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Density of analytic polynomials in abstract Hardy spaces

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Summary. Let X be a separable Banach function space on the unit circle $\mathbb{T}$ and let $H[X]$ be the abstract Hardy space built upon X. We show that the set of analytic polynomials is dense in $H[X]$ if the Hardy–Littlewood maximal operator is bounded on the associate space X'. This result is specified to the case of variable Lebesgue spaces.

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Banach function space; rearrangement-invariant space; variable Lebesgue space; abstract Hardy space; analytic polynomial; Fejér kernel

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1. Introduction

For $1 \leqslant p \leqslant \infty$, let $L^p := L^p(\mathbb{T})$ be the Lebesgue space on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ in the complex plane $\mathbb{C}$. For $f \in L^1$, let

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\varphi}) e^{-in\varphi} \, \mathrm{d}\varphi, \quad n \in \mathbb{Z},$$

be the sequence of the Fourier coefficients of f. The classical Hardy spaces H^p are given by

$$H^p := \left\{ f \in L^p : \widehat{f}(n) = 0, \ n < 0 \right\}.$$

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A function of the form

$$q(t) = \sum_{k=0}^{n} \alpha_k t^k, \quad t \in \mathbb{T},\ \alpha_0, \ldots, \alpha_n \in \mathbb{C},$$

is said to be an analytic polynomial on $\mathbb{T}$. The set of all analytic polynomials is denoted by $\mathcal{P}_A$. It is well known that the set $\mathcal{P}_A$ is dense in H^p whenever $1 \leqslant p < \infty$ (see, e.g., [3, Chap. III, Corollary 1.7(a)]).

Let X be a Banach space continuously embedded in L^1. Following [17, p. 877], we will consider the abstract Hardy space $H[X]$ built upon the space X, which is defined by

$$H[X] := \left\{ f \in X : \widehat{f}(n) = 0,\ n < 0 \right\}.$$

It is clear that if $1 \leqslant p \leqslant \infty$, then $H[L^p]$ is the classical Hardy space H^p. The aim of this note is to find sufficient conditions for the density of the set $\mathcal{P}_A$ in the space $H[X]$ when X falls into the class of so-called Banach function spaces.

We equip $\mathbb{T}$ with the normalized Lebesgue measure $\mathrm{d}m(t) = |dt|/(2\pi)$. Let L^0 be the space of all measurable complex-valued functions on $\mathbb{T}$. As usual, we do not distinguish functions, which are equal almost everywhere (for the latter we use the standard abbreviation a.e.). Let L^0_+ be the subset of functions in L^0 whose values lie in $[0, \infty]$. The characteristic function of a measurable set $E \subset \mathbb{T}$ is denoted by χ_E.

Following [1, Chap. 1, Definition 1.1], a mapping $\rho\colon L^0_+ \to [0, \infty]$ is called a Banach function norm if for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$, for all constants $a \geqslant 0$, and for all measurable subsets E of $\mathbb{T}$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \leqslant \rho(f) + \rho(g)$;

(A2) $0 \leqslant g \leqslant f$ a.e. $\Rightarrow \rho(g) \leqslant \rho(f)$ (the lattice property);

(A3) $0 \leqslant f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property);

(A4) $m(E) < \infty \Rightarrow \rho(\chi_E) < \infty$;

(A5) $\int_E f(t)\,\mathrm{d}m(t) \leqslant C_E \rho(f)$ with the constant $C_E \in (0, \infty)$ that may depend on E and ρ, but is independent of f.

When functions differing only on a set of measure zero are identified, the set X of all functions $f \in L^0$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$. The set X under the natural linear space operations and under this norm becomes a Banach space (see [1, Chap. 1, Theorems 1.4 and 1.6]). If ρ is a Banach function norm, its associate norm ρ' is defined on L^0_+ by

$$\rho'(g) := \sup\left\{ \int_{\mathbb{T}} f(t) g(t)\,\mathrm{d}m(t) : f \in L^0_+,\ \rho(f) \leqslant 1 \right\}, \quad g \in L^0_+.$$

The associate norm itself is a Banach function norm [1, Chap. 1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the associate space

(Köthe dual) of X. The associate space X' can be viewed as a subspace of the (Banach) dual space X^*.

