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Characterization of inclusion among Riesz–Medvedev variation spaces

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Summary. We present a characterization of inclusion among Riesz–Medvedev bounded variation spaces, i.e., we shall present necessary and sufficient conditions for the Young functions φ_1 and φ_2 so that $RV_{\varphi_1}[a,b] \subset RV_{\varphi_2}[a,b]$ or $RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$.

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1. Introduction

J. Musielak and W. Orlicz [8] established necessary and sufficient conditions for Young functions φ_1 and φ_2 (cf. [2]) so that $BV_{\varphi_1}[a,b] \subset BV_{\varphi_2}[a,b]$ where $BV_{\varphi}[a,b]$ is the class of all functions $x : [a,b] \to \mathbb{R}$ of bounded Riesz–Medvedev variation in the sense of *Wiener*. More precisely, $BV_{\varphi_1}[a,b] \subset BV_{\varphi_2}[a,b]$ if and only if there exist positive constants C and T such that $\varphi_2(t) \leqslant C\varphi_1(t)$ for all $0 < t \leqslant T$.

According to W. Orlicz's classical result [3], it is known that $BV_{\varphi_1}[a,b] \subset BV_{\varphi_2}[a,b]$ if and only if $\ell_{\varphi_1} \subset \ell_{\varphi_2}$ where ℓ_{φ} is the class of all sequences $\{a_n\}_{n\geqslant 1}$ such that

$$\sum_{n=1}^{\infty} \varphi(|a_n|) < \infty.$$

H. Herda [4] found necessary and sufficient conditions for Young functions φ_1 and φ_2 so that $BV^*_{\varphi_1}[a,b] \subset BV^*_{\varphi_2}[a,b]$, where $BV^*_{\varphi}[a,b]$ is the space generated by the class

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$BV_\varphi[a,b]$. More precisely, $BV^*_{\varphi_1}[a,b] \subset BV^*_{\varphi_2}[a,b]$ if and only if there exist positive constants C and T such that $\varphi_2(t) \leqslant \varphi_1(Ct)$ for all $0 < t \leqslant T$. By a classical result due to Orlicz [3], it follows that $\ell^*_{\varphi_1} \subset \ell^*_{\varphi_2}$ if and only if $BV^*_{\varphi_1}[a,b] \subset BV^*_{\varphi_2}[a,b]$, where ℓ^*_φ is the space generated by the class ℓ_φ.

In the present work, we shall establish a similar characterization for the class $RV_\varphi[a,b]$ of all functions $x\colon [a,b] \to \mathbb{R}$ of bounded φ-variation in the sense of Riesz. Moreover, we shall show that $RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$ if and only if there exists positive constant C and T such that $\varphi_2(t) \leqslant \varphi_1(Ct)$ for all $t \geqslant T$. From W. Orlicz's classical result [5, p. 63] we conclude that $BV^*_{\varphi_1}[a,b] \subset BV^*_{\varphi_2}[a,b]$ if and only if $L_{\varphi_1}[a,b] \subset L_{\varphi_2}[a,b]$, where $L_\varphi[a,b]$ is the so-called Orlicz class. Concerning the space $RV^*_\varphi[a,b]$ generated by the class $RV_\varphi[a,b]$, we establish the fact that $RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$ if and only if there exist positive constants C and T such that $\varphi_2(t) \leqslant \varphi_1(Ct)$ for all $t \geqslant T$. As a corollary one obtains that $RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$ if and only if $L^*_{\varphi_1}[a,b] \subset L^*_{\varphi_2}[a,b]$, where $L^*_\varphi[a,b]$ is the Orlicz space generated by the class $L_\varphi[a,b]$.

