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A note on the order of polynomial-like iterative equations

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Summary. We show that, under reasonable assumptions, two negative roots can be eliminated from the characteristic equation of a polynomial-like iterative equation. This result gives a new case where we may lower the order of such an equation.

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1. Introduction

Let n be a positive integer and $I \subset \mathbb{R}$ an interval. We are interested in the so-called polynomial-like iterative equations, namely, equations of the form

$$a_n f^n(x) + \cdots + a_1 f(x) + a_0 x = 0, \tag{1}$$

where f^k stands for the k-fold iterate of a self-mapping unknown function $f\colon I \to I$, and the coefficients $a_n, \dots, a_1, a_0$ are fixed real numbers with $a_0 \neq 0$. In general, it is difficult to find all continuous functions satisfying equation (1) even in the case $n = 3$; a partial solution in this case was given in [7] and the complete solution for $n = 2$ was presented in [3,5]. One method of finding solutions to equation (1) involves lowering its order. Such results were obtained in [1,6,9] (see Theorem 1 below); the present paper contains a new result in

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this spirit. For a similar investigation concerning non-homogenous polynomial-like iterative equations where the zero on the right-hand side is replaced by an arbitrary continuous function (see, e.g., [8]).

We shall recall some basic properties of solutions to polynomial-like equations. Assume that a continuous function $f\colon I \to I$ satisfies equation (1). It can be easily shown that f is injective (see, e.g. [1, Lemma 2.1]) and therefore monotone. Assuming that $f(x) = rx$ we obtain the so-called characteristic equation of (1):

$$a_n r^n + \cdots + a_1 r + a_0 = 0. \tag{2}$$

This equation may also be considered as the characteristic equation of the recurrence relation

$$a_n x_{j+n} + \cdots + a_1 x_{j+1} + a_0 x_j = 0 \tag{3}$$

in which the sequence $(x_j)_{j\in\mathbb{N}_0}$ is obtained in the following way: We choose $x_0 \in I$ arbitrarily and put $x_j = f(x_{j-1})$ for $j \in \mathbb{N}$. It is easy to see that f satisfies (1) if and only if $(x_j)_{j\in\mathbb{N}_0}$ satisfies (3).

Since the function f is monotone, the sequence $(x_j)_{j\in\mathbb{N}_0}$ is either monotone (in the case of increasing f) or anti-monotone (in the case of decreasing f). By *anti-monotone* we mean that the expression $(-1)^j(x_{j+1} - x_j)$ does not change its sign when j runs through $\mathbb{N}_0$. Consider $y_0 \in I$ and define a sequence $(y_j)_{j\in\mathbb{N}_0}$ in the same way as we did for x_0. Similarly, the sequence $(x_j - y_j)_{j\in\mathbb{N}_0}$ has a constant sign, in the case of increasing f, or alternates in sign, in the case of decreasing f. Let us note that the sequence $(x_{j+1} - y_j)_{j\in\mathbb{N}_0}$ also has the same property.

In the case where f is surjective (and hence bijective) we can consider the dual equation

$$a_0 f^n(x) + \cdots + a_{n-1} f(x) + a_n x = 0. \tag{4}$$

Putting $f^{-n}(x)$ in place of x we see that f satisfies (1) if and only if f^{-1} satisfies (4). We can also extend the above defined sequence $(x_j)_{j\in\mathbb{N}_0}$ to the whole $\mathbb{Z}$ by setting $x_{-j} = f^{-1}(x_{-j+1})$ for $j \in \mathbb{N}$. Then relation (3) is satisfied for all $j \in \mathbb{Z}$.

For the theory of linear recurrence relations we refer the reader, for instance, to [2, § 3.2]. We shall recall only the most significant theorem in this matter. In order to do this and simplify the writing we introduce the following notation: For a given polynomial $c_n r^n + \cdots + c_1 r + c_0$ we denote by $\mathcal{R}(c_n, \dots, c_0)$ the collection $\{(r_1, k_1), \dots, (r_p, k_p)\}$ of all pairs of pairwise distinct (complex) roots $r_1, \dots, r_p$ and their multiplicities $k_1, \dots, k_p$, respectively. Here and throughout the present paper by a polynomial we mean a polynomial with real coefficients. Note that in the introduced notation $k_1 + \cdots + k_p$ equals the degree of $c_n r^n + \cdots + c_1 r + c_0$ and by writing $(\mu, k), (\overline{\mu}, k) \in \mathcal{R}(c_n, \ldots, c_0)$ we mean μ to be non-real.