The distribution function m_f of an a.e. finite function $f \in L^0$ is defined by

$$m_f(\lambda) := m\{t \in \mathbb{T} : |f(t)| > \lambda\}, \quad \lambda \geqslant 0.$$

Two a.e. finite functions $f, g \in L^0$ are said to be equimeasurable if

$$m_f(\lambda) = m_g(\lambda), \quad \lambda \geqslant 0.$$

The non-increasing rearrangement of an a.e. finite function $f \in L^0$ is defined by

$$f^*(x) := \inf\{\lambda : m_f(\lambda) \leqslant x\}, \quad x \geqslant 0.$$

We refer to [1, Chap. 2, Section 1] and [11, Chap. II, Section 2] for properties of distribution functions and non-increasing rearrangements. A Banach function space X is called rearrangement-invariant if for every pair of a.e. finite equimeasurable functions $f, g \in L^0$, one has the following property: if $f \in X$, then $g \in X$ and the equality $\|f\|_X = \|g\|_X$ holds. Lebesgue spaces L^p, $1 \leqslant p \leqslant \infty$, as well as more general Orlicz spaces, Lorentz spaces, and Marcinkiewicz spaces are classical examples of rearrangement-invariant Banach function spaces (see [1, 11]). For more recent examples of rearrangement-invariant spaces, like Cesàro, Copson, and Tandori spaces, we refer to the paper by Maligranda and Leśnik [13].

One of our motivations for this work is the recent progress in the study of harmonic analysis in the setting of variable Lebesgue spaces [4, 6, 10]. Let $\mathfrak{P}(\mathbb{T})$ be the set of all measurable functions $p: \mathbb{T} \to [1, \infty]$. For $p \in \mathfrak{P}(\mathbb{T})$, put

$$\mathbb{T}_\infty^{p(\cdot)} := \{t \in \mathbb{T} : p(t) = \infty\}.$$

For a measurable function $f: \mathbb{T} \to \mathbb{C}$, consider

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{T} \setminus \mathbb{T}_\infty^{p(\cdot)}} |f(t)|^{p(t)} \mathrm{d}m(t) + \|f\|_{L^\infty(\mathbb{T}_\infty^{p(\cdot)})}.$$

In accordance with [4, Definition 2.9], the variable Lebesgue space $L^{p(\cdot)}$ is defined as the set of all measurable functions $f: \mathbb{T} \to \mathbb{C}$ such that $\rho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. This space is a Banach function space with respect to the Luxemburg–Nakano norm given by

$$\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\}$$

(see, e.g., [4, Theorems 2.17, 2.71 and Section 2.10.3]). If $p \in \mathfrak{P}(\mathbb{T})$ is constant, then $L^{p(\cdot)}$ is nothing but the standard Lebesgue space L^p. If $p \in \mathfrak{P}(\mathbb{T})$ is not constant, then $L^{p(\cdot)}$ is not rearrangement-invariant [4, Example 3.14]. Variable Lebesgue spaces are often called Nakano spaces. We refer to Maligranda's paper [14] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces. The associate space of $L^{p(\cdot)}$ is isomorphic to the space $L^{p'(\cdot)}$, where $p' \in \mathfrak{P}(\mathbb{T})$ is defined by the equation $1/p(t) + 1/p'(t) = 1$ for a.e. $t \in \mathbb{T}$ with the usual convention $1/\infty := 0$ [6, Theorem 3.2.13]. For $p \in \mathfrak{P}(\mathbb{T})$, put

$$p_- := \operatorname*{ess\,inf}_{t\in\mathbb{T}} p(t), \quad p_+ := \operatorname*{ess\,sup}_{t\in\mathbb{T}} p(t).$$

The variable Lebesgue space $L^{p(\cdot)}$ is separable if and only if $p_+ < \infty$ (see, e.g., [4, Theorem 2.78]).

The following result is considered folklore.

