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Singular integrals depending on two parameters

1. Introduction. Let $L_{2\pi}$ be the class of all functions $2\pi$-periodic and Lebesgue-integrable in the interval $<-\pi, \pi>$. Write $L(a, b)$ for the class of functions Lebesgue-integrable in $<a, b>$. Denote, once for all, by $K(t, \xi)$ a function defined for all $t$ and $\xi \in E$ (where $E$ is a given set of numbers), $2\pi$-periodic, even, bounded and measurable with respect to $t$ for every fixed $\xi \in E$. Suppose that $\xi_0$ is an accumulation point of $E$.

It is easily observed that Fejér's theorem concerning the convergence of singular integrals

\[ U(x, \xi, f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)K(t-x, \xi)dt \quad (f \in L_{2\pi}) \]

can be extended to the convergence $(x, \xi) \to (x_0, \xi_0)$ on an arbitrary set of points of the plane. Namely,

1.1. If

\[ \lim_{\xi \to \xi_0} \int_{-\pi}^{\pi} K(t, \xi)dt = 1 \quad (\xi \in E), \]

\[ \int_{-\pi}^{\pi} |K(t, \xi)|dt \leq C \quad \text{on } E \quad (C = \text{const}), \]

\[ \lim_{\xi \to \xi_0} \sup_{\delta < t < \pi} |K(t, \xi)| = 0 \quad \text{for } \delta > 0 \quad (\delta \leq \pi, \xi \in E), \]

then we have

\[ \lim_{(x, \xi) \to (x_0, \xi_0)} U(x, \xi, f) = f(x_0) \quad (\xi \in E) \]

at every $x_0$ at which $f$ is continuous (cf. [4], I, p. 89, [1], p. 418).

Relation (5) holds at the points of differentiability of $\int_0^x f(t)dt$, under the assumptions of Romanovski's theorem, if the convergence $(x, \xi) \to (x_0, \xi_0)$ is restricted to some set of points of the plane. Also a theorem of the Faddeev type and a result concerning the convergence of derivatives $\partial^r U(x, \xi, f)/\partial x^r$ can be stated. The present note is devoted to these problems.
2. A generalization of Natanson's lemma. We shall prove the following fundamental lemma (cf. [3], p. 243).

2.1. Let \( \varphi(t) \) be a function of bounded variation in every interval \( (a + \eta, b) \) \( (0 < \eta < b - a) \), such that \( \int_a^b v(s)ds < \infty \), where

\[
v(s) = \text{var} \varphi(t) \quad (a \leq s < b), \quad v(b) = 0.
\]

Then, if

\[
M = \sup_{0 < h < b - a} \left| \frac{1}{h} \int_a^{a+h} f(t)dt \right| < \infty \quad (f \in L(a, b)),
\]

the improper Lebesgue integral \( I = \int_a^b f(t) \varphi(t)dt \) exists and

\[
|I| \leq M \int_a^b [v(s) + |\varphi(b)|]ds.
\]

Proof. Write

\[
F(t) = \int_a^t f(u)du \quad (a \leq t \leq b),
\]

\[
I_{a,\beta} = \int_a^\beta f(t)\varphi(t)dt \quad (a < \alpha < \beta \leq b).
\]

Integrating by parts, we obtain

\[
I_{a,\beta} = \int_a^\beta \varphi(t)dF(t) = \varphi(t)F(t)|_a^\beta - \int_a^\beta F(t)d\varphi(t).
\]

Since

\[
\left| \int_a^\beta F(t)d\varphi(t) \right| \leq M \int_a^\beta (t - a)d[-v(t)] = M \left[ (a - t)v(t)|_a^\beta + \int_a^\beta v(t)dt \right],
\]

\[
(a - a)|\varphi(a) - \varphi(b)| \leq (a - a)v(a) \leq \int_a^b v(t)dt,
\]

the integral \( \int_a^b F(t)d\varphi(t) \) exists and

\[
\left| \int_a^b F(t)d\varphi(t) \right| \leq M \int_a^b v(t)dt.
\]