2. Preliminary

In this section we introduce some definitions and notation. A function $\varphi\colon [0,\infty) \to [0,\infty)$ is called a Young function if it satisfies the following conditions: $\varphi(t) = 0$ if and only if $t = 0$, φ is continuous and non-decreasing on $[0,\infty)$, and $\varphi(t) \to \infty$ as $t \to \infty$. In our considerations the condition (∞_1) is given by:

$$\limsup_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

The Young function φ satisfies the condition Δ_2 for large t when for some constants $k > 0$, $t_0 > 0$ we have $\varphi(2t) \leqslant k\varphi(t)$ for $t \geqslant t_0$. If this inequality holds for $t \geqslant 0$ we say the condition Δ_2 is satisfied for all t. Throughout this paper $\mathcal{F}$ will denote the vector space of real-valued functions x defined in a finite interval $[a,b]$. Let $x \in \mathcal{F}$. For a Young function φ, we can define the Riesz–Medvedev variation as the number

$$v^R_\varphi\big(x;[a,b]\big) := \sup_\pi \sum_{i=1}^n \varphi\left(\frac{|x(t_i) - x(t_{i-1})|}{|t_i - t_{i-1}|}\right)|t_i - t_{i-1}|,$$

where supremun is taken over all partitions $\pi : a = t_0 < \cdots < t_n = b$ of the interval $[a,b]$.

In the literature well known is also the so-called Wiener–Young variation (cf. [2])

$$v_\varphi\big(x;[a,b]\big) := \sup_\pi \sum_{i=1}^n \varphi\Big(\big|x(t_i) - x(t_{i-1})\big|\Big),$$

where supremun is taken over all partitions π of $[a,b]$.

The function $x \in \mathcal{F}$ is said to have bounded (or finite) Riesz–Medvedev variation if $v_\varphi^R(x;[a,b]) < \infty$. By $RV_\varphi[a,b]$ we denote the class of all functions $x \in \mathcal{F}$ such that $v_\varphi^R(x;[a,b]) < \infty$. By $BV_\varphi[a,b]$ we denote the class of all functions $x \in \mathcal{F}$ such that $v_\varphi(x;[a,b]) < \infty$. The space $RV_\varphi^*[a,b]$ of all functions $x \in \mathcal{F}$ such that $v_\varphi^R(\lambda x;[a,b]) < \infty$ for some $\lambda > 0$ is the space generated by the class $RV_\varphi[a,b]$. Similarly, the space $BV_\varphi^*[a,b]$ of all functions $x \in \mathcal{F}$ such that $v_\varphi(\lambda x;[a,b]) < \infty$ for some $\lambda > 0$ is the space generated by the class $BV_\varphi[a,b]$.

The spaces $RV_\varphi^*[a,b]$ and $BV_\varphi^*[a,b]$ appeared for the first time in papers [1] and [8], respectively.

When $\varphi(t) = t^p$ for some $p \geqslant 1$, then we have the classical space $RV_p[a,b]$ of functions of bounded Riesz p-variation.

Note that the assumption (∞_1) in the case of a convex function φ is just

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

Moreover, as was observed in [6], if φ is a convex Young function and condition (∞_1) is not satisfied, i.e.

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = r < \infty,$$

then $RV_\varphi^*[a,b] = BV[a,b]$ where $BV[a,b]$ denotes the usual space of functions of bounded variation.

If φ is a convex Young function, then the space $RV_\varphi^*[a,b]$ is a *Banach* space with the norm

$$\|x\|_\varphi^R := |x(a)| + \|x\|_\varphi^0,$$

where

$$\|x\|_\varphi^0 := \inf\Big\{\varepsilon > 0 \,:\, v_\varphi^R\Big(\frac{x}{\varepsilon};[a,b]\Big) \leqslant 1\Big\}.$$

Similarly, we can obtain a *Banach* space $BV_\varphi^*[a,b]$ with the following norm:

$$\|x\|_\varphi := |x(a)| + \|x\|_\varphi^0,$$

where

$$\|x\|_\varphi^0 := \inf\Big\{\varepsilon > 0 \,:\, v_\varphi\Big(\frac{x}{\varepsilon};[a,b]\Big) \leqslant 1\Big\}.$$

For a convex Young function φ which satisfies (∞_1), some useful properties of Riesz–Medvedev variation are stated in the following lemma.

