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On second κ-variation

Jurancy Ereú, Lorena López, and Nelson Merentes

Summary. We present the notion of bounded second κ-variation for real functions defined on an interval $[a,b]$. We introduce the class $\kappa BV^2([a,b])$ of all functions of bounded second κ-variation on $[a,b]$. We show several properties of this class and present a sufficient condition under which a composition operator acts between these spaces.

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functions of bounded second variation; functions of bounded variation; κ-function

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1. introduction

The concept of a function of bounded variation was introduced in 1881 by Camile Jordan [11] who carried out a rigorous analysis of the proof given by Dirichlet [8] on the convergence of the Fourier series of a monotone function. Jordan, who exploited the fact that the concept was already implicit in the work of Dirichlet, characterized the functions of bounded variation (on an interval) as those that can be expressed as the diference of two increasing (decreasing) functions.

The interest generated by the notion of a function of bounded variation led to some important generalizations of that concept, mainly, to help the search for larger classes

Jurancy Ereú, Departamento de Matemáticas, Decanato de Ciencias y Tecnología, Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Venezuela (*e-mail*: jereu@ucla.edu.ve)

Lorena López, Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida, Venezuela (*e-mail*: lomalopez@ula.ve)

Nelson Merentes, Escuela de Matemáticas, Universidad Central de Venezuela, Caracas, Venezuela (*e-mail*: nmerucv@gmail.com)

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of functions with pointwise convergent Fourier series (see e.g., [5, 16, 18, 21–24]). As in the classical case, these generalizations found many applications in the study of certain (partial) differential and integral equations (see e.g., [4]) and also in the theory of linear and nonlinear composition operators: namely, to find necessary and sufficient conditions guaranteeing that a composition operator maps a given space of functions of generalized bounded variation (or a subset of it) into itself. Other conditions, called *acting conditions* involve: boundedness, continuity, compactness (in the linear case), the local or global Lipschitz condition (see, e.g., [2, 6, 9, 12, 17]). In a recent book by J. Appell, J. Banaś and N. Merentes [1], a thorough study of the structure of some similar spaces is presented as well as the acting conditions of superposition operators on such spaces.

In this work, we introduce the concept of a function of bounded second κ-variation, a natural generalization of De la Vallée Poussin's classical notion of a function of bounded second variation and we show that the largest classes can be equipped with a norm which turns them into Banach spaces. In particular, we show that the functions in the unit ball of such a space are uniformly majorized by a certain fixed function. Moreover, using techniques similar to those used in [10], we give a sufficient condition for a composition operator to act between those spaces.

2. Preliminaries

Given an interval $[a,b] \subset \mathbb{R}$, we will use the notation $\Pi([a,b])$ to denote the set of all partitions of $[a,b]$, whereas $\Pi_3([a,b])$ will denote the subset of $\Pi([a,b])$ consisting of partitions of $[a,b]$ with at least three points.

Recall that a function $u : [a,b] \to \mathbb{R}$ is said to be of bounded variation (in the sense of Jordan) if

$$V\big(u;[a,b]\big) := \sup_{\xi} \sum_{i=1}^{n} \big|u(t_i) - u(t_{i-1})\big| < \infty,$$

where the supremum is taken over the set of all partitions $\xi = \{a = t_o < t_1 < \cdots < t_n = b\} \in \Pi([a,b])$.

The notion of bounded second variation in the sense of De La Vallée Poussin is defined as follows: A function $u : [a,b] \to \mathbb{R}$ is of bounded second variation if and only if

$$V^2\big(u;[a,b]\big) := \sup_{\pi \in \Pi_3([a,b])} \sum_{i=0}^{m-2} \big|u[t_{i+1}, t_{i+2}] - u[t_i, t_{i+1}]\big| < \infty,$$

where

$$u[t_{i+1}, t_{i+2}] := \frac{u(t_{i+2}) - u(t_{i+1})}{t_{i+2} - t_{i+1}}, \quad i = 0, \dots, m-2.$$

The class of all functions of bounded second variation on $[a,b]$ in the sense of De La Vallée Poussin is denoted by $BV^2([a,b])$.

The following properties of functions in $BV^2([a,b])$ are known (see [17,19] and [20]).

2.1. Proposition. *Let $u \in BV^2([a,b])$.*

(i) *If $v \in BV^2([a,b])$ and λ is any real constant, then*

$$V^2\big(\lambda u;[a,b]\big) = |\lambda| V^2\big(u;[a,b]\big)$$

and

$$V^2\big(u+v;[a,b]\big) \leqslant V^2(u;[a,b]) + V^2\big(v;[a,b]\big).$$

(ii) (Monotonicity) *If $a < c < d < b$, then $V^2(u;[c,d]) \leqslant V^2(u;[a,b])$.*

(iii) (Semi-additivity) *If $a < c < b$, then $u \in BV^2([a,c])$, $u \in BV^2([c,b])$ and $V^2(u;[a,b]) \geqslant V^2(u;[a,c]) + V^2(u;[c,b])$.*