1.0.1. Theorem. *Let X be a separable rearrangement-invariant Banach function space on $\mathbb{T}$. Then the set of analytic polynomials $\mathcal{P}_A$ is dense in the abstract Hardy space $H[X]$. Moreover, for every $f \in H[X]$, there is a sequence of analytic polynomials $\{p_n\}$ such that $\|p_n\|_X \leqslant \|f\|_X$ for all $n \in \mathbb{N}$ and $p_n \to f$ in the norm of X as $n \to \infty$.*

Surprisingly enough, we could not find in the literature either Theorem 1.0.1 explicitly stated or any result on the density of $\mathcal{P}_A$ in abstract Hardy spaces $H[X]$ in the case when X is an arbitrary Banach function space beyond the class of rearrangement-invariant spaces. The aim of this note is to fill in this gap.

Given $f \in L^1$, the Hardy–Littlewood maximal function is defined by

$$(Mf)(t) := \sup_{I \ni t} \frac{1}{m(I)} \int_I |f(\tau)|\, \mathrm{d}m(\tau), \quad t \in \mathbb{T},$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$ containing $t \in \mathbb{T}$. The operator $f \mapsto Mf$ is called the Hardy–Littlewood maximal operator.

1.0.2. Theorem (Main result). *Suppose X is a separable Banach function space on $\mathbb{T}$. If the Hardy–Littlewood maximal operator M is bounded on the associate space X', then the set of analytic polynomials $\mathcal{P}_A$ is dense in the abstract Hardy space $H[X]$.*

To illustrate this result in the case of variable Lebesgue spaces, we will need the following classes of variable exponents. Following [4, Definition 2.2], one says that $r\colon \mathbb{T} \to \mathbb{R}$ is locally log-Hölder continuous if there exists a constant $C_0 > 0$ such that

$$|r(x) - r(y)| = C_0/(-\log|x-y|) \quad \text{for all } x, y \in \mathbb{T},\ |x-y| < 1/2.$$

The class of all locally log-Hölder continuous functions is denoted by $LH_0(\mathbb{T})$. If $p_+ < \infty$, then $p \in LH_0(\mathbb{T})$ if and only if $1/p \in LH_0(\mathbb{T})$. By [4, Theorem 4.7], if $p \in \mathfrak{P}(\mathbb{T})$ is such that $1 < p_-$ and $1/p \in LH_0(\mathbb{T})$, then the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}$. This condition was initially referred to as "almost necessary" (see [4, Section 4.6.1] for further details). However, Lerner [12] constructed an example of a discontinuous variable exponent such that the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}$.

Kapanadze and Kopaliani [7] developed Lerner's ideas further. They considered the following class of variable exponents. Recall that a function $f \in L^1$ belongs to the space BMO if

$$\|f\|_* := \sup_{I \subset \mathbb{T}} \frac{1}{m(I)} \int_I |f(t) - f_I| \, \mathrm{d}m(t) < \infty,$$

where f_I is the integral average of f on the arc I and the supremum is taken over all arcs $I \subset \mathbb{T}$. For $f \in BMO$, put

$$\gamma(f, r) := \sup_{m(I) \leqslant r} \frac{1}{m(I)} \int_I |f(t) - f_I| \, \mathrm{d}m(t).$$

Let $VMO^{1/|\log|}$ be the set of functions $f \in BMO$ such that

$$\gamma(f, r) = o(1/|\log r|) \quad \text{as } r \to 0.$$

Note that $VMO^{1/|\log|}$ contains discontinuous functions. We will say that $p \in \mathfrak{P}(\mathbb{T})$ belongs to the Kapanadze-Kopaliani class $\mathfrak{K}(\mathbb{T})$ if $1 < p_- \leqslant p_+ < \infty$ and $p \in VMO^{1/|\log|}$. It is shown in [7, Theorem 2.1] that if $p \in \mathfrak{K}(\mathbb{T})$, then the Hardy–Littelwood maximal operator M is bounded on the variable Lebesgue space $L^{p(\cdot)}$.