1. Theorem. *Assume that*

$$\mathcal{R}(a_n, \ldots, a_0) = \{(\lambda_1, l_1), \ldots, (\lambda_p, l_p), (\mu_1, m_1), (\overline{\mu_1}, m_1), \ldots, (\mu_q, m_q), (\overline{\mu_q}, m_q)\}.$$

Then a real-valued sequence $(x_j)_{j \in \mathbb{N}_0}$ is a solution to (3) if and only if it is given by

$$x_j = \sum_{k=1}^{p} A_k(j)\lambda_k^j + \sum_{k=1}^{q} \big(B_k(j)\cos j\phi_k + C_k(j)\sin j\phi_k\big)|\mu_k|^j \quad \text{for } j \in \mathbb{N}_0,$$

where A_k is a polynomial whose degree equals at most $l_k - 1$ for $k = 1, \ldots, p$ and B_k, C_k are polynomials whose degrees equal at most $m_k - 1$, with ϕ_k being an argument of μ_k, for $k = 1, \ldots, q$.

It is worth mentioning that the above theorem is also valid for sequences defined on the whole of $\mathbb{Z}$. We shall use this fact in the proof of our main result.

2. The main result

It was observed by Matkowski and Zhang in [4] that if a polynomial $b_m r^m + \cdots + b_1 r + b_0$ divides $a_n r^n + \cdots + a_1 r + a_0$ and f satisfies

$$b_m f^m(x) + \cdots + b_1 f(x) + b_0 x = 0, \tag{5}$$

then f satisfies also (1). One method of solving equation (1) involves a partial converse of this statement. More precisely, we want to find a divisor of the polynomial $a_n r^n + \cdots + a_1 r + a_0$ such that a corresponding polynomial-like iterative equation of lower order is satisfied. Some known results concerning the elimination of non-real roots or real roots of opposite sign are listed below.

1. Theorem.

(i) [1, Thm. 3.3] (*cf.* [6, Thm. 5] *and* [9, Thm. 1]) *Assume that*

$$\mathcal{R}(a_n, \ldots, a_0) = \{(\lambda_1, l_1), \ldots, (\lambda_p, l_p), (\mu_1, k_1), (\overline{\mu_1}, k_1), \ldots, (\mu_q, k_q), (\overline{\mu_q}, k_q)\}.$$

If $|\lambda_1| \leqslant \ldots \leqslant |\lambda_p| < |\mu_1| \leqslant \ldots \leqslant |\mu_q|$, then a continuous function $f\colon I \to I$ satisfies equation (1) if and only if it satisfies (5) with

$$\mathcal{R}(b_m, \ldots, b_0) = \{(\lambda_1, l_1) \ldots, (\lambda_p, l_p)\}.$$

(ii) [1, Thms. 4.1 and 4.2] *Assume that*

$$\mathcal{R}(a_n, \dots, a_0) = \{(r_1, k_1), (r_2, k_2), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

Let also $|r_1| < |\lambda_1| \leqslant \dots \leqslant |\lambda_p| < |r_2|$ *and* r_1, r_2 *be real with* $r_1 r_2 < 0$; *say* $r_i > 0$ *and* $r_j < 0$. *Then a continuous increasing surjection* $f\colon I \to I$ *satisfies equation* (1) *if and only if it satisfies* (5) *with*

$$\mathcal{R}(b_m, \dots, b_0) = \{(r_i, k_i), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

If $r_i \neq 1$, *then a continuous decreasing surjection* $f\colon I \to I$ *satisfies equation* (1) *if and only if it satisfies* (5) *with*

$$\mathcal{R}(b_m, \dots, b_0) = \{(r_j, k_j), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

If $r_i = 1$, *then a continuous decreasing surjection* $f\colon I \to I$ *satisfies equation* (1) *if and only if it satisfies* (5) *with*

$$\mathcal{R}(b_m, \dots, b_0) = \{(1,1), (r_j, k_j), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

These results were proved by examining the asymptotic behaviour of the sequence of consecutive iterates of the unknown function at a given point. Using a similar approach we obtain our new result concerning the elimination of negative roots, which reads as follows.

2. Theorem. *Assume that*

$$\mathcal{R}(a_n, \dots, a_0) = \{(r_1, k_1), (r_2, k_2), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\}.$$

If $|r_2| < |\lambda_1| \leqslant \dots \leqslant |\lambda_p| < |r_1|$ *and* r_1, r_2 *are real with* $r_1 < -1 < r_2 < 0$, *then a continuous surjection* $f\colon I \to I$ *satisfies equation* (1) *if and only if it satisfies equation* (5) *with*

$$\mathcal{R}(b_m, \dots, b_0) = \{(r_i, k_i), (\lambda_1, l_1), \dots, (\lambda_p, l_p)\},$$

where $i = 1$ *or* $i = 2$.

Proof. Choose $x \in I$ arbitrarily. Define a sequence $(x_j)_{j\in\mathbb{Z}}$ by putting $x_0 = x$, $x_j = f(x_{j-1})$ and $x_{-j} = f^{-1}(x_{-j+1})$ for $j \in \mathbb{N}$. Then relation (3) is satisfied for all $j \in \mathbb{Z}$. Therefore, by Theorem 1, we have

$$x_j = A(j) r_1^j + F(j) + B(j) r_2^j \quad \text{for } j \in \mathbb{Z},$$

where A, B are polynomials and F stands for the part of the solution to (3) for which the roots $\lambda_1, \dots, \lambda_p$ are responsible. We shall show that either $A \equiv 0$ or $B \equiv 0$.

