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Copies of $\mathbb{R}^{\mathbb{N}}$ in F-lattices

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Summary. Modifying ideas presented in [14] we prove that a complete metrizable locally solid Riesz space E contains a linear subspace linearly homeomorphic to $\mathbb{R}^{\mathbb{N}}$ iff E contains a sublattice order isomorphic to $\mathbb{R}^{\mathbb{N}}$.

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One of the natural and classical problems considered in the Banach lattice theory is the following:

($*$) *Suppose that a Banach lattice E contains a linear subspace isomorphic (=linearly homeomorphic) to a Banach lattice F. Does E contain a sublattice (=Riesz subspace) E_1 order isomorphic to F (= there exists a linear positive bijection $T\colon E_1 \to F$ such that T^{-1} is positive too)?*

An affirmative answer is well known for $F = c_0$ (see [2] or [7]) and for $F = \ell^\infty$ whenever E belongs to the class of σ-Dedekind complete Banach lattices (see [2] or [7]). The case $F = \ell^1$ is more subtle. Of course, the space $C[0,1]$ contains copies of all separable Banach spaces, but no sublattice in $C[0,1]$ is order isomorphic to ℓ^1. On the other hand, a Banach lattice E contains a complemented subspace isomorphic to ℓ^1 if and only if E contains a sublattice order isomorphic to ℓ^1 (see [2] or [7]). Restricting considerations to Banach lattices E with order continuous norm we can strike out the word “complemented” in the previous equivalence (see [13]). The problem formulated in ($*$) was investigated in

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a general case when E is a topological Riesz space (= locally solid Riesz space). It turns out that the answer is affirmative again for $F \in \{c_0, \ell^\infty\}$ when E belongs to the class TRS-λ of the so-called topological Riesz spaces of λ-measurable functions satisfying some completion properties (see [4, Theorems 5.5 and 5.8]). It is worth adding that the class TRS-λ contains complete topological Riesz spaces with separating continuous dual (for details, see [4, Sections 3 and 5]). Moreover, it was shown in [12] that the assumption of topological completeness is not necessary when E is a σ-Dedekind complete locally convex topological Riesz space and $F = \ell^\infty$.

Sequence spaces ℓ^1, c_0, ℓ^∞ are important because the presence or absence of their copies determines several important order and topological properties of Banach lattices (the Lebesgue property = order continuity of the norm, the Lebesgue and Levi properties = KB-space, reflexivity – see [2, 7, 11]). Clearly, ℓ^1, c_0, ℓ^∞ are ideals in the space of real sequences $\mathbb{R}^{\mathbb{N}}$. Products of the form $\mathbb{R}^\Gamma$ are very specific from the point of view of their natural pointwise convergence topology τ_p, which is minimal, i.e., $\mathbb{R}^\Gamma$ does not admit any Hausdorff linear topology weaker than τ_p, and, as is well known, ($\mathbb{R}^\Gamma$, τ_p) is a unique (up to isomorphism) minimal real locally convex topological space. The authors of [14] obtained an interesting result related to the problem $(*)$. Namely, they proved that a function F-lattice E (= complete metrizable locally solid Riesz space where E is an ideal in the space $L^0(\mu)$ of equivalence classes of measurable functions with the σ-finite measure μ) contains a linear subspace linearly homeomorphic to $\mathbb{R}^{\mathbb{N}}$ if and only if E contains a sublattice order isomorphic to $\mathbb{R}^{\mathbb{N}}$. It turns out that the above equivalence remains valid in the whole class of F-lattices (=complete metrizable locally solid Riesz spaces) – it is enough to introduce several modifications to the proof of the Main Theorem formulated in [14]. We obtain a general form where the assumption about the Dedekind completeness of E is superfluous. Below we present the detailed exposition. Our terminology and notation are standard – they are practically the same as in [1, 2, 7] and [5].

1. Theorem. *If $E = (E, \tau)$ denotes a complete metrizable locally solid Riesz space then the following statements are equivalent.*

(i) *E contains a linear subspace linearly homeomorphic to $\mathbb{R}^{\mathbb{N}}$.*

(ii) *E contains a sublattice order isomorphic to $\mathbb{R}^{\mathbb{N}}$.*

Proof. Let $\|\cdot\|$ be a monotone F-norm determining the topology τ. Suppose first that E is Dedekind complete. According to Bessaga–Pełczyński–Rolewicz theorem ([3, Theorem 9] or [9, Proposition 4.2.7]) there exists a sequence (x_n) with $\sup_{t\in\mathbb{R}} \|tx_n\| = r(x_n) \to 0$. Since $r(x_n) = r(|x_n|)$ we can assume x_n's are positive. The topological completeness of E implies $\{x_n : n \in \mathbb{N}\}$ is included in the principal ideal generated by some $x \in E_+$ (we can take $x = \sum t_n x_n$ where the numbers $0 < t_n$ satisfy the condition $\|t_n x_n\| < 2^{-n}$). By the classical Freudenthal spectral theorem ([6, Theorems 40.2 and 40.3]) $x_n = \sup_m y_{n,m}$