1.0.3. Corollary. *Suppose $p \in \mathfrak{P}(\mathbb{T})$. If $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$ or if $p' \in \mathfrak{K}(\mathbb{T})$, then the set of analytic polynomials $\mathcal{P}_A$ is dense in the abstract Hardy space $H[L^{p(\cdot)}]$ built upon the variable Lebesgue space $L^{p(\cdot)}$.*

The paper is organized as follows. In Section 2, we prove that the separability of a Banach function space X is equivalent to the density of the set of trigonometric polynomials $\mathcal{P}$ in X and to the density of the set of all continuous functions C in X. Further, we recall a pointwise estimate of the Fejér means $f * K_n$, where K_n is the n-th Fejér kernel, by the Hardy–Littlewood maximal function Mf. In Section 3 we show that the norms of the operators $F_n f = f * K_n$ are uniformly bounded on a Banach function space X if X is rearrangement-invariant or if the Hardy–Littlewood maximal operator is bounded on X'. Moreover, if X is rearrangement-invariant, then $\|F_n\|_{\mathcal{B}(X)} \leqslant 1$ for all $n \in \mathbb{N}$. Further, we prove that under the assumptions of Theorem 1.0.1 or 1.0.2, $\|f * K_n - f\|_X \to 0$ as $n \to \infty$. It remains to observe that $f * K_n \in \mathcal{P}_A$ if $f \in H[X]$, which will complete the proof of Theorems 1.0.1 and 1.0.2.

2. Preliminaries

2.1. Elementary lemma. We start with the following elementary lemma, whose proof can be found, e.g., in [3, Chap. III, Proposition 1.6(a)]. Here and in what follows, the space of all bounded linear operators on a Banach space E will be denoted by $\mathcal{B}(E)$.

2.1.1. Lemma. *Let E be a Banach space and let $\{T_n\}$ be a sequence of bounded linear operators on E such that*

$$\sup_{n\in\mathbb{N}} \|T_n\|_{\mathcal{B}(E)} < \infty.$$

If D is a dense subset of E and for all $x \in D$,

$$\|T_n x - x\|_E \to 0 \quad \text{as } n \to \infty, \tag{1}$$

then (1) holds for all $x \in E$.

2.2. Density of continuous function and trigonometric polynomials in Banach function spaces. A function of the form

$$q(t) = \sum_{k=-n}^{n} \alpha_k t^k, \quad t \in \mathbb{T},\ \alpha_{-n}, \dots, \alpha_n \in \mathbb{C},$$

is said to be a trigonometric (or Laurent) polynomial on $\mathbb{T}$. The set of all trigonometric polynomials is denoted by $\mathcal{P}$.

2.2.1. Lemma. *Let X be a Banach function space on $\mathbb{T}$. The following statements are equivalent:*

(i) *the set $\mathcal{P}$ of all trigonometric polynomials is dense in X;*

(ii) *the space C of all continuous functions on $\mathbb{T}$ is dense in X;*

(iii) *the Banach function space X is separable.*

Proof. The proof is developed in analogy with [8, Lemma 1.3].

(i) $\Rightarrow$ (ii) is trivial because $\mathcal{P} \subset C \subset X$.

(ii) $\Rightarrow$ (iii). Since C is separable and $C \subset X$ is dense in X, we conclude that X is separable.

(iii) $\Rightarrow$ (i). Assume that X is separable and $\mathcal{P}$ is not dense in X. Then by a corollary of the Hahn–Banach theorem (see, e.g., [2, Chap. 7, Theorem 4.2]), there exists a nonzero functional $\Lambda \in X^*$ such that $\Lambda(p) = 0$ for all $p \in \mathcal{P}$. Since X is separable, it follows from [1, Chap. 1, Corollaries 4.3 and 5.6] that the Banach dual X^* of X is canonically isometrically isomorphic to the associate space X'. Hence there exists a nonzero function $h \in X' \subset L^1$ such that

$$\int_{\mathbb{T}} p(t)h(t)\,\mathrm{d}m(t) = 0 \quad \text{for all } p \in \mathcal{P}.$$