For an indirect proof, suppose that both polynomials A and B are non-zero. Denote by s and t the degrees of A and B, respectively. Similarly, let a and b be the leading coefficients of A and B.

Since

$$(-1)^j(x_{j+1}-x_j) = \big(A(j+1)r_1 - A(j)\big)|r_1|^j \\ + (-1)^j\big(F(j+1)-F(j)\big) + \big(B(j+1)r_2 - B(j)\big)|r_2|^j,$$

we have

$$\lim_{j\to-\infty}\frac{(-1)^j(x_{j+1}-x_j)}{|j|^t\cdot|r_2|^j} = (-1)^t(r_2-1)b \tag{6}$$

and

$$\lim_{j\to\infty}\frac{(-1)^j(x_{j+1}-x_j)}{j^s\cdot|r_1|^j} = (r_1-1)a. \tag{7}$$

This shows that the sequence $(x_j)_{j\in\mathbb{Z}}$ cannot be monotone (in fact, this shows that it cannot be monotone when either $A\not\equiv 0$ or $B\not\equiv 0$); consequently, f cannot be increasing. Thus f is decreasing.

According to the above observation the expression $(-1)^j(x_{j+1}-x_j)$ has a constant sign when j runs through $\mathbb{Z}$. Combining this fact with equations (6) and (7), we conclude that a and $(-1)^t b$ have the same sign. Further, since f^2 is increasing, the expression

$$x_{2j+2}-x_{2j} = \big(A(2j+2)r_1^2 - A(2j)\big)|r_1|^{2j} \\ + F(2j+2) - F(2j) + \big(B(2j+2)r_2^2 - B(2j)\big)|r_2|^{2j}$$

also has a constant sign. Similarly, we have

$$\lim_{j\to-\infty}\frac{x_{2j+2}-x_{2j}}{|2j|^t\cdot|r_2|^{2j}} = (-1)^t(r_2^2-1)b$$

and

$$\lim_{j\to\infty}\frac{x_{2j+2}-x_{2j}}{(2j)^s\cdot|r_1|^{2j}} = (r_1^2-1)a.$$

As a result, a and $(-1)^t b$ are of opposite sign: a contradiction. Therefore, $A\equiv 0$ or $B\equiv 0$. Using Theorem 1 once again we conclude that the assertion holds for a fixed $x\in I$. It remains to show that elimination of the root r_1 or r_2 does not depend on x.

Consider $y\in I$ and define a sequence $(y_j)_{j\in\mathbb{Z}}$ in the same way as we did for x. Suppose, for the sake of a contradiction, that $x_j = A(j)r_1^j + F(j)$ and $y_j = G(j) + B(j)r_2^j$ for $j\in\mathbb{Z}$ with non-zero polynomials A and B (F and G stand for the terms for which the roots $\lambda_1,\dots,\lambda_p$ are responsible). As before, let s, t be the degrees and a, b be the leading coefficients of A

and B, respectively. Since f monotonically decreases, the sequence $(x_j - y_j)_{j\in\mathbb{Z}}$ alternates in sign. Thus the expression

$$(-1)^j(x_j - y_j) = A(j)|r_1|^j + (-1)^j\big(F(j) - G(j)\big) - B(j)|r_2|^j$$

has a constant sign. Futher, we have

$$\lim_{j\to-\infty} \frac{(-1)^j(x_j - y_j)}{|j|^t \cdot |r_2|^j} = (-1)^{t+1}b$$

and

$$\lim_{j\to\infty} \frac{(-1)^j(x_j - y_j)}{j^s \cdot |r_1|^j} = a,$$

which means that a and $(-1)^{t+1}b$ have the same sign. Repeating this reasoning with the sequence $(x_{j+1} - y_j)_{j\in\mathbb{Z}}$ we conclude that a and $(-1)^{t+1}b$ have opposite signs. The obtained contradiction ends the proof. □

3. Remark. Since the equation $2f^2(x) + 5f(x) + 2x = 0$ is satisfied by $f(x) = -2x$ and $f(x) = -\frac{1}{2}x$, in general it cannot be decided which root, whether r_1 or r_2, may be eliminated. Therefore, Theorem 2 states that equation (1) is actually equivalent to an alternative of two equations of lower order.

4. Remark. It is worth mentioning that if $I = \mathbb{R}$, then f is necessarily bijective (see [9, Lemma 1]). Therefore, the assumption of surjectivity in Theorems 1 and 2 is satisfied automatically in this case.

5. Remark. Using the quoted results and Theorem 2, the order of equation (1) can be essentially lowered in many important cases. However, some cases still remain open. For instance, it is unknown whether all non-real roots may be eliminated from the characteristic equation (2) without additional assumptions (cf. [1, Section 6] and [9, Section 6]).

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