ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE VIII (1964)

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A remark on Ehrenfeucht's criterion for irreducibility of polynomials

Let $f_1(w_1), \ldots, f_r(w_r)$ be polynomials with complex coefficients, having positive degrees $n_1, \ldots, n_r$. Put $F = f_1 + \ldots + f_r$. A. Ehrenfeucht proved in [1] that F is irreducible over the field of complex numbers if $(n_1, \ldots, n_r) = 1$. The purpose of this note is to give a simple proof of this result in the case $r = 2$, and to prove that when $r > 2$, F is always irreducible.

Assume that $f_1(w_1) + f_2(w_2) = P \cdot Q$, where P and Q have positive degrees. Then

$$f_1(w_1^{n_2}) + f_2(w_2^{n_1}) = P(w_1^{n_2}, w_2^{n_1}) \cdot Q(w_1^{n_2}, w_2^{n_1}) = P_1 \cdot Q_1,$$

where P_1 and Q_1 have positive degrees. Let P_2 (Q_2) be the homogeneous part of P_1 (Q_1) having greatest degree. Then $a_1 w_1^{n_1 n_2} + a_2 w_2^{n_1 n_2} = P_2 Q_2$, where a_1 and a_2 are the leading coefficients of f_1 and f_2. P_2 must obviously have at least two nonzero terms, $p' w_1^{k' n_2} w_2^{l' n_1}$ and $p'' w_1^{k'' n_2} w_2^{l'' n_1}$. As P_2 is homogeneous, $k' n_2 + l' n_1 = k'' n_2 + l'' n_1$, i.e. $(k' - k'') n_2 = (l'' - l') n_1$, where we assume $k' > k''$. Now $n_1 \mid k' - k''$, as $(n_1, n_2) = 1$ and, accordingly $k' \geqslant n_1$. But, the degree of P_2 is $k' n_2 + l' n_2 \geqslant k' n_2$, and we have thus arrived at a contradiction.

Assume next that $r \geqslant 3$. Putting $N = \{n_1, \ldots, n_r\}$, we obtain, as above, an equation

$$a_1 w_1^N + \ldots + a_r w_r^N = P_2 \cdot Q_2$$

where P_2 and Q_2 are homogeneous polynomials of degrees M and $N - M$, $0 < M < N$. We put $w_3 = 1$, $w_4 = \ldots = w_r = 0$, and obtain

$$a_1 w_1^N + a_2 w_2^N + a_3 = (p_0 w_1^M + \ldots + p_M)(q_0 w_1^{N-M} + \ldots + q_{N-M}),$$

where $p_0, \ldots, q_{N-M}$ are polynomials in w_2, of degree $< N$. If we substitute here for w_2 a solution of the equation $a_2 z^N + a_3 = 0$, we see that $p_1, \ldots, p_M, q_1, \ldots, q_{N-M}$ must vanish. But a complex polynomials, of degree $< N$, which vanishes for N different values of the variable, must be the zero polynomial. Thus we must have $a_1 w_1^N + a_2 w_2^N + a_3 = p_0 q_0 w_1^N$ which is clearly impossible.

Reference

[1] A. Ehrenfeucht, *Kryterium absolutnej nierozkładalności wielomianów*, Prace Mat. 2 (1956), pp. 167-169.