Taking $p(t) = t^n$ for $n \in \mathbb{Z}$, we obtain that all Fourier coefficients of $h \in L^1$ vanish, which implies that $h = 0$ a.e. on $\mathbb{T}$ by the uniqueness theorem of the Fourier series (see, e.g., [9, Chap. I, Theorem 2.7]). This contradiction proves that $\mathcal{P}$ is dense in X. □

2.3. Pointwise estimate for the Fejér means. Recall that L^1 is a commutative Banach algebra under the convolution multiplication defined for $f, g \in L^1$ by

$$(f * g)(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta - i\varphi})g(e^{i\varphi})\,\mathrm{d}\varphi, \quad e^{i\theta} \in \mathbb{T}.$$

For $n \in \mathbb{N}$, the function

$$K_n(e^{i\theta}) := \sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{i\theta k} = \frac{1}{n+1}\left(\frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}\right)^2, \quad e^{i\theta} \in \mathbb{T},$$

is called the n-th Fejér kernel. It is well-known that $\|K_n\|_{L^1} \leqslant 1$. For $f \in L^1$, the n-th Fejér mean of f is defined as the convolution $f * K_n$. Then

$$(f * K_n)(e^{i\theta}) = \sum_{k=-n}^{n} \widehat{f}(k)\left(1 - \frac{|k|}{n+1}\right)e^{i\theta k}, \quad e^{i\theta} \in \mathbb{T} \tag{2}$$

(see, e.g., [9, Chap. I]). This means that if $f \in L^1$, then $f * K_n \in \mathcal{P}$. Moreover, if $f \in H^1 = H[L^1]$, then $f * K_n \in \mathcal{P}_A$.

2.3.1. Lemma. *For every $f \in L^1$ and $t \in \mathbb{T}$,*

$$\sup_{n\in\mathbb{N}}\left|(f * K_n)(t)\right| \leqslant \frac{\pi^2}{2}(Mf)(t). \tag{3}$$

Proof. Since $|\sin\varphi| \geqslant 2|\varphi|/\pi$ for $|\varphi| \leqslant \pi/2$, we have for $\theta \in [-\pi, \pi]$,

$$\begin{aligned}
K_n(e^{i\theta}) &\leqslant \frac{\pi^2}{n+1}\frac{\sin^2\left(\frac{n+1}{2}\theta\right)}{\theta^2} \\
&= \frac{\pi^2}{4}(n+1)\frac{\sin^2\left(\frac{n+1}{2}\theta\right)}{\left(\frac{n+1}{2}\theta\right)^2} \\
&\leqslant \frac{\pi^2}{4}(n+1)\min\left\{1, \left(\frac{n+1}{2}\theta\right)^{-2}\right\} \\
&\leqslant \frac{\pi^2}{2}\frac{n+1}{1+\left(\frac{n+1}{2}\theta\right)^2} =: \Psi_n(\theta).
\end{aligned} \tag{4}$$

It is easy to see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi_n(\theta)\,\mathrm{d}\theta \leqslant \frac{\pi^2}{2}, \quad n\in\mathbb{N}. \tag{5}$$

From [15, Lemma 2.11] and estimates (4–5) we immediately get estimate (3). □

3. Proofs of the main results

3.1. Norm estimates for the Fejér means.

First we consider the case of rearrangement-invariant Banach function spaces.

3.1.1. Lemma. *Let X be a rearrangement-invariant Banach function space on $\mathbb{T}$. Then for each $n\in\mathbb{N}$, the operator $F_n f = f * K_n$ is bounded on X and*

$$\sup_{n\in\mathbb{N}}\|F_n\|_{\mathcal{B}(X)} \leqslant 1.$$

Proof. By [1, Chap. 3, Lemma 6.1], for every $f\in X$ and every $n\in\mathbb{N}$,

$$\|f * K_n\|_X \leqslant \|K_n\|_{L^1}\|f\|_X.$$

It remains to recall that $\|K_n\|_{L_1}\leqslant 1$ for all $n\in\mathbb{N}$. □

Now we will show the corresponding results for Banach function spaces for which the Hardy–Littlewood maximal operator is bounded on X'.