2.1. Lemma. *Let φ be a convex Young function.*

(i) (*Musielak–Orlicz* [8]) *If $x \in RV_\varphi^*[a,b]$ and $\|x\|_\varphi^0 > 0$, then*

$$v_\varphi^R\left(\frac{x}{\|x\|_\varphi^0};[a,b]\right) \leqslant 1.$$

(ii) (*Maligranda–Orlicz* [6]) *If* $x \in RV_\varphi^*[a,b]$, *then* x *is bounded on* $[a,b]$ *and*

$$\sup_{t\in[a,b]} |x(t)| \leqslant C_\varphi(h)\|x\|_\varphi^0,$$

where

$$C_\varphi(h) := \max\left\{\min\left\{\frac{1}{\varphi(1)}, \frac{1}{h\varphi(1/h)}\right\}, \frac{h}{\varphi^{-1}(1/h)}\right\},\ h = b-a.$$

Moreover, if additionally φ *satisfies the condition*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty,$$

then

$$\|x\|_\varphi = \sup_{t\in[a,b]} |x(t)| + \|x\|_\varphi^R \quad or \quad \|x\|_\varphi = 2C_\varphi(h)\|x\|_\varphi^R$$

is a normed Banach algebra.

(iii) (*Medvedev* [7]) $v_\varphi^R(x;[a,b]) < \infty$ *if and only if* x *is absolutely continuous on* $[a,b]$ *and*

$$\int_a^b \varphi(|x'(t)|)dt < \infty.$$

In this case we also have the equality

$$v_\varphi^R(x;[a,b]) = \int_a^b \varphi(|x'(t)|)dt.$$

The purpose of this paper is to solve the inclusion problem for the class $RV_\varphi[a,b]$ and the space $RV_\varphi^*[a,b]$. Before presenting our main results in Theorems 3.1–3.5 below, we briefly review what is known in the literature of the class $BV_\varphi[a,b]$ and the space $BV_\varphi^*[a,b]$.

(i) Musielak–Orlicz [8] proved that $BV_{\varphi_1}[a,b] \subset BV_{\varphi_2}[a,b]$ if and only if there exist positive constants C and T such that $\varphi_2(t) \leqslant C\varphi_1(t)$ for all $0 < t \leqslant T$.

(ii) Musielak–Orlicz [8] proved that $BV_\varphi[a,b]$ is a vector space if and only if the function φ satisfies the condition Δ_2 for small t, i.e. if there exist positive constants k and t_0 such that $\varphi(2t) \leqslant k\varphi(t)$ for all $0 < t \leqslant t_0$.

(iii) Herda [4] proved that $BV_{\varphi_1}^*[a,b] \subset BV_{\varphi_2}^*[a,b]$ if and only if there exist positive constants C and T such that $\varphi_2(t) \leqslant \varphi_1(Ct)$ for all $0 < t \leqslant T$.

(iv) According to W. Orlicz's classical results [3], it is known that $BV_{\varphi_1}[a,b] \subset BV_{\varphi_2}[a,b]$ if and only if $\ell_{\varphi_1}[a,b] \subset \ell_{\varphi_2}[a,b]$, where $\ell_\varphi[a,b]$ is the class of all sequences $\{a_n\}_{n\geqslant 1}$ such that

$$\sum_{n=1}^\infty \varphi(|a_n|) < \infty.$$

(v) $BV^*_{\varphi_1}[a,b] \subset BV^*_{\varphi_2}[a,b]$ if and only if $\ell^*_{\varphi_1}[a,b] \subset \ell^*_{\varphi_2}[a,b]$,where $\ell^*_{\varphi}[a,b]$ is the space generated by the class $\ell_\varphi[a,b]$.

3. Main results

In this section we shall present necessary and sufficient conditions for the Young functions φ_1 and φ_2 so that $RV_{\varphi_1}[a,b] \subset RV_{\varphi_2}[a,b]$ or $RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$.

