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Singular integrals depending on two parameters

1. Introduction. Let $L_{2\pi}$ be the class of all functions 2π-periodic and Lebesgue-integrable in the interval $\langle -\pi, \pi\rangle$. Write $L\langle a, b\rangle$ for the class of functions Lebesgue-integrable in $\langle a, b\rangle$. Denote, once for all, by $K(t, \xi)$ a function defined for all t and $\xi \in E$ (where E is a given set of numbers), 2π-periodic, even, bounded and measurable with respect to t for every fixed $\xi \in E$. Suppose that ξ_0 is an accumulation point of E.

It is easily observed that Fejér's theorem concerning the convergence of singular integrals

$$U(x, \xi, f) = \int_{-\pi}^{\pi} f(t) K(t-x, \xi)\, dt \qquad (f \in L_{2\pi}) \tag{1}$$

can be extended to the convergence $(x, \xi) \to (x_0, \xi_0)$ on an arbitrary set of points of the plane. Namely,

1.1. *If*

$$\lim_{\xi \to \xi_0} \int_{-\pi}^{\pi} K(t, \xi)\, dt = 1 \qquad (\xi \in E), \tag{2}$$

$$\int_{-\pi}^{\pi} |K(t, \xi)|\, dt \leqslant C \qquad \text{on } E \ (C = \text{const}), \tag{3}$$

$$\lim_{\xi \to \xi_0} \sup_{\delta \leqslant t \leqslant \pi} |K(t, \xi)| = 0 \qquad \text{for } \delta > 0 \ (\delta \leqslant \pi, \ \xi \in E), \tag{4}$$

then we have

$$\lim_{(x, \xi) \to (x_0, \xi_0)} U(x, \xi, f) = f(x_0) \qquad (\xi \in E) \tag{5}$$

at every x_0 at which f is continuous (cf. [4], I, p. 89, [1], p. 418).

Relation (5) holds at the points of differentiability of $\int_0^x f(t)\, dt$, under the assumptions of Romanovski's theorem, if the convergence $(x, \xi) \to (x_0, \xi_0)$ is restricted to some set of points of the plane. Also a theorem of the Faddeev type and a result concerning the convergence of derivatives $\partial^r U(x, \xi, f)/\partial x^r$ can be stated. The present note is devoted to these problems.

2. A generalization of Natanson's lemma. We shall prove the following fundamental lemma (cf. [3], p. 243).

2.1. *Let $\varphi(t)$ be a function of bounded variation in every interval $\langle a+\eta, b\rangle$ $(0<\eta<b-a)$, such that $\int_a^b v(s)ds<\infty$, where*

$$v(s) = \operatorname*{var}_{s\leqslant t\leqslant b} \varphi(t) \ (a\leqslant s<b), \qquad v(b)=0.$$

Then, if

$$M = \sup_{0<h\leqslant b-a}\left|\frac{1}{h}\int_a^{a+h} f(t)dt\right| < \infty \qquad (f\in L\langle a, b\rangle),$$

the improper Lebesgue integral $I=\int_{a+}^b f(t)\varphi(t)dt$ exists and

$$|I| \leqslant M\int_a^b [v(s)+|\varphi(b)|]ds.$$

Proof. Write

$$F(t) = \int_a^t f(u)du \qquad (a\leqslant t\leqslant b),$$

$$I_{\alpha,\beta} = \int_\alpha^\beta f(t)\varphi(t)dt \qquad (a<\alpha<\beta\leqslant b).$$

Integrating by parts, we obtain

$$I_{\alpha,\beta} = \int_\alpha^\beta \varphi(t)dF(t) = \varphi(t)F(t)\Big|_\alpha^\beta - \int_\alpha^\beta F(t)d\varphi(t).$$

Since

$$\Big|\int_\alpha^\beta F(t)d\varphi(t)\Big| \leqslant M\int_\alpha^\beta (t-a)d[-v(t)] = M\Big[(a-t)v(t)\Big|_\alpha^\beta + \int_\alpha^\beta v(t)dt\Big],$$

$$(\alpha-a)|\varphi(\alpha)-\varphi(b)| \leqslant (\alpha-a)v(\alpha) \leqslant \int_a^\alpha v(t)dt,$$

the integral $\int_{a+}^b F(t)d\varphi(t)$ exists and

$$\Big|\int_{a+}^b F(t)d\varphi(t)\Big| \leqslant M\int_a^b v(t)dt.$$

Our conclusion is now evident.