Our conclusion is now evident.
Remark. If \( \psi(t) \) is such that \( \int_{a}^{b} w(s) \, ds < \infty \), where
\[
 w(s) = \var\psi(t) \quad (a < s \leq b), \quad w(a) = 0,
\]
then
\[
\left| \int_{a}^{b} f(t) \psi(t) \, dt \right| \leq N \int_{a}^{b} [w(s) + |\psi(a)|] \, ds,
\]
with
\[
N = \sup_{0 < h < b-a} \left| \frac{1}{h} \int_{b-h}^{b} f(t) \, dt \right| < \infty.
\]

3. Theorems of the Romanovski and the Faddeev type. Some results on the convergence of integrals (1) will now be given.

3.1. Suppose that the function \( K(t, \xi) \) is non-negative and non-increasing in \( t \) on \( \langle 0, \pi \rangle \) for every \( \xi \in E \), that it satisfies condition (2) and that
\[
\lim_{\xi \to 0} K(\delta, \xi) = 0 \quad \text{for} \quad 0 < \delta \leq \pi \quad (\xi \in E).
\]
Let
\[
\lim_{h \to 0} \frac{1}{h} \int_{x_0 - h}^{x_0 + h} f(t) \, dt = f(x_0) \quad (f \in L_{2\pi})
\]
at some \( x_0 \). Then integrals (1) tend to \( f(x_0) \) as \( (x, \xi) \to (x_0, \xi_0) \) on any plane set \( Z \) in which the function
\[
\lambda(x, \xi) = (x-x_0)K(0, \xi) \quad (\xi \in E)
\]
is bounded (cf. [3], p. 245, [4], I, p. 101, (7.9)).

Proof. Of course, it is sufficient to show that
\[
I(x, \xi) = \int_{-\pi}^{\pi} [f(t)-f(x_0)]K(t-x, \xi) \, dt \to 0
\]
as \( (x, \xi) \to (x_0, \xi_0) \) on \( Z \).

We consider only the case \(-\pi < x_0 \leq 0\). By (7), given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that
\[
\left| \frac{1}{h} \int_{x_0}^{x_0+h} [f(t)-f(x_0)] \, dt \right| \leq \varepsilon \quad \text{when} \quad 0 < h \leq \delta.
\]
Suppose that \( \delta < \pi - x_0, \quad 0 < x_0 - x < \frac{1}{2} \delta \), and write
\[
I(x, \xi) = \left( \int_{-\pi}^{x_0-\delta} + \int_{x_0-\delta}^{x_0+\delta} + \int_{x_0+\delta}^{\pi} \right) [f(t)-f(x_0)] K(t-x, \xi) \, dt = I_1 + I_2 + I_3.
\]
It is easy to see that
\[ |I_1| \leq K(x_0 - x - \delta, \xi) \int_{-\pi}^{\pi} |f(t) - f(x_0)| dt \leq K(\frac{1}{2} \delta, \xi) \int_{-\pi}^{\pi} |f(t) - f(x_0)| dt \]
and
\[ |I_3| \leq k(\frac{1}{2} \delta, \xi) \int_{x_0 - \delta}^{x_0 + \delta} |f(t) - f(x_0)| dt \leq K(\frac{1}{2} \delta, \xi) \int_{-\pi}^{\pi} |f(t) - f(x_0)| dt. \]
Hence, by (6),
\[ \lim_{(x, \xi) \to (x_0, \xi_0)} I_1 = 0 = \lim_{(x, \xi) \to (x_0, \xi_0)} I_3 \quad \text{as} \quad (x, \xi) \to (x_0, \xi_0), \quad \xi \in E. \]
In view of 2.1,
\[ |I_2| \leq \epsilon \int_{x_0 - \delta}^{x_0 + \delta} \left[ \var K(t - x, \xi) + K(x_0 - x, \xi) \right] dt + \epsilon \int_{x_0 - \delta}^{x_0 + \delta} \left[ \var K(t - x, \xi) + K(x_0 - x, \xi) \right] dt \leq \epsilon \left[ \int_{-\pi}^{\pi} K(s, \xi) ds + 2(x_0 - x)K(0, \xi) \right]. \]
Therefore, if the points \((x, \xi) \in Z\) are sufficiently near to \((x_0, \xi_0)\), we have
\[ |I_2| \leq 2\epsilon (Q + 1), \]
where \(Q = \sup |(x-x_0)K(0, \xi)| (x, \xi) \in Z). \]
Thus, the proof is finished. Similarly, the following result can be obtained.