3.1.2. Theorem. *Let X be a Banach function space on $\mathbb{T}$ such that the Hardy–Littlewood maximal operator M is bounded on its associate space X'. Then for each $n\in\mathbb{N}$, the operator $F_n f = f * K_n$ is bounded on X and*

$$\sup_{n\in\mathbb{N}}\|F_n\|_{\mathcal{B}(X)} \leqslant \pi^2\|M\|_{X'\to X'}.$$

Proof. The idea of the proof is borrowed from the proof of [4, Theorem 5.1]. Fix $f\in X$ and $n\in\mathbb{N}$. Since $K_n\geqslant 0$, we have $|f * K_n|\leqslant |f| * K_n$. Then from the Lorentz-Luxemburg theorem (see, e.g., [1, Chap. 1, Theorem 2.7]) we deduce that

$$\begin{aligned}\|f * K_n\|_X &\leqslant \big\| |f| * K_n \big\|_X = \big\| |f| * K_n \big\|_{X''}\\ &= \sup\left\{\int_{\mathbb{T}}\big(|f| * K_n\big)(t)\big|g(t)\big|\,\mathrm{d}m(t) \,:\, g\in X',\ \|g\|_{X'}\leqslant 1\right\}.\end{aligned}$$

Hence there exists a function $h\in X'$ such that $h\geqslant 0$, $\|h\|_{X'}\leqslant 1$, and

$$\|f * K_n\|_X \leqslant 2\int_{\mathbb{T}}\big(|f| * K_n\big)(t)h(t)\,\mathrm{d}m(t). \tag{6}$$

Taking into account that $K_n(e^{i\theta}) = K_n(e^{-i\theta})$ for all $\theta \in \mathbb{R}$, by Fubini's theorem, we get

$$\int_{\mathbb{T}} \big(|f| * K_n\big)(t)h(t)\,\mathrm{d}m(t) = \int_{\mathbb{T}} (h * K_n)(t)\big|f(t)\big|\,\mathrm{d}m(t).$$

From this identity and Hölder's inequality for X (see, e.g., [1, Chap. 1, Theorem 2.4]), we obtain

$$\int_{\mathbb{T}} \big(|f| * K_n\big)(t)h(t)\,\mathrm{d}m(t) \leqslant \|f\|_X \|h * K_n\|_{X'}. \tag{7}$$

Applying Lemma 2.3.1 to $h \in X' \subset L^1$, by the lattice property, we see that

$$\|h * K_n\|_{X'} \leqslant \frac{\pi^2}{2}\|Mh\|_{X'}. \tag{8}$$

Combining estimates (6–8) and taking into account that M is bounded on X' and that $\|h\|_{X'} \leqslant 1$, we arrive at

$$\|f * K_n\|_X \leqslant \pi^2 \|M\|_{X' \to X'} \|f\|_X.$$

Hence

$$\sup_{n \in \mathbb{N}} \|F_n\|_{\mathcal{B}(X)} = \sup_{n \in \mathbb{N}} \sup_{f \in X \setminus \{0\}} \frac{\|f * K_n\|_X}{\|f\|_X} \leqslant \pi^2 \|M\|_{X' \to X'} < \infty,$$

which completes the proof. □

3.2. Convergence of the Fejér means in the norm. The following statement is at the heart of the proof of the main results.

3.2.1. Theorem. *Suppose X is a separable Banach function space on $\mathbb{T}$. If X is rearrangement-invariant or the Hardy–Littlewood maximal operator is bounded on the associate space X', then for every $f \in X$,*

$$\lim_{n \to \infty} \|f * K_n - f\|_X = 0. \tag{9}$$

Proof. It is well known that for every $f \in C$,

$$\lim_{n \to \infty} \|f * K_n - f\|_C = 0$$

(see, e.g., [3, Chap. III, Theorem 1.1(a)] or [9, Theorem 2.11]). From the definition of the Banach function space X it follows that $C \subset X \subset L^1$, where both embeddings are continuous. Then, for every $f \in C$, (9) is fulfilled. From Lemma 2.2.1 we know that the set C is dense in the space X. By Lemma 3.1.1 and Theorem 3.1.2,

$$\sup_{n \in \mathbb{N}} \|F_n\|_{\mathcal{B}(X)} < \infty,$$

where $F_n f = f * K_n$. It remains to apply Lemma 2.1.1. □

This statement for rearrangement-invariant Banach function spaces is contained, e.g., in [5, p. 268]. Notice that the assumption of the separability of X is hidden there.