(iv) *$u[y_0,y_1]$ is bounded for all $y_0, y_1 \in [a,b]$.*

(v) *u is Lipschitz and therefore absolutely continuous on $[a,b]$.*

(vi) *$u \in BV^2([a,b])$ if and only if $u = u_1 - u_2$, where u_1, u_2 are convex functions.*

(vii) *A necessary and sufficient condition for a function F to be the integral of a function $f \in BV([a,b])$ is that $F \in BV^2([a,b])$.*

(viii) *If u is twice differentiable with u'' integrable on $[a,b]$, then $u \in BV^2([a,b])$ and $V^2(u;[a,b]) = \int_a^b |u''(t)|dt$.*

3. κ functions

In 1975, B. Korenblum [14] introduced the notion of κ-variation of a function, while studying the problem of representation of harmonic functions defined on the unit disk of the complex plane by means of generalized Poisson integrals involving the so-called premeasures defined on sub-intervals of $[0,2\pi]$. This notion differs from the classical notion, and other known *variations*, in that Korenblum's concept maximizes ratios between Jordan's sums and the so-called κ-*entropies* generated by a distortion function[1] κ which measures lengths in the domain of functions. A weak point of this notion, as we will show later, is that the associated κ-variation functionals need not be monotone with respect to enlargements of partitions or additive on union of intervals. An advantage, on the other hand, is that

[1] Actually, the letter κ stands for Carleson (Lennart), according to Korenblum, [14].

a function of bounded κ-variation can be decomposed into a difference of two so-called κ-decreasing functions (see [7]).

3.1. Definition. A function $\kappa\colon[0,1]\to[0,1]$ is called a *distortion function* or *κ-function* if it is continuous, nondecreasing, concave on $[0,1]$, and such that $\kappa(0)=0$, $\kappa(1)=1$ and

$$\lim_{x\to 0^+}\frac{\kappa(x)}{x}=\infty.$$

That is, κ has an infinite slope at the origin.

Note that any distortion function κ satisfies

$$\frac{\kappa(x+y)-\kappa(y)}{(x+y)-y}\leqslant\frac{\kappa(x)-\kappa(0)}{x-0},\qquad x,y,x+y\in[0,1]$$

and therefore it is subadditive; that is, if $x,y\in[0,1]$ are such that $x+y\in[0,1]$ then

$$\kappa(x+y)\leqslant\kappa(x)+\kappa(y). \tag{1}$$

In particular, if $x\in[0,1]$, $n,m\in\mathbb{N}$, and $\mu\in[0,\infty)$ are such that $nx\in[0,1]$, $\mu x\in[0,1]$, then

$$\frac{1}{m}\kappa(x)\leqslant\kappa\left(\frac{x}{m}\right),\quad \kappa(nx)\leqslant n\kappa(x)\quad\text{and}\quad \kappa(\mu x)\leqslant[[\mu+1]]\kappa(x),$$

where $[[a]]$ denotes the *integer part* of the positive real number a, that is,

$$[[a]]:=\max\{n\in\mathbb{N}:n\leqslant a\}.$$

The set of all distortion functions will be denoted by $\mathcal{K}$.

Throughout the paper, unless explicitly stated otherwise, we will assume that κ is a distortion function and $[a,b]\subset\mathbb{R}$ is a closed interval.

3.2. Definition ([15]). Let $\kappa\in\mathcal{K}$ and let $\xi=\{t_j\}_0^n\in\pi[a,b]$. The quantity

$$\kappa_\xi:=\kappa(\xi;[a,b])=\sum_{j=1}^n\kappa\left(\frac{t_j-t_{j-1}}{b-a}\right)$$

is called the *κ-entropy* of ξ relative to $[a,b]$.

Examples ([15]):

- $\kappa(s):=s(1-\log s)$. The corresponding entropy is called the *Shannon entropy*.
- $\kappa(s):=s^\alpha\ \ (0<\alpha<1)$. The corresponding entropy is called the *Lipschitz entropy*.
- $\kappa(s):=(1-\frac{1}{2}\log s)^{-1}$. The corresponding entropy is called the *Dini entropy*.

Notice that for any $\xi = \{t_j\}_{j=0}^n \in \pi[a,b]$ we have

$$1 = \kappa(1) = \kappa\left(\sum_{j=1}^{n} \frac{t_j - t_{j-1}}{b-a}\right) \leqslant \kappa_\xi. \tag{2}$$

Let $\kappa \in \mathcal{K}$. A function $u \in \mathbb{R}^{[a,b]}$ is said to be of *bounded κ-variation* if there is a positive constant C such that, for every partition $\xi = \{t_j\}_{j=0}^n \in \pi[a,b]$ of $[a,b]$, the following inequality holds

$$\sum_{j=1}^{n} \left|u(t_j) - u(t_{j-1})\right| \leqslant C\kappa\big(\xi;[a,b]\big). \tag{3}$$

The total κ-variation of u in $[a,b]$ is defined as

$$\kappa V(u) := \kappa V(u;[a,b])$$
$$:= \inf\left\{C : \sum_{j=1}^{n} \left|u(t_j) - u(t_{j-1})\right| \leqslant C\kappa\big(\xi;[a,b]\big) \,:\, \xi = \{t_j\}_{j=0}^n \in \pi[a,b]\right\}.$$

