we have

$$G_2'(x) = g_1(x) - c \quad \text{a.e. in } (a, \infty),$$

and by (14) and (15)

$$G'(x) = g_0(x) + G'(x_0) - g_1(x_0) \quad \text{a.e. in } (a, \infty),$$

whence in view of (18)

$$g_1(x) = g_0(x) + G'(x_0) - g_1(x_0) + c \quad \text{a.e. in } (a, \infty).$$

Setting in the above relation $x = x_0$ and taking into account (15), we obtain

$$G'(x_0) - g_1(x_0) + c = 0,$$

i.e.

$$g_1(x) = g_0(x) \quad \text{a.e. in } (a, \infty).$$

It remains to prove formula (6). By Lemmas 1 and 2 we have

$$G'(x) = \Phi(x_0) - \sum_{n=0}^{\infty} \Big\{\varphi(x+n) - \int_n^{n+1} \varphi(x_0+t)\,dt\Big\} \quad \text{a.e. in } (a, \infty).$$

Hence

$$g(x) = G'(x) - G'(x_0) + y_0 = y_0 - \sum_{n=0}^{\infty} \{\varphi(x+n) - \varphi(x_0+n)\} \quad \text{a.e. in } (a, \infty).$$

This completes the proof.

§ 3. Now we assume Theorem 2 to be true and we shall prove Theorem 1.

Let $\varphi(x)$ be a concave function defined in (a, ∞) and fulfilling condition (2). It is then differentiable a.e. in (a, ∞) and the function

$$\Phi(x) = \varphi'(x)$$

is decreasing. Moreover, we have for $x \in (n, n+1)$ for which $\varphi'(x)$ exists (cf. [1])

$$\varphi(n) - \varphi(n-1) \geqslant \varphi'(x) \geqslant \varphi(n+2) - \varphi(n+1),$$

whence by (2) $\lim_{x\to\infty}\Phi(x) = 0$. By Theorem 2 equation (13) has an increasing solution

$$G(x) = Y - \sum_{n=0}^{\infty}\{\Phi(x+n)-\Phi(x_1+n)\} \tag{19}$$

(x_1 is a point from (a, ∞)). Here we take

$$Y = \varphi(x_0)+\int_{x_0}^{x_0+1}\sum_{n=0}^{\infty}\{\Phi(x+n)-\Phi(x_1+n)\}\,dx \tag{20}$$

$$= \varphi(x_0)+\sum_{n=0}^{\infty}\{\varphi(x_0+n+1)-\varphi(x_0+n)-\Phi(x_1+n)\}.$$

(The point x_1 may be chosen fairly arbitrarily, in particular it may be taken outside (x_0, x_0+1). Since the function $\Phi(x)$ is decreasing, the terms $\Phi(x+n)-\Phi(x_1+n)$ have a constant sign in (x_0, x_0+1) and consequently we may integrate term by term; cf. [9], p. 294.)

The function

$$g(x) = y_0+\int_{x_0}^{x} G(t)\,dt$$

is convex in (a, ∞). Integrating (13) we get

$$g(x+1)-g(x)-\int_{x_0}^{x_0+1} G(t)\,dt = \varphi(x)-\varphi(x_0),$$

i.e. according to (20)

$$g(x+1)-g(x) = \varphi(x).$$

On the other hand, let $g_1(x)$ be a convex solution of (1), fulfilling condition (3). Then $G_1(x) \overset{\mathrm{df}}{=} g_1'(x)$ is an increasing solution of equation (13) and by Theorem 2

$$G_1(x) = G(x)+C \quad \text{a.e. in } (a, \infty),$$

where C is a constant. Since $g_1(x)$, being convex, is absolutely continuous, we have

$$g_1(x) = y_0+\int_{x_0}^{x} G(t)\,dt+C(x-x_0) = g(x)+C(x-x_0). \tag{21}$$

Moreover, the functions $g(x)$ and $g_1(x)$ both satisfy equation (1), whence

$$g(x+1)-g(x) = g_1(x+1)-g_1(x). \tag{22}$$

Inserting (21) into (22) we obtain $C = 0$, i.e. $g_1(x) = g(x)$.

Finally, integrating (19) (we may integrate term by term) and taking into account (20), we obtain formula (4). This completes the proof.

References

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