Now we formulate the corollary of the above theorem in the case of variable Lebesgue spaces.

3.2.2. Corollary. *Suppose $p \in \mathfrak{P}(\mathbb{T})$. If $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$ or if $p' \in \mathfrak{K}(\mathbb{T})$, then for every $f \in L^{p(\cdot)}$,*

$$\lim_{n\to\infty} \|f * K_n - f\|_{L^{p(\cdot)}} = 0.$$

For variable exponents $p \in \mathfrak{P}(\mathbb{T})$ satisfying $p_+ < \infty$ and $p \in LH_0(\mathbb{T})$, this result was obtained by Sharapudinov [16, Section 3.1]. For $p' \in \mathfrak{K}(\mathbb{T})$, the above corollary is new.

3.3. Proofs of Theorems 1.0.1 and 1.0.2. If $f \in H[X]$, then $p_n = f * K_n \in \mathcal{P}_A$ for all $n \in \mathbb{N}$ in view of (2). By Theorem 3.2.1, $\|p_n - f\|_X \to 0$ as $n \to \infty$. Thus the set $\mathcal{P}_A$ is dense in in the abstract Hardy space $H[X]$ built upon X.

Moreover, if X is a rearrangement-invariant Banach function space, then from Lemma 3.1.1 it follows that $\|p_n\|_X \leqslant \|f\|_X$ for all $n \in \mathbb{N}$.

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References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston 1988.

[2] Yu. M. Berezansky, Z. G. Sheftel, and G. F. Us, *Functional Analysis, Vol. 1*, Birkhäuser, Basel 1996.

[3] J. B. Conway, *The Theory of Subnormal Operators*, American Mathematical Society, Providence, RI 1991.

[4] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces*, Birkhäuser, Basel 2013.

[5] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer, Berlin 1993.

[6] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin 2011.

[7] E. Kapanadze and T. Kopaliani, *A note on maximal operator on $L^{p(t)}(\Omega)$ spaces*, Georgian Math. J. 15 (2008), 307–316.

[8] A. Yu. Karlovich, *On the essential norm of the Cauchy singular integral operator in weighted rearrangement--invariant spaces*, Integr. Equ. Oper. Theory 38 (2000), 28–50.

[9] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dower Publications, Inc., New York 1976.

[10] V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko, *Integral Operators in Non-Standard Function Spaces. Volume 1: Variable Exponent Lebesgue and Amalgam Spaces*, Birkhäuser Verlag, Basel 2016.

[11] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Amer. Math. Soc., Providence 1982.

[12] A. Lerner, *Some remarks on the Hardy–Littlewood maximal function on variable L^p spaces*, Math. Z. 251 (2005), 509–521.

[13] K. Leśnik and L. Maligranda, *Interpolation of abstract Cesàro, Copson and Tandori spaces*, Indag. Math., New Ser. 27 (2016), 764–785.

[14] L. Maligranda, *Hidegoro Nakano* (*1909–1974*) *– on the centenary of his birth*, (Kitakyushu, September 14–17, 2009), Proceedings of the 3rd international symposium on Banach and function spaces (ISBFS 2009) (M. Kato, ed.), Yokohama Publishers, Yokohama, 2011, 99–171.

[15] C. Muscalu and W. Schlag, *Classical and Multilinear Harmonic Analysis. Vol. I.*, Cambridge Studies in Advanced Mathematics, vol. 137, Cambridge University Press, Cambridge 2013.

[16] I. I. Sharapudinov, *Uniform boundedness in* L^p $(p = p(x))$ *of some families of convolution operators*, Math. Notes 59 (1996), 205–212.

[17] Q. Xu, *Notes on interpolation of Hardy spaces*, Ann. Inst. Fourier 42 (1992), 875–889.