The set of all functions of bounded κ–variation on $[a,b]$ will be denoted by $\kappa BV[a,b]$. It is readily seen that this is a linear space. If equipped with the norm

$$\|u\|_{\kappa BV[a,b]} := |u(a)| + \kappa V\big(u;[a,b]\big)$$

the space $\kappa BV[a,b]$ becomes a Banach algebra (see [1, 3, 13]); in fact, it readily follows from the definitions that if $u, v \in \kappa BV[a,b]$ then

$$\|uv\|_{\kappa BV[a,b]} \leqslant \|u\|_\infty \|v\|_{\kappa BV[a,b]} + \|v\|_\infty \|u\|_{\kappa BV[a,b]}.$$

Considering the partition $\xi := \{a,s,b\}$, for fixed s, it is readily seen that every $u \in \kappa BV[a,b]$ is bounded with

$$\|u\|_\infty \leqslant 2\|u\|_{\kappa BV}.$$

On the other hand, from (2) it follows easily that every function of bounded (Jordan) variation on $[a,b]$ is in $\kappa BV[a,b]$, and

$$\kappa V(u) \leqslant V(u).$$

Also, from (2) and the fact that the trivial partition $\{a,b\}$ is optimal for (3), it follows that if u is a monotone function on $[a,b]$ then

$$\kappa V(u) = V(u) = \left|u(b) - u(a)\right|.$$

In his seminal 1975 paper [14], B. Korenblum presented a Jordan like decomposition result for the functions in the space $\kappa BV[a,b]$, when κ is the Shannon distortion. Later, Cyphert and Kelingos [7] generalized that result by showing that for any $\kappa \in \mathcal{K}$ the functions in $\kappa BV[a,b]$ can be expressed as a difference of two κ-decreasing functions.

A function $u \in \mathbb{R}^{[a,b]}$ is said to be *κ-decreasing* if there is a constant $A \geqslant 0$ such that for every interval $I = [x,y] \subset [a,b]$, $u(y) - u(x) \leqslant A\kappa\left(\frac{y-x}{b-a}\right)$.

From the decomposition theorem it follows that every function $u \in \kappa BV[a,b]$ is regulated, that is, it has one sided limits $u(t+)$ and $u(t-)$ at every point $t \in (a,b)$ and the limits $u(a+)$ and $u(b-)$ exist. Thus, we have the following chain of inclusions

$$BV[a,b] \subseteq \kappa BV[a,b] \subseteq \mathfrak{R}[a,b], \tag{4}$$

where $\mathfrak{R}[a,b]$ denotes the set of all regulated functions on $[a,b]$.

Notice that inclusions (4) are strict (see, for instance [2,7]).

4. Unidimensional second κ- variation

4.1. Definition. A function $u\colon [a,b] \to \mathbb{R}$ it is said of *bounded second κ-variation in* $[a,b]$, where κ is a function distortion, if

$$\kappa V^2(u;[a,b]) := \sup_{\xi \in \Pi_3([a,b])} \frac{\sum_{i=0}^{n-2} \left| u[t_{i+1}, t_{i+2}] - u[t_i, t_{i+1}] \right|}{\sum_{i=1}^{n} \kappa\left(\frac{t_i - t_{i-1}}{b-a}\right)} < \infty,$$

where

$$u[t_{i+1}, t_{i+2}] := \frac{u(t_{i+2}) - u(t_{i+1})}{t_{i+2} - t_{i+1}}, \qquad i = 0, \ldots, n-2$$

and the supremum is taken over the set of all partitions $\xi \in \Pi_3([a,b])$.

The class of all functions of bounded second κ-variation in $[a,b]$ is denoted as $\kappa BV^2[a,b]$

Since κ is subaditive, all function of bounded second variation in $[a,b]$ are of bounded second κ-variation in $[a,b]$ and

$$\kappa V^2\big(u;[a,b]\big) \leqslant V^2\big(u;[a,b]\big).$$

4.2. Proposition. *If $u, v \in \kappa BV^2([a,b])$ and λ is any real constant, then*

$$\kappa V^2\big(\lambda u;[a,b]\big) = |\lambda| \kappa V^2\big(u;[a,b]\big)$$

and

$$\kappa V^2\big(u+v;[a,b]\big) \leqslant \kappa V^2\big(u;[a,b]\big) + \kappa V^2\big(v;[a,b]\big).$$

Proof. These properties are immediate consequence of Definition 4.1, the triangle inequality and the homogeneity of the absolute value. □

To keep notation simple, we will also denote by $\kappa V^2(u;\cdot)$ the *functional*

$$\kappa V^2(u;[a,b]) := \sup_{\xi\in\Pi_3([a,b])} \frac{\sum_{i=0}^{n-2} \left| u[t_{i+1},t_{i+2}] - u[t_i,t_{i+1}] \right|}{\sum_{i=1}^{n} \kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)},$$

where

$$u[t_{i+1},t_{i+2}] := \frac{u(t_{i+2})-u(t_{i+1})}{t_{i+2}-t_{i+1}}, \quad i=0,\ldots,n-2,$$

and the supremum is taken over the set of all partitions $\xi \in \Pi_3([a,b])$.