Remark. If $\psi(t)$ is such that $\int_a^b w(s)ds < \infty$, where

$$w(s) = \operatorname*{var}_{a \leqslant t \leqslant s} \psi(t) \qquad (a < s \leqslant b), \qquad w(a) = 0,$$

then

$$\left|\int_a^{b-} f(t)\psi(t)dt\right| \leqslant N \int_a^b [w(s)+|\psi(a)|]ds,$$

with

$$N = \sup_{0<h\leqslant b-a} \left|\frac{1}{h}\int_{b-h}^{b} f(t)dt\right| < \infty.$$

3. Theorems of the Romanovski and the Faddeev type. Some results on the convergence of integrals (1) will now be given.

3.1. *Suppose that the function $K(t, \xi)$ is non-negative and non-increasing in t on $\langle 0, \pi\rangle$ for every $\xi \in E$, that it satisfies condition* (2) *and that*

$$\lim_{\xi\to\xi_0} K(\delta, \xi) = 0 \quad \textit{for } 0 < \delta \leqslant \pi \ (\xi \in E). \tag{6}$$

Let

$$\lim_{h\to 0} \frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt = f(x_0) \qquad (f \in L_{2\pi}) \tag{7}$$

at some x_0. Then integrals (1) *tend to $f(x_0)$ as $(x, \xi) \to (x_0, \xi_0)$ on any plane set Z in which the function*

$$\lambda(x, \xi) = (x-x_0)K(0, \xi) \qquad (\xi \in E)$$

is bounded (cf. [3], p. 245, [4], I, p. 101, (7.9)).

Proof. Of course, it is sufficient to show that

$$I(x, \xi) = \int_{-\pi}^{\pi} [f(t)-f(x_0)]K(t-x, \xi)dt \to 0$$

as $(x, \xi) \to (x_0, \xi_0)$ on Z.

We consider only the case $-\pi < x_0 \leqslant 0$. By (7), given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{1}{h}\int_{x_0}^{x_0\pm h} [f(t)-f(x_0)]dt\right| \leqslant \varepsilon \qquad \text{when} \qquad 0 < h \leqslant \delta.$$

Suppose that $\delta < \pi+x_0$, $0 < x_0-x < \frac{1}{2}\delta$, and write

$$I(x, \xi) = \Big(\int_{-\pi}^{x_0-\delta} + \int_{x_0-\delta}^{x_0+\delta} + \int_{x_0+\delta}^{\pi}\Big)[f(t)-f(x_0)]K(t-x, \xi)dt = I_1+I_2+I_3.$$

It is easy to see that

$$|I_1| \leqslant K(x_0-x-\delta, \xi) \int_{-\pi}^{x_0-\delta} |f(t)-f(x_0)|\,dt \leqslant K(\tfrac{1}{2}\delta, \xi) \int_{-\pi}^{\pi} |f(t)-f(x_0)|\,dt$$

and

$$|I_3| \leqslant k(\tfrac{1}{2}\delta, \xi) \int_{x_0+\delta}^{\pi} |f(t)-f(x_0)|\,dt \leqslant K(\tfrac{1}{2}\delta, \xi) \int_{-\pi}^{\pi} |f(t)-f(x_0)|\,dt.$$

Hence, by (6),

$$\lim_{(x,\xi)} I_1 = 0 = \lim_{(x,\xi)} I_3 \quad \text{as} \quad (x, \xi) \to (x_0, \xi_0), \quad \xi \in E.$$

In view of 2.1,

$$\begin{aligned} |I_2| &\leqslant \varepsilon \int_{x_0-\delta}^{x_0} [\operatorname*{var}_{x_0-\delta \leqslant t \leqslant s} K(t-x, \xi)+K(x_0-x-\delta, \xi)]\,ds + \\ &\quad + \varepsilon \int_{x_0}^{x_0+\delta} [\operatorname*{var}_{s \leqslant t \leqslant x_0+\delta} K(t-x, \xi)+K(x_0-x+\delta, \xi)]\,ds \\ &\leqslant \varepsilon \Big[\int_{-\pi}^{\pi} K(s, \xi)\,ds + 2(x_0-x)K(0, \xi)\Big]. \end{aligned}$$

Therefore, if the points $(x, \xi) \in Z$ are sufficiently near to (x_0, ξ_0), we have

$$|I_2| \leqslant 2\varepsilon(Q+1),$$

where $Q = \sup |(x-x_0)K(0, \xi)|$ $((x, \xi) \in Z)$.