3.1. Theorem. *Let φ_1, φ_2 be convex Young functions which satisfy the condition (∞_1). Then $RV_{\varphi_1}[a,b] \subset RV_{\varphi_2}[a,b]$ if and only if there exist positive constants C and T such that*

$$\varphi_2(t) \leqslant C\varphi_1(t) \quad \textit{for all } t \geqslant T. \tag{1}$$

Proof. Suppose that there exist positive constants C and T such that inequality (1) holds. Let $\pi : a = t_0 < \cdots < t_n = b$ be a partition of $[a,b]$. Let $x \in RV_{\varphi_1}[a,b]$ and define the set e_T by

$$e_T := \left\{ i = 0,1,\ldots,n \, : \, \frac{|x(t_i) - x(t_{i-1})|}{|t_i - t_{i-1}|} \geqslant T \right\}.$$

Then the following estimate can be obtained

$$\begin{aligned}\sum_{i=1}^{n} \varphi_2\left(\frac{|x(t_i) - x(t_{i-1})|}{|t_i - t_{i-1}|}\right)|t_i - t_{i-1}| &\leqslant C\sum_{i=1}^{n} \varphi_1\left(\frac{|x(t_i) - x(t_{i-1})|}{|t_i - t_{i-1}|}\right) + \varphi_2(T)\sum_{i=1}^{n}(t_i - t_{i-1}) \\ &\leqslant Cv^R_{\varphi_1}\big(x;[a,b]\big) + \varphi_2(T)(b-a).\end{aligned}$$

Thus

$$v^R_{\varphi_2}\big(x;[a,b]\big) \leqslant Cv^R_{\varphi_1}\big(x;[a,b]\big) + \varphi_2(T)(b-a),$$

consequently $RV_{\varphi_1}[a,b] \subset RV_{\varphi_2}[a,b]$.

Suppose now that $RV_{\varphi_1}[a,b] \subset RV_{\varphi_2}[a,b]$ and that φ_1 and φ_2 does not satisfy the inequality (1), that is, there exists a sequence $\{t_n\}_{n\geqslant 1}$ such that $t_n \uparrow \infty$ as $n \to \infty$ and

$$\varphi_2(t_n) > 2^n\varphi_1(t_n), \quad n \in \mathbb{N}. \tag{2}$$

Without loss of generality, we can assume that $[a,b] = [0,1]$. We shall prove that there exists $x \in RV_{\varphi_1}[0,1]$ such that $x \notin RV_{\varphi_2}[0,1]$. By considering subsequences, if necessary, we may define the sequence $\{a_n\}_{n\geqslant 1}$ on $[0,1]$ in the following way:

$$a_n = \frac{1}{2^n\varphi_1(t_{n+1})}, \quad n \in \mathbb{N}.$$

Then $a_n \downarrow 0$ as $n \to \infty$, and the series

$$\sum_{n=1}^{\infty} \frac{t_{n+1} - t_n}{2^n\varphi_1(t_{n+1})} \tag{3}$$

is convergent. Indeed, since the function φ_1 is convex and the function $t \mapsto t/\varphi_1(t)$ is non-increasing, we have

$$\frac{|t_{n+1}-t_n|}{2^n\varphi_1(t_{n+1})} \leqslant \frac{1}{2^n}\left(\frac{t_{n+1}}{\varphi_1(t_{n+1})}+\frac{t_n}{\varphi_1(t_{n+1})}\right) \leqslant \frac{1}{2^n}\frac{t_{n+1}}{\varphi_1(t_{n+1})} \leqslant \frac{1}{2^n}\frac{t_2}{\varphi_1(t_2)},$$

for all $n \in \mathbb{N}$. Consequently the series (3) is convergent.

Consider the set sequence $\{I_n\}_{n\geqslant 1}$ given by

$$I_1 = [a_1, 1] \quad \text{and} \quad I_n = [a_n, a_{n-1}), \qquad n = 2, 3, \ldots$$

Then $I_m \cap I_n = \varnothing$ $(n, m \in \mathbb{N},\ n \neq m)$ and

$$\bigcup_{n=1}^{\infty} I_n = (0, 1].$$

Define the function $x\colon [0,1] \to \mathbb{R}$ in the following way:

$$x(\tau) = \begin{cases} t_n\tau - \displaystyle\sum_{i=n}^{\infty}\frac{t_{i+1}-t_i}{2^i\varphi_1(t_{i+1})}, & \tau \in I_n, \\ 0, & \tau = 0. \end{cases}$$