In the rest of this section we examine what we may call *quasi-monotone* properties of the functional $\kappa V^2(u;\cdot)$. The designation "quasi-monotone" refers to the fact that, given two real-valued functionals A, B defined on a subspace $X \subset \mathbb{R}^{[a,b]}$, there is a positive number M such that $A(u) \leqslant M\,B(u)$ for all $u \in X$.

4.3. Proposition. (Quasi-Monotonicity) *If $a < c < d < b$, then there exists $\lambda > 0$ such as $\kappa V^2\big(u;[c,d]\big) \leqslant (\lambda+1)\kappa V^2(u;[a,b])$.*

Proof. Let $a<c<d<b$ and let $\xi = \{t_i\}_0^n \in \Pi_3([c,d])$. Then

(i) $\{s_j\}_0^m = \{t_i\}_0^n \cup \{a,b\} \in \Pi_3([a,b])$,

(ii) since $b-a > d-c$ and κ is increasing, $\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right) \leqslant \kappa\left(\frac{t_i-t_{i-1}}{d-c}\right)$, for each $i=1,\ldots,n$, thus

$$\sum_{i=1}^{n} \kappa\left(\frac{t_i-t_{i-1}}{b-a}\right) \leqslant \sum_{i=1}^{n} \kappa\left(\frac{t_i-t_{i-1}}{d-c}\right). \tag{5}$$

(iii) On the other hand, by (1)

$$\sum_{i=1}^{n} \kappa\left(\frac{t_i-t_{i-1}}{b-a}\right) \geqslant \kappa\left(\sum_{i=1}^{n}\frac{t_i-t_{i-1}}{b-a}\right) = \kappa\left(\frac{d-c}{b-a}\right). \tag{6}$$

Making $\lambda := \dfrac{\kappa\left(\frac{c-a}{b-a}\right)+\kappa\left(\frac{b-d}{b-a}\right)}{\kappa\left(\frac{d-c}{b-a}\right)}$ we get from inequality (6) that

$$\begin{aligned}\lambda\sum_{i=1}^{n} \kappa\left(\frac{t_i-t_{i-1}}{b-a}\right) &\geqslant \lambda\kappa\left(\frac{d-c}{b-a}\right) \\ &= \frac{\kappa\left(\frac{c-a}{b-a}\right)+\kappa\left(\frac{b-d}{b-a}\right)}{\kappa\left(\frac{d-c}{b-a}\right)}\kappa\left(\frac{d-c}{b-a}\right) \\ &= \kappa\left(\frac{c-a}{b-a}\right)+\kappa\left(\frac{b-d}{b-a}\right).\end{aligned} \tag{7}$$

Now, using (5) and (7) we have

$$
\begin{aligned}
\frac{\sum_{i=0}^{n-2}\left|u[t_{i+1},t_{i+2}]-u[t_i,t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{d-c}\right)} &= \frac{\sum_{i=0}^{n-2}\left|u[t_{i+1},t_{i+2}]-u[t_i,t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{d-c}\right)}\cdot\frac{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)}\\
&\leqslant \frac{\sum_{i=0}^{n-2}\left|u[t_{i+1},t_{i+2}]-u[t_i,t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)}\\
&= \frac{(\lambda+1)\sum_{i=0}^{n-2}\left|u[t_{i+1},t_{i+2}]-u[t_i,t_{i+1}]\right|}{\lambda\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)+\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)}\\
&\leqslant (\lambda+1)\frac{\left|u[s_1,s_2]-u[s_0,s_1]\right|}{\kappa\left(\frac{c-a}{b-a}\right)+\kappa\left(\frac{b-d}{b-a}\right)+\sum_{j=2}^{n-1}\kappa\left(\frac{s_j-s_{j-1}}{b-a}\right)}\\
&\quad+(\lambda+1)\frac{\sum_{j=1}^{n-3}\left|u[s_{j+1},s_{j+2}]-u[s_j,s_{j+1}]\right|+\left|u[s_{n-1},s_n]-u[s_{n-2},s_{n-3}]\right|}{\kappa\left(\frac{c-a}{b-a}\right)+\kappa\left(\frac{b-d}{b-a}\right)+\sum_{j=2}^{n-1}\kappa\left(\frac{s_j-s_{j-1}}{b-a}\right)}\\
&\leqslant (\lambda+1)\frac{\sum_{j=0}^{n-2}\left|u[s_{j+1},s_{j+2}]-u[s_j,s_{j+1}]\right|}{\sum_{j=1}^{n}\kappa\left(\frac{s_j-s_{j-1}}{b-a}\right)}\\
&\leqslant (\lambda+1)\kappa V^2(u;[a,b]).
\end{aligned}
$$