3.2. Let \(K^*(t, \xi)\) be non-negative and non-increasing in \(t\) on \((0, \pi)\) for \(\xi \in E\), and satisfy the same assumptions as \(K(t, \xi)\) in §1, above 1.1, and let
\[ |K(t, \xi)| \leq K^*(t, \xi) \quad (t \in (0, \pi), \quad \xi \in E). \]
Suppose that conditions (2), (4) for \(K(t, \xi)\) and condition (3) for \(K^*(t, \xi)\) hold. Then the relation
\[ \lim_{h \to 0} \frac{1}{h} \int_{x_0}^{x_0 + h} |f(t) - f(x_0)| dt = 0 \quad (f \in L_2) \]
implies (5), i.e.
\[ \lim_{(x, \xi) \to (x_0, \xi_0)} \int_{-\pi}^{\pi} f(t)K(t - x, \xi) dt = f(x_0), \]
as \((x, \xi) \to (x_0, \xi_0)\) on a plane set \(Z^*\) in which the function
\[ \lambda^*(x, \xi) = (x-x_0)K^*(0, \xi) \quad (\xi \in E) \]
is bounded (cf. [3], p. 246).
4. Differentiation of singular integrals. The following theorem hold.

4.1. Let the function $K(t, \xi)$ and its derivatives $\partial^r K(t, \xi)/\partial t^r$ ($\nu = 1, 2, \ldots, r$) be continuous with respect to $t$ on $(-\infty, \infty)$ for every fixed $\xi \in E$. Suppose that conditions (2), (3), (4) together with (8) and (9) lim sup $\sup_{t \in E} \left| \frac{\partial^r K(t, \xi)}{\partial t^r} \right| dt < \infty$

and

(9) $\lim_{\delta \to 0} \sup_{\delta < t < \pi} \left| \frac{\partial^r K(t, \xi)}{\partial t^r} \right| = 0 \quad \text{for all } \delta > 0 \ (\delta \leq \pi, \xi \in E)$

are satisfied. Denote by $S$ a set of points $(x, \xi) (\xi \in E)$ such that

\[ |x - x_0|^r \int_0^\pi \sin^{-r} t \left| \frac{\partial^r K(t, \xi)}{\partial t^r} \right| dt \leq C_r \quad (r = 1, 2, \ldots, r), \]

$x_0$ being fixed, where $C_r$ are certain positive constants. Then, if the function $f \in L^2_{2\pi}$ possesses at $x_0$ a finite derivative $f^{(\nu)}(x_0)$, we have

(11) $\lim_{(x, \xi) \to (x_0, \xi_0)} \frac{\partial^r U(x, \xi, f)}{\partial x^r} = \lim_{(x, \xi) \to (x_0, \xi_0)} \int_{-\pi}^\pi f(t) \frac{\partial^r K(t - x, \xi)}{\partial t^r} dt = f^{(\nu)}(x_0)$

as $(x, \xi) \to (x_0, \xi_0)$ on $S$ (cf. [4], II, pp. 60-61, [1], pp. 419-420).

Proof. We construct a function

\[ g(t) = \sum_{n=0}^r a_n \sin^n(t - x_0) \quad (-\pi < x_0 < \pi) \]

such that

\[ g^{(p)}(x_0) = f^{(p)}(x_0) \quad (p = 0, 1, \ldots, r). \]

In this case, the coefficients $a_n$ are certain linear combinations of the derivatives $f^{(m)}(x_0)$ ($m \leq n$), e.g. $a_0 = f(x_0)$, $a_1 = f'(x_0)$, $a_2 = \frac{1}{2!} f''(x_0)$, $a_3 = \frac{1}{3!} [f'(x_0) + f'''(x_0)]$, $a_4 = \frac{1}{4!} [4f''(x_0) + f^{IV}(x_0)]$, $a_5 = \frac{1}{5!} [9f''(x_0) + 10f'''(x_0) + f^{V}(x_0)]$ etc. It is easy to observe that $\omega(t) = [f(t) - g(t)]/\sin^r(t - x_0) \to 0$ as $t \to x_0$ ([2], pp. 25-27).