for some sequences $(y_{n,m})_{m=1}^{\infty} \subset \operatorname{lin} C(x) \cap E_+$ where $C(x)$ is the set of components of x. More precisely, every $y_{n,m}$ is a linear combination of pairwise disjoint components with positive scalar coefficients (compare [11, Proposition 0.2.7]). Clearly, $r(\cdot)$ is subadditive monotone and $r(y) = r(ty)$ for every $y \in E$ and a number $t \neq 0$, and so we are able to find a sequence $(p_n) \subset C(x)$ such that $0 < \|p_n\| \leqslant r(p_n) \leqslant r(x_n) \to 0$. Hence, passing to a subsequence if necessary, without loss of generality we may assume that $\|p_{n+1}\| < \frac{1}{3}\|p_n\|$. The last inequality implies $\sum_{n=1}^{\infty} \|p_n\| < \infty$. Put $q_\ell = p_\ell \wedge (x - \bigvee_{k=\ell+1}^{\infty} p_k)$. Components q_ℓ are non-zero because

$$\begin{aligned}
\|p_\ell\| &= \|p_\ell \wedge x\| \\
&= \Big\|p_\ell \wedge \Big(\Big(x - \bigvee_{k=\ell+1}^{\infty} p_k\Big) \vee \bigvee_{k=\ell+1}^{\infty} p_k\Big)\Big\| \\
&\leqslant \|q_\ell + \bigvee_{k=\ell+1}^{\infty} p_k\| \\
&\leqslant \|q_\ell\| + \sum_{j=1}^{\infty} \|p_{j+\ell}\| \\
&\leqslant \|q_\ell\| + \sum_{j=1}^{\infty} \frac{1}{3^j}\|p_\ell\| \\
&= \|q_\ell\| + \frac{1}{2}\|p_\ell\|,
\end{aligned}$$

i.e., $\|q_\ell\| \geqslant \frac{1}{2}\|p_\ell\| > 0$. It is clear that $\overline{\operatorname{lin}}\{q_n : n \in \mathbb{N}\}$ contains an order copy of $\mathbb{R}^{\mathbb{N}}$ because $r(q_n) \to 0$ and q_ℓ's are pairwise disjoint: $q_{\ell+m} \wedge q_\ell \leqslant p_{\ell+m} \wedge (x - p_{\ell+m}) = 0$.

Let E be an arbitrary complete metrizable locally solid Riesz space and denote by E^δ the Dedekind completion of E. The formula $\|x^\delta\| = \inf\{\|x\| : x \geqslant |x^\delta|,\ x \in E\}$ extends the F-norm on E to a complete F-norm on E^δ (see [10] or [8] – the latter paper concerns norms but the proof of Theorem A formulated there works, without changes, for F-norms). The space E is isometrically embedded in E^δ, thus a linearly homeomorphic copy of $\mathbb{R}^{\mathbb{N}}$ is included in E^δ. By the previous part of our proof, E^δ contains a sequence (q_n^δ) of pairwise disjoint elements such that $\sum_{n=1}^{\infty} r(q_n^\delta) < \infty$, i.e., $\overline{\operatorname{lin}}\{q_n^\delta : n \in \mathbb{N}\}$ is an order isomorphic copy of $\mathbb{R}^{\mathbb{N}}$ in E^δ. Since E is order dense in its Dedekind completion, then we can find $0 < y_n \in [0, q_n^\delta] \cap E$. Clearly, $\sum_{n=1}^{\infty} r(y_n) < \infty$ and $\overline{\operatorname{lin}}\{y_n : n \in \mathbb{N}\}$ is an order copy of $\mathbb{R}^{\mathbb{N}}$ in E. □

The second part of the proof implies immediately the following corollary.

2. Corollary. *Let F be a closed order dense sublattice in a complete metrizable locally solid Riesz space E. Then F contains a sublattice order isomorphic to* $\mathbb{R}^{\mathbb{N}}$ *if and only if E contains a linear subspace linearly homeomorphic to* $\mathbb{R}^{\mathbb{N}}$.

3. Remark. I have to explain that the foregoing theorem was put forward by Wójtowicz, the first author of [14], on the web page www.math.uni.wroc.pl/~measure07/abstracts/wojtowicz.pdf, where one can find an abstract of Wójtowicz's talk "A subsequence disjoint refinement property for Boolean algebras and applications" presented at the conference "Measure Theory. Edward Marczewski Centennial Conference" in 2007. Unfortunately, as far as I know, Wójtowicz has never published a proof. The recently published paper [14] containing an essentially weaker version of the theorem indicates either that the information on the web page was not precise (some assumptions were omitted) or that Wójtowicz's previous proof from 2007 concerning a general form of our theorem contained a gap. The circumstances described above have led me to a decision to prepare this short paper.

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