And since this inequality is valid for all partition $\{t_i\}_{i=0}^{n}\in\Pi_3([c,d])$, we finally get

$$\kappa V^2\big(u;[c,d]\big)\leqslant(\lambda+1)\kappa V^2\big(u;[a,b]\big).$$

□

4.4. Theorem. *A function $u\colon[a,b]\to\mathbb{R}$ satisfies $\kappa V^2(u;[a,b])=0$ if and only if there are constants A,B such that $u(t)=At+B$.*

Proof. If $\kappa V^2(u,[a,b])=0$ for $a<t<b$, then by considering the partition $\xi:=\{a,t,b\}=\{t_1,t_2,t_3\}$ we have

$$\frac{\left|u[t,b]-u[a,t]\right|}{\sum_{i=1}^{2}\kappa\left(\frac{t_{i+1}-t_i}{b-a}\right)}=0\Rightarrow\frac{u(b)-u(t)}{b-t}=\frac{u(t)-u(a)}{t-a}$$

from which it readily follows that

$$u(t) = \frac{\big(u(b) - u(a)\big)t + bu(a) - au(b)}{b - a}. \tag{8}$$

Clearly, if $t = a$ or $t = b$ the expression (8) is satisfied.

On the other hand, if $u(t) = At + B$ we have

$$\begin{aligned}\big|u[t_{i+1}, t_{i+2}] - u[t_i, t_{i+1}]\big| &= \frac{u(t_{i+2}) - u(t_{i+1})}{t_{i+2} - t_{i+1}} - \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \\ &= \frac{A(t_{i+2} - t_{i+1})}{t_{i+2} - t_{i+1}} - \frac{A(t_{i+1} - t_i)}{t_{i+1} - t_i} = 0.\end{aligned}$$

Therefore $\kappa V^2(u; [a, b]) = 0$. □

4.5. Definition. For any $u \in \kappa BV^2([a, b])$ define

$$\|u\| := \Sigma|u|[a, b] + \kappa V^2\big(u; [a, b]\big),$$

where $\Sigma|u|[a, b] := |u(a)| + |u(b)|$.

4.6. Corollary. *$\|\cdot\|$ is a norm on $\kappa BV^2([a, b])$.*

Proof. Let $u \in \kappa BV^2([a, b])$. By definition $\|u\| \geqslant 0$ and clearly $u = 0$ implies $\|u\| = 0$. On the other hand, if $\|u\| = 0$, then $\kappa V^2(u; [a, b]) = 0$ and $\Sigma|u|[a, b] = 0$. It follows, by (8), that $u \equiv 0$.

On the other hand, the properties:

(P_2) $\forall\, \alpha \in \mathbb{R} : \|\alpha u\| = |\alpha|\|u\|$ and

(P_3) $\|u + v\| \leqslant \|u\| + \|v\|$, $(u, v \in \kappa BV^2([a, b]))$

follow readily from the definition and the properties of the (real) functionals $|\cdot|$ and sup. □

In the following proposition we show that the functions in the unit ball of the space $(\kappa BV^2([a, b]), \|\cdot\|)$ are uniformly majorized by a certain fixed continuous function. This property will be fundamental in proving that $\kappa BV^2([a, b])$ is a Banach space.

4.7. Proposition. *There is a continous function $p_\kappa : [a, b] \to \mathbb{R}$ such that for all $u \in \kappa BV^2([a, b])$*

$$|u(t)| \leqslant p_\kappa(t)\, \|u\| \quad \text{for all } t \in [a, b].$$

In particular, $\kappa BV^2([a, b])$ is a subspace of $\mathcal{B}([a, b])$, the Banach space of all bounded functions on $[a, b]$ with the sup *norm.*

Proof. Let u be in $\kappa BV^2([a,b])$. Put $\delta := b-a$. Then, by Definitions (4.1) and (4.5), for all $t \in (a,b)$ we have the following inequality

$$\frac{\left|\frac{u(b)-u(t)}{b-t} - \frac{u(t)-u(a)}{t-a}\right|}{\kappa\left(\frac{b-t}{\delta}\right) + \kappa\left(\frac{t-a}{\delta}\right)} \leqslant \|u\|;$$

which implies

$$\left|\frac{\delta}{(b-t)(t-a)}u(t)\right| \leqslant \|u\| \left(\kappa\left(\frac{b-t}{\delta}\right) + \kappa\left(\frac{t-a}{\delta}\right)\right) + \left|\frac{u(b)}{b-t}\right| + \left|\frac{u(a)}{t-a}\right| \tag{9}$$

or

$$|u(t)| \leqslant \frac{(b-t)(t-a)\|u\| \left(\kappa\left(\frac{b-t}{\delta}\right) + \kappa\left(\frac{t-a}{\delta}\right)\right)}{\delta} + \left|\frac{(t-a)u(b)}{\delta}\right| + \left|\frac{(b-t)u(a)}{\delta}\right|.$$

Taking into account (9) and the fact that $\Sigma|u|[a,b] \leqslant \|u\|$, we obtain

$$|u(t)| \leqslant \left[\frac{(b-t)(t-a)\left(\kappa\left(\frac{b-t}{\delta}\right) + \kappa\left(\frac{t-a}{\delta}\right)\right)}{\delta} + \frac{(t-a)}{\delta} + \frac{(b-t)}{\delta}\right]\|u\|. \tag{10}$$