Thus, the proof is finished. Similarly, the following result can be obtained.

3.2. *Let $K^*(t, \xi)$ be non-negative and non-increasing in t on $\langle 0, \pi\rangle$ for $\xi \in E$, and satisfy the same assumptions as $K(t, \xi)$ in § 1, above* 1.1, *and let*

$$|K(t, \xi)| \leqslant K^*(t, \xi) \qquad (t \in \langle 0, \pi\rangle, \ \xi \in E).$$

Suppose that conditions (2), (4) *for $K(t, \xi)$ and condition* (3) *for $K^*(t, \xi)$ hold. Then the relation*

$$\lim_{h \to 0} \frac{1}{h} \int_{x_0}^{x_0+h} |f(t)-f(x_0)|\,dt = 0 \qquad (f \in L_{2\pi})$$

implies (5), *i.e.*

$$\lim_{(x,\xi)} \int_{-\pi}^{\pi} f(t)K(t-x, \xi)\,dt = f(x_0),$$

as $(x, \xi) \to (x_0, \xi_0)$ on a plane set Z^* in which the function

$$\lambda^*(x, \xi) = (x-x_0)K^*(0, \xi) \qquad (\xi \in E)$$

is bounded (cf. [3], p. 246).

4. Differentiation of singular integrals. The following theorem hold.

4.1. *Let the function* $K(t, \xi)$ *and its derivatives* $\partial^\nu K(t, \xi)/\partial t^\nu$ $(\nu = 1, 2, \ldots, r)$ *be continuous with respect to* t *on* $(-\infty, \infty)$ *for every fixed* $\xi \in E$. *Suppose that conditions* (2), (3), (4) *together with*

$$\sup_{\xi \in E} \int_0^\pi \sin^r t \left| \frac{\partial^r K(t, \xi)}{\partial t^r} \right| dt < \infty \tag{8}$$

and

$$\lim_{\xi \to \xi_0} \sup_{\delta \leqslant t \leqslant \pi} \left| \frac{\partial^r K(t, \xi)}{\partial t^r} \right| = 0 \quad \textit{for all } \delta > 0 \ (\delta \leqslant \pi, \ \xi \in E) \tag{9}$$

are satisfied. Denote by S *a set of points* (x, ξ) $(\xi \in E)$ *such that*

$$|x - x_0|^\nu \int_0^\pi \sin^{r-\nu} t \left| \frac{\partial^r K(t, \xi)}{\partial t^r} \right| dt \leqslant C_\nu \quad (\nu = 1, 2, \ldots, r), \tag{10}$$

x_0 *being fixed, where* C_ν *are certain positive constants. Then, if the function* $f \in L_{2\pi}$ *possesses at* x_0 *a finite derivative* $f^{(r)}(x_0)$, *we have*

$$\lim_{(x, \xi)} \frac{\partial^r U(x, \xi, f)}{\partial x^r} = \lim_{(x, \xi)} \int_{-\pi}^{\pi} f(t) \frac{\partial^r K(t - x, \xi)}{\partial x^r} dt = f^{(r)}(x_0) \tag{11}$$

as $(x, \xi) \to (x_0, \xi_0)$ *on* S (cf. [4], II, pp. 60-61, [1], pp. 419-420).

Proof. We construct a function

$$g(t) = \sum_{n=0}^{r} a_n \sin^n (t - x_0) \qquad (-\pi < x_0 < \pi)$$

such that

$$g^{(p)}(x_0) = f^{(p)}(x_0) \qquad (p = 0, 1, \ldots, r).$$

In this case, the coefficients a_n are certain linear combinations of the derivatives $f^{(m)}(x_0)$ $(m \leqslant n)$, e.g. $a_0 = f(x_0)$, $a_1 = f'(x_0)$, $a_2 = \frac{1}{2!} f''(x_0)$, $a_3 = \frac{1}{3!}[f'(x_0) + f'''(x_0)]$, $a_4 = \frac{1}{4!}[4f''(x_0) + f^{IV}(x_0)]$, $a_5 = \frac{1}{5!}[9f'(x_0) + 10f'''(x_0) + f^{V}(x_0)]$ etc. It is easy to observe that $\omega(t) = [f(t) - g(t)]/\sin^r(t - x_0) \to 0$ as $t \to x_0$ ([2], pp. 25-27).