We claim that $x \in RV_{\varphi_1}[0,1]$, but $x \notin RV_{\varphi_2}[0,1]$. Indeed, the function x is continuous and differentiable in the interior of I_n, $n \geqslant 1$, and since

$$\begin{aligned} \lim_{\tau\uparrow a_n} x(\tau) &= \lim_{\tau\uparrow a_n}\left(t_{n+1}\tau - \sum_{i=n+1}^{\infty}\frac{t_{i+1}-t_i}{2^i\varphi_1(t_{i+1})}\right) \\ &= \frac{t_{n+1}}{2^n\varphi_1(t_{n+1})} - \sum_{i=n+1}^{\infty}\frac{t_{i+1}-t_i}{2^i\varphi_1(t_{i+1})} \\ &= \frac{t_n}{2^n\varphi_1(t_{n+1})} - \sum_{i=n}^{\infty}\frac{t_i-t_{i-1}}{2^{i-1}\varphi_1(t_i)} \\ &= \lim_{\tau\downarrow a_n} x(\tau), \end{aligned}$$

the function x is continuous on $(0,1]$. Next we shall prove that the function x is continuous at $\tau = 0$.

Let $\{s_n\}_{n\geqslant 1}$ be a sequence in $(0,1]$ such that $s_n \downarrow 0$ as $n \to \infty$. Since $a_n \downarrow 0$ as $n \to \infty$, for all $n \in \mathbb{N}$ there exists a positive number m_n such that $s_n \in I_{m_n}$, and consequently

$$\begin{aligned} 0 \leqslant \lim_{n\to\infty} x(s_n) &= \lim_{n\to\infty}\left(t_{m_n}a_{m_n-1} - \sum_{i=m_n}^{\infty}\frac{t_{i+1}-t_i}{2^i\varphi_1(t_{i+1})}\right) \\ &= \lim_{n\to\infty}\frac{t_{m_n}}{2^{m_n}\varphi_1(t_{m_n})} - \sum_{i=m_n}^{\infty}\frac{t_{i+1}-t_i}{2^i\varphi_1(t_{i+1})} \end{aligned}$$

$$= \lim_{n\to\infty} \frac{t_{m_n}}{2^{m_n}\varphi_1(t_{m_n})} - \lim_{n\to\infty} \sum_{i=m_n}^{\infty} \frac{t_{i+1}-t_i}{2^i\varphi_1(t_{i+1})} = 0.$$

Thus $x(s_n) \to 0$ as $n \to \infty$ for every sequence $\{s_n\}_{n\geqslant 1}$ in $(0,1]$ such that $s_n \downarrow 0$. Consequently x is continuous at $\tau = 0$.

Since x is continuous on $[0,1]$ and differentiable in the interior of I_n, $n \geqslant 1$, x is absolutely continuous on $[0,1]$.

Next we shall prove that $x \in RV_{\varphi_1}[0,1]$. By the definition of the function x and Lemma 2.1, we get the following estimate:

$$\begin{aligned} v^R_{\varphi_1}(x;[0,1]) &= \sum_{n=1}^{\infty} \varphi_1(t_n)(a_{n+1}-a_n) \\ &= \sum_{n=1}^{\infty} \varphi_1(t_n)\left(\frac{1}{2^{n-1}\varphi_1(t_n)} - \frac{1}{2^n\varphi_1(t_{n+1})}\right) \\ &\leqslant \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} + \sum_{n=1}^{\infty} \frac{\varphi_1(t_n)}{2^n\varphi_1(t_{n+1})} \leqslant 3. \end{aligned}$$

Consequently $x \in RV_{\varphi_1}[0,1]$.