Finally, by regrouping the right hand side of this inequality we may define

$$p_\kappa(t) := 1 + \frac{(b-t)(t-a)\left(\kappa\left(\frac{b-t}{\delta}\right) + \kappa\left(\frac{t-a}{\delta}\right)\right)}{\delta}.$$

On the other hand, if t_0 is an extreme point of the interval $[a,b]$ then $p_\kappa(t_0) = 1$ and $|u(t_0)| \leqslant \Sigma|u|[a,b] \leqslant p_\kappa(t)\|u\|$. Then (10) actually holds for every $t \in [a,b]$. This finishes the proof. □

4.8. Corollary. *$\kappa BV^2([a,b])$ is a Banach space.*

Proof. Suppose $\{u_r\}_{r\geqslant 1}$ is a Cauchy sequence in $\kappa BV^2([a,b])$ and let p_κ be the continuous function given by Proposition 4.7. Then, for all $t \in [a,b]$ and all $r,s \in \mathbb{N}$, we have

$$|(u_r - u_s)(t)| \leqslant \sup_{t\in[a,b]} p_\kappa(t)\,\|u_r - u_s\|.$$

Thus, $\{u_r\}_{r\geqslant 1}$ is a Cauchy sequence in $\mathcal{B}([a,b])$ and therefore there is $u \in \mathcal{B}([a,b])$ such that $\|u_r - u\|_\infty \to 0$. Fix $\epsilon > 0$. Since $\{u_r\}_{r\geqslant 1}$ is a Cauchy sequence in $\kappa BV^2([a,b])$, there is $\rho \in \mathbb{N}$ such that for all $r, s > \rho$ and all $\xi_0 = \{t_i^0\}_0^{n_0} \in \Pi_3[a,b]$:

$$\begin{aligned}\epsilon &> \kappa V^2((u_r - u_s);[a,b]) \\ &= \sup_{\pi\in\Pi_3([a,b])} \frac{\sum_{i=0}^{n-2}\left|(u_r-u_s)[t_{i+1},t_{i+2}]-(u_r-u_s)[t_i,t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)} \\ &\geqslant \frac{\sum_{i=0}^{n_0-2}\left|(u_r-u_s)[t_{i+1}^0,t_{i+2}^0]-(u_r-u_s)[t_i^0,t_{i+1}^0]\right|}{\sum_{i=1}^{n_0}\kappa\left(\frac{t_i^0-t_{i-1}^0}{b-a}\right)}.\end{aligned}$$

It follows that, for all $r > \rho$ and all $\xi_0 = \{t_i^0\}_1^{n_0} \in \Pi_3[a,b]$:

$$\epsilon \geqslant \lim_{s\to\infty} \frac{\sum_{i=0}^{n_0-2}\left|(u_r-u_s)[t_{i+1}^0,t_{i+2}^0]-(u_r-u_s)[t_i^0,t_{i+1}^0]\right|}{\sum_{i=1}^{n_0}\kappa\left(\frac{t_i^0-t_{i-1}^0}{b-a}\right)}.$$

Consequently, for all $r > \rho$

$$\kappa V^2\big(u_r - u;[a,b]\big) = \sup_{\pi\in\Pi_3([a,b])} \frac{\sum_{i=0}^{n-2}\left|(u_r-u)[t_{i+1},t_{i+2}]-(u_r-u)[t_i,t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)} \leqslant \epsilon$$

which in turn implies that $u \in \kappa BV^2([a,b])$ and (since $u_r \to u$ pointwise in $[a,b]$)

$$\lim_{r\to\infty} \|u_r - u\| = 0.$$

We conclude that $\kappa BV^2([a,b])$ is a Banach space.

□

5. Composing functions of bounded second κ-variation

A *linear composition operator* is defined as follows: Suppose that D and E are given sets, X is a linear subspace of $\mathbb{R}^E$ and f is a map from D to E. The composition operator $C_f : X \to \mathbb{R}^D$ is defined by

$$C_f(g) := g \circ f.$$

In our next result we give a sufficient condition for an operator C_f to map $\kappa BV[a,b]$ into $\kappa BV[c,d]$. Recall that a function $f\colon[c,d]\to\mathbb{R}$ is said to be Lipschitz continuous if and only if

$$L(f) := \sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right| \,:\, x,y\in[c,d],\ x\neq y\right\} < \infty.$$

5.1. Theorem. *Suppose that $f\colon[c,d]\to[a,b]$ is an injective Lipschitz continuous function. Then C_f maps $\kappa BV^2[a,b]$ into $\kappa BV^2[c,d]$ and is bounded (continuous).*

Proof. It follows from the hypothesis that f is continuous and injective, hence it is strictly monotone on $[c,d]$. Assume that f is increasing and that $g\in\kappa BV^2[a,b]$.