Obviously,

$$\frac{\partial^r U(x, \xi, g)}{\partial x^r} = (-1)^r \int_{-\pi}^{\pi} g(t) \frac{\partial^r K(t - x, \xi)}{\partial t^r} dt = \int_{-\pi}^{\pi} g^{(r)}(t) K(t - x, \xi) dt.$$

Hence, by 1.1,

$$\lim_{(x,\xi)} \frac{\partial^r U(x, \xi, g)}{\partial x^r} = g^{(r)}(x_0) = f^{(r)}(x_0) \quad \text{as} \quad (x, \xi) \to (x_0, \xi_0).$$

Since

$$\frac{\partial^r U(x, \xi, f)}{\partial x^r} = \frac{\partial^r U(x, \xi, g)}{\partial x^r} + \frac{\partial^r U(x, \xi, f-g)}{\partial x^r},$$

it is sufficient to show that the second term on the right-hand side of the last identity tends to zero as $(x, \xi) \to (x_0, \xi_0)$ on S. But

$$H(x, \xi) = \frac{\partial^r U(x, \xi, f-g)}{\partial x^r} = (-1)^r \int_{-\pi}^{\pi} \omega(t) \sin^r(t-x_0) \frac{\partial^r K(t-x, \xi)}{\partial t^r} dt$$

and $\lim_{t \to x_0} \omega(t) = 0$. Thus, for any $\varepsilon > 0$ there is a $\delta > 0$ ($\delta \leqslant \min(\pi - x_0, \pi + x_0)$) such that

$$|H(x, \xi)| \leqslant \varepsilon \int_{x_0-\delta}^{x_0+\delta} \left| \sin^r(t-x_0) \frac{\partial^r K(t-x, \xi)}{\partial t^r} \right| dt +$$

$$+ \left| \left(\int_{-\pi}^{x_0-\delta} + \int_{x_0+\delta}^{\pi} \right) \omega(t) \sin^r(t-x_0) \frac{\partial^r K(t-x, \xi)}{\partial t^r} dt \right|$$

$$= \varepsilon I_1(x, \xi) + |I_2(x, \xi) + I_3(x, \xi)|.$$

Evidently,

$$I_1(x, \xi) \leqslant \int_{-\pi}^{\pi} \left| \sin^r(t+x-x_0) \frac{\partial^r K(t, \xi)}{\partial t^r} \right| dt$$

$$\leqslant \int_{-\pi}^{\pi} \left| \sin^r t \frac{\partial^r K(t, \xi)}{\partial t^r} \right| dt + \int_{-\pi}^{\pi} |\sin^r(t+x-x_0) - \sin^r t| \left| \frac{\partial^r K(t, \xi)}{\partial t^r} \right| dt$$

$$= J_1(\xi) + J_2(x, \xi).$$

In view of (8), there is a constant C such that $J_1(\xi) \leqslant C$ ($\xi \in E$).

Applying the formulas $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$, $\sin\alpha - \sin\beta = 2\cos[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2]$, and taking into account (10), we can show that $J_2(x, \xi) \leqslant P$ on S, where P is a constant; whence $I_1(x, \xi) < C + P$ on S.

Let $|x-x_0| < \frac{1}{2}\delta$. Since $\gamma(t) = \omega(t)\sin^r(t-x_0)$ is Lebesgue-integrable in $\langle -\pi, \pi\rangle$,

$$|I_3(x, \xi)| \leqslant \int_{x_0+\delta}^{\pi} |\gamma(t)| \left| \frac{\partial^r K(t-x, \xi)}{\partial t^r} \right| dt \leqslant \sup_{\delta/2 \leqslant u \leqslant \pi} \left| \frac{\partial^r K(u, \xi)}{\partial u^r} \right| \int_{-\pi}^{\pi} |\gamma(t)| dt.$$

Hence, by (9), $\lim I_3(x, \xi) = 0$ as $(x, \xi) \to (x_0, \xi_0)$. Similarly, $\lim I_2(x, \xi) = 0$. The proof is completed.

Finally, we remark that from 1.1, 3.1 and 4.1 theorems 1, 3, 2 of [1], pp. 417-23, follow, respectively. An analogical result can be obtained for Cesàro's integrals ([4], II, pp. 60-61).

References

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