As a result of the definition of the function x, inequality (2) and Lemma 2.1, we get the following estimate:

$$\begin{aligned} v^R_{\varphi_2}(x;\ [0,1]) &= \sum_{n=1}^{\infty} \varphi_2(t_n)(a_{n+1}-a_n) \\ &= \sum_{n=1}^{\infty} \frac{\varphi_2(t_n)}{2^{n-1}}\left(\frac{1}{\varphi_1(t_n)} - \frac{1}{2\varphi_1(t_{n+1})}\right) \\ &\geqslant \sum_{n=1}^{\infty} 2\varphi_1(t_n)\left(\frac{1}{\varphi_1(t_n)} - \frac{1}{2\varphi_1(t_{n+1})}\right) \\ &\geqslant \sum_{n=1}^{\infty}\left(2 - \frac{\varphi_1(t_n)}{\varphi_1(t_{n+1})}\right) = \infty. \end{aligned}$$

Thus $x \in RV_{\varphi_1}[0,1]$ and $x \notin RV_{\varphi_2}[0,1]$, which is a contradiction. □

From W. Orlicz's classical result [5, p. 63], we obtain the following corollary.

3.2. Corollary. *Let φ_1, φ_2 be convex Young functions which satisfy the condition (∞_1). Then $RV_{\varphi_1}[a,b] \subset RV_{\varphi_2}[a,b]$ if and only if $L_{\varphi_1}[a,b] \subset L_{\varphi_2}[a,b]$ where $L_\varphi[a,b]$ is the so-called Orlicz class.*

In the following theorem we shall establish a necessary and sufficient condition for Young functions φ_1 and φ_2 so that

$$RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b].$$

3.3. Theorem. *Let φ_1, φ_2 be convex Young functions which satisfy the condition (∞_1). Then $RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$ if and only if there exist positive constants C and T such that*

$$\varphi_2(t) \leqslant \varphi_1(Ct), \quad t \geqslant T. \tag{4}$$

Proof. Suppose that there exist positive constants C and T such that inequality (4) holds. Let $x \in RV^*_{\varphi_1}[a,b]$. Then by Lemma 2.1 and the definition of the space $RV^*_{\varphi_1}[a,b]$ we have that the function x is absolutely continuous on $[a,b]$ and there exists $\lambda > 0$ such that

$$v^R_{\varphi_1}\big(\lambda x;[a,b]\big) = \int_a^b \varphi_1\big(\lambda |x'(t)|\big)dt < \infty.$$

Define the set $e_{T,\lambda}$ by

$$e_{T,\lambda} := \big\{ t \in [a,b] \,:\, \lambda |x'(t)| > CT \big\}.$$

The following estimate can be obtained

$$\begin{aligned}\int_a^b \varphi_2\Big(\frac{\lambda}{C}|x'(t)|\Big)dt &= \int_{e_{T,\lambda}} \varphi_2\Big(\frac{\lambda}{C}|x'(t)|\Big)dt + \int_{[a,b]\setminus e_{T,\lambda}} \varphi_2\Big(\frac{\lambda}{C}|x'(t)|\Big)dt \\ &\leqslant \int_{e_{T,\lambda}} \varphi_1\Big(\lambda|x'(t)|\Big)dt + \varphi_2(T)(b-a) < \infty,\end{aligned}$$

consequently $x \in RV^*_{\varphi_2}[a,b]$.

Suppose now that $RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$ and that φ_1 and φ_2 do not satisfy inequality (4), hence there exists a sequence $\{t_n\}_{n\geqslant 1}$ such that $t_n \uparrow \infty$ as $n \to \infty$, and

$$\varphi_2(t_n) > 2^n \varphi_1(t_n) \quad n \in \mathbb{N}. \tag{5}$$

Without loss of generality we can assume that $[a,b] = [0,1]$. We shall prove that there exists $x \in RV^*_{\varphi_1}[a,b]$ such that $x \notin RV^*_{\varphi_2}[a,b]$. Define a sequence $\{a_n\}_{n\geqslant 1}$ in $[0,1]$ by

$$a_n = \frac{1}{2^n \varphi_1\big((n+1)^2 t_{n+1}\big)} \quad n \in \mathbb{N}.$$