Let $\xi=\{t_i\}_0^n\in\Pi_3([c,d])$. Since f is Lipschitz, there is a real number $L\geqslant 0$ such that $f(y)-f(x)\leqslant L(y-x)$ for all $x,y\in[c,d]$ such that $x<y$. Hence for each $i\in\{1,\dots,n\}$

$$\begin{aligned}\frac{f(t_i)-f(t_{i-1})}{b-a} &\leqslant \frac{L\,(d-c)}{b-a}\,\frac{t_i-t_{i-1}}{d-c}\\ &\leqslant \left([[L(d-c)(b-a)^{-1}]]+1\right)\frac{t_i-t_{i-1}}{d-c}\\ &= N\left(\frac{t_i-t_{i-1}}{d-c}\right),\end{aligned}$$

where $N := [[\,L(d-c)(b-a)^{-1}]]+1$.

It follows from the monotonicity and subadditivity of κ that

$$\begin{aligned}\kappa\left(\frac{t_i-t_{i-1}}{d-c}\right) &\geqslant \kappa\left(\frac{1}{N}\,\frac{f(t_i)-f(t_{i-1})}{b-a}\right)\\ &\geqslant \frac{1}{N}\kappa\left(\frac{f(t_i)-f(t_{i-1})}{b-a}\right).\end{aligned}$$

Hence

$$\frac{\sum\limits_{i=0}^{n-2}\left|(g\circ f)[t_{i+1},t_{i+2}]-(g\circ f)[t_i,t_{i+1}]\right|}{\sum\limits_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{d-c}\right)} \leqslant N\frac{\sum\limits_{i=0}^{n-2}\left|(g\circ f)[t_{i+1},t_{i+2}]-(g\circ f)[t_i,t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{f(t_i)-f(t_{i-1})}{b-a}\right)}.$$

By using a "$\lambda+1$-argument" as in the proof of Proposition 4.2, with

$$\lambda := \frac{\kappa\left(\frac{f(c)-a}{b-a}\right)+\kappa\left(\frac{b-f(d)}{b-a}\right)}{\kappa\left(\frac{f(d)-f(c)}{b-a}\right)},$$

we obtain

$$\frac{\sum_{i=0}^{n-2}\left|(g\circ f)[t_{i+1},t_{i+2}]-(g\circ f)[t_i,t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{d-c}\right)}\leqslant(\lambda+1)N\frac{\sum_{j=0}^{m-2}\left|g[s_{j+1},s_{j+2}]-g[s_j,s_{j+1}]\right|}{\sum_{i=1}^{m}\kappa\left(\frac{s_j-s_{j-1}}{b-a}\right)}$$
$$\leqslant(\lambda+1)N\,\kappa V^2\big(g;[a,b]\big),$$

where $\eta=\{s_j\}_{j=0}^m=\{f(t_i)\}_{i=0}^n\cup\{a,b\}$.

Therefore,

$$\kappa V^2\big(g\circ f;[c,d]\big)\leqslant(\lambda+1)N\kappa V^2\big(g;[a,b]\big). \tag{11}$$

From (11) it follows that C_f maps $\kappa BV^2[a,b]$ into $\kappa BV^2[c,d]$ and that there is a real number $M\geqslant 0$ such that

$$\|C_f(g)\|\leqslant M\|g\|$$

for all $g\in\kappa BV^2[a,b]$, that is, C_f is bounded.

The case in which f is decreasing is treated similarly. □

6. A Representation Theorem

In this section we present a partial version of Riesz's Lemma which says that a function is of second bounded variation if and only if it is the integral of a function of bounded variation [18]. We show that the integral of a function of bounded κ-variation is a function of second bounded κ-variation.

6.1. Theorem. *If $f\in\kappa BV([a,b])$ and $F(\sigma):=\int_a^\sigma f(t)dt$, then*

$$\kappa V^2\big(F,[a,b]\big)\leqslant 2\kappa V\big(f,[a,b]\big)$$

and $F\in\kappa BV^2([a,b])$.

Proof. Note that if $F(\sigma):=\int_a^\sigma f(t)\,dt$ then

$$\frac{F(\sigma_2)-F(\sigma_1)}{\Delta\sigma}=\int_0^1 f(\sigma_1+t\Delta\sigma)\,dt \tag{12}$$

with $\Delta\sigma:=\sigma_2-\sigma_1$. Let $\xi:=\{t_i\}_{i=0}^n\in\Pi_3([a,b])$. By (12)

$$\left[\frac{F(t_{i+2})-F(t_{i+1})}{(t_{i+2}-t_{i+1})}-\frac{F(t_{i+1})-F(t_i)}{(t_{i+1}-t_i)}\right]=\int_0^1 f(t_{i+1}+t(t_{i+2}-t_{i+1}))\,dt$$
$$-\int_0^1 f(t_i+t(t_{i+1}-t_i))\,dt$$

and

$$\sum_{i=0}^{n-2}\left|F[t_{i+1},t_{i+2}]-F[t_i,t_{i+1}]\right| = \sum_{i=0}^{n-2}\left|\int_0^1 f\big(t_{i+1}+t(t_{i+2}-t_{i+1})\big) - f\big(t_i+t(t_{i+1}-t_i)\big)\,dt\right|$$

and

$$\frac{\sum_{i=0}^{n-2}\left|F[t_{i+1},t_{i+2}]-F[t_i,t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)} \leqslant \frac{\int_0^1\sum_{i=0}^{n-2}\left|f(t_{i+1}+t(t_{i+2}-t_{i+1}))-f(t_i+t(t_{i+1}-t_i))\right|dt}{\sum_{i=1}^{n}\kappa\left(\frac{t_i-t_{i-1}}{b-a}\right)}.$$