Obviously,

\[ \frac{\partial^r U(x, \xi, g)}{\partial x^r} = (-1)^r \int_{-\pi}^\pi g(t) \frac{\partial^r K(t - x, \xi)}{\partial t^r} dt = \int_{-\pi}^\pi g^{(r)}(t) K(t - x, \xi) dt. \]
Hence, by 1.1,
\[
\lim_{(x, \xi) \to (x_0, \xi_0)} \frac{\partial^r U(x, \xi, g)}{\partial x^r} = g(r)(x_0) = f(r)(x_0) \quad \text{as} \quad (x, \xi) \to (x_0, \xi_0).
\]

Since
\[
\frac{\partial^r U(x, \xi, f)}{\partial x^r} = \frac{\partial^r U(x, \xi, g)}{\partial x^r} + \frac{\partial^r U(x, \xi, f-g)}{\partial x^r},
\]

it is sufficient to show that the second term on the right-hand side of the last identity tends to zero as \((x, \xi) \to (x_0, \xi_0)\) on \(S\). But
\[
H(x, \xi) = \frac{\partial^r U(x, \xi, f-g)}{\partial x^r} = (-1)^r \int_{-\pi}^{\pi} \omega(t) \sin^r(t-x_0) \frac{\partial^r K(t-x, \xi)}{\partial t^r} \, dt
\]
and \(\lim_{t \to x_0} \omega(t) = 0\). Thus, for any \(\epsilon > 0\) there is a \(\delta > 0\) \((\delta \leq \min(\pi - x_0, \pi + x_0))\) such that
\[
|H(x, \xi)| \leq \epsilon \int_{x_0-\delta}^{x_0+\delta} \left| \sin^r(t-x_0) \frac{\partial^r K(t-x, \xi)}{\partial t^r} \right| dt +
\]
\[
+ \left| \int_{-\pi}^{x_0-\delta} + \int_{x_0+\delta}^{\pi} \omega(t) \sin^r(t-x_0) \frac{\partial^r K(t-x, \xi)}{\partial t^r} \, dt \right|
\]
\[
= \epsilon I_1(x, \xi) + |I_2(x, \xi)|.
\]

Evidently,
\[
I_1(x, \xi) \leq \int_{-\pi}^{\pi} \left| \sin^r(t+x-x_0) \frac{\partial^r K(t, \xi)}{\partial t^r} \right| \, dt
\]
\[
\leq \int_{-\pi}^{\pi} \left| \sin^r t \right| \frac{\partial^r K(t, \xi)}{\partial t^r} \, dt + \int_{-\pi}^{\pi} \left| \sin^r(t+x-x_0) - \sin^r t \right| \frac{\partial^r K(t, \xi)}{\partial t^r} \, dt
\]
\[
= J_1(\xi) + J_2(x, \xi).
\]

In view of (8), there is a constant \(C\) such that \(J_1(\xi) \leq C (\xi \epsilon E)\). Applying the formulas \(a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})\), \(\sin a - \sin b = 2 \cos [(a+b)/2] \sin [(a-b)/2]\), and taking into account (10), we can show that \(J_2(x, \xi) \leq P\) on \(S\), where \(P\) is a constant; whence \(I_1(x, \xi) < C + P\) on \(S\).
Let \( |x - x_0| < \frac{1}{2} \delta \). Since \( \gamma(t) = \omega(t) \sin(\tau(t - x_0)) \) is Lebesgue-integrable in \((-\pi, \pi)\),

\[
|I_3(x, \xi)| \leq \int_{x_0 - \delta}^{x_0 + \delta} |\gamma(t)| \left| \frac{\partial^r K(t - x, \xi)}{\partial t^r} \right| dt \leq \sup_{\delta \pi < u < \pi} \left| \frac{\partial^r K(u, \xi)}{\partial u^r} \right| \int_{-\pi}^{\pi} |\gamma(t)| dt.
\]

Hence, by (9), \( \lim I_3(x, \xi) = 0 \) as \( (x, \xi) \to (x_0, \xi_0) \). Similarly, \( \lim I_2(x, \xi) = 0 \). The proof is completed.

Finally, we remark that from 1.1, 3.1 and 4.1 theorems 1, 3, 2 of [1], pp. 417-23, follow, respectively. An analogical result can be obtained for Cesàro's integrals ([4], II, pp. 60-61).

References