Then $a_n \downarrow 0$ as $n \to \infty$, and the series

$$\sum_{n=1}^{\infty} a_n\big((n+1)t_{n+1} - nt_n\big)$$

is convergent. Indeed, since φ_1 is a convex function and the function $t \mapsto t/\varphi_1(t)$ is non--increasing, the following estimate can be obtained:

$$\begin{aligned}a_n\big((n+1)t_{n+1} - nt_n\big) &\leqslant 2(n+1)t_{n+1}a_n \\ &= \frac{(n+1)t_{n+1}}{2^{n-1}\varphi_1((n+1)^2 t_{n+1})}\end{aligned}$$

$$\leqslant \frac{(n+1)t_{n+1}}{2^{n-1}(n+1)^2\varphi_1(t_{n+1})}$$
$$\leqslant \frac{t_{n+1}}{2^{n-1}\varphi_1(t_{n+1})} \leqslant \frac{1}{2^{n-1}}\frac{t_2}{\varphi_1(t_2)}.$$

Thus the series

$$\sum_{n=1}^{\infty} a_n\big((n+1)t_{n+1} - nt_n\big)$$

is convergent.

Consider the set sequence I_n, $n \geqslant 1$, given by

$$I_1 = [a_1, 1] \quad \text{and} \quad I_n = [a_n, a_{n-1}), \qquad n = 2, 3, \dots$$

Then $I_m \cap I_n = \varnothing$ $(n, m \in \mathbb{N},\ n \neq m)$ and

$$\bigcup_{n=1}^{\infty} I_n = (0, 1].$$

Define the function $x\colon [0,1] \to \mathbb{R}$ in the following way:

$$x(\tau) = \begin{cases} nt_n\tau - \sum\limits_{i=n}^{\infty} a_i((i+1)t_{i+1} - it_i), & \tau \in I_n \\ 0, & \tau = 0. \end{cases}$$

We claim that $x \in RV^*_{\varphi_1}[0,1]$ but $x \notin RV^*_{\varphi_2}[0,1]$. Indeed, the function x is continuous and differentiable in the interior of I_n, $n \geqslant 1$, and since

$$\lim_{\tau\uparrow a_n} x(\tau) = \lim_{\tau\uparrow a_n}\Big((n+1)t_{n+1}\tau - \sum_{i=n+1}^{\infty} a_i\big((i+1)t_{i+1} - it_i\big)\Big)$$
$$= nt_{n+1}a_n - \sum_{i=n}^{\infty} a_{i-1}\big(it_i - (i-1)t_{i-1}\big)$$
$$= \lim_{\tau\downarrow a_n} x(\tau),$$

the function x is continuous on $(0,1]$. Next we shall prove that the function x is continuous at $\tau = 0$.

Let $\{s_n\}_{n\geqslant 1}$ be a sequence in $(0,1]$ such that $s_n \downarrow 0$ as $n \to \infty$. Since $a_n \downarrow 0$ as $n \to \infty$, for all $n \in \mathbb{N}$ there exists a positive number m_n such that $s_n \in I_{m_n}$ and consequently

$$0 \leqslant \lim_{n\to\infty} x(s_n) \leqslant \lim_{n\to\infty}\Big(m_n t_{m_n} a_{m_n-1} - \sum_{i=m_n}^{\infty} a_i\big((i+1)t_{i+1} - it_i\big)\Big)$$
$$= \lim_{n\to\infty} \frac{m_n t_{m_n}}{2^{m_n-1}\varphi_1(m_n t_{m_n-1})} = 0.$$

Thus $x(s_n) \to 0$ as $n \to \infty$ for all sequences $\{s_n\}_{n\geqslant 1}$ in $(0,1]$ such that $s_n \downarrow 0$. Consequently, x is continuous at $\tau = 0$.

Since x is continuous on $[0,1]$ and differentiable in the interior of I_n, $n \geqslant 1$, x is absolutely continuous on $[0,1]$.

Next we shall prove that $x \in RV^*_{\varphi_1}[0,1]$. By the definition of the function x and Lemma 2.1, we get the following estimate:

$$\begin{aligned} v^R_{\varphi_1}\big(x;[0,1]\big) &= \sum_{n=1}^{\infty} \varphi_1(nt_n)(a_{n-1}-a_n) \\ &= \sum_{n=1}^{\infty} \varphi_1(nt_n)\left(\frac{1}{2^{n-1}\varphi_1(n^2t_n)} - \frac{1}{