On the other hand, since κ is a nondecreasing function, for $0<t<1$ we have

$$\begin{aligned}
&\kappa\left(\frac{t_0+t(t_1-t_0)-t_0}{b-a}\right)+\sum_{i=0}^{n-2}\kappa\left(\frac{t_{i+1}+t(t_{i+2}-t_{i+1})-(t_i+t(t_{i+1}-t_i))}{b-a}\right)\\
&\quad+\kappa\left(\frac{t_n-(t_{n-1}+t(t_n-t_{n-1}))}{b-a}\right)\\
&=\kappa\left(\frac{t_0+t(t_1-t_0)-t_0}{b-a}\right)+\sum_{i=0}^{n-2}\kappa\left(\frac{(1-t)(t_{i+1}-t_i)+t(t_{i+2}-t_{i+1})}{b-a}\right)\\
&\quad+\kappa\left(\frac{t_n-(t_{n-1}+t(t_n-t_{n-1}))}{b-a}\right)\\
&\leqslant\kappa\left(\frac{t_1-t_0}{b-a}\right)+\sum_{i=0}^{n-2}\kappa\left(\frac{(t_{i+1}-t_i)+(t_{i+2}-t_{i+1})}{b-a}\right)+\kappa\left(\frac{t_n-t_{n-1}}{b-a}\right)\\
&\leqslant\kappa\left(\frac{t_1-t_0}{b-a}\right)+\sum_{i=0}^{n-2}\kappa\left(\frac{t_{i+2}-t_{i+1}}{b-a}\right)+\sum_{i=0}^{n-2}\kappa\left(\frac{t_{i+1}-t_i}{b-a}\right)+\kappa\left(\frac{t_n-t_{n-1}}{b-a}\right)\\
&\leqslant 2\sum_{i=0}^{n-1}\kappa\left(\frac{t_{i+1}-t_i}{b-a}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{\sum_{i=0}^{n-1}\kappa\left(\frac{t_{i+1}-t_i}{b-a}\right)}\\
&\leqslant\frac{2}{\kappa\left(\frac{t_0+t(t_1-t_0)-t_0}{b-a}\right)+\sum_{i=0}^{n-2}\kappa\left(\frac{t_{i+1}+t(t_{i+2}-t_{i+1})-(t_i+t(t_{i+1}-t_i))}{b-a}\right)+\kappa\left(\frac{t_n-(t_{n-1}+t(t_n-t_{n-1}))}{b-a}\right)}.
\end{aligned}$$

We put

$$g(t) = \kappa\left(\frac{t_0 + t(t_1 - t_0) - t_0}{b-a}\right) + \sum_{i=0}^{n-2}\kappa\left(\frac{t_{i+1} + t(t_{i+2} - t_{i+1}) - (t_i + t(t_{i+1} - t_i))}{b-a}\right) + \kappa\left(\frac{t_n - (t_{n-1} + t(t_n - t_{n-1}))}{b-a}\right)$$

and hence

$$\begin{aligned}
\frac{\sum_{i=0}^{n-2}\left|F[t_{i+1}, t_{i+2}] - F[t_i, t_{i+1}]\right|}{\sum_{i=1}^{n}\kappa\left(\frac{t_i - t_{i-1}}{b-a}\right)} &\leqslant \frac{\int_0^1 \sum_{i=0}^{n-2}\left|f(t_{i+1} + t(t_{i+2} - t_{i+1})) - f(t_i + t(t_{i+1} - t_i))\right| dt}{\sum_{i=1}^{n}\kappa\left(\frac{t_i - t_{i-1}}{b-a}\right)} \\
&\leqslant 2\int_0^1 \frac{1}{g(t)}\sum_{i=0}^{n-2}\left|f(t_{i+1} + t(t_{i+2} - t_{i+1})) - f(t_i + t(t_{i+1} - t_i))\right| dt \\
&\leqslant 2\int_0^1 \frac{1}{g(t)}\left|f(t_0 + t(t_1 - t_0)) - f(t_0)\right| dt \\
&\quad + 2\int_0^1 \frac{1}{g(t)}\sum_{i=0}^{n-2}\left|f\big(t_{i+1} + t(t_{i+2} - t_{i+1})\big) - f\big(t_i + t(t_{i+1} - t_i)\big)\right| dt \\
&\quad + 2\int_0^1 \frac{1}{g(t)}\left|f\big(t_n\big) - f\big(t_{n-1} + t(t_n - t_{n-1})\big)\right| dt \\
&\leqslant 2\kappa V\big(f, [a,b]\big) < \infty.
\end{aligned}$$

Therefore $\kappa V^2(F; [a,b]) \leqslant 2\kappa V(f, [a,b])$ and $F \in \kappa BV^2([a,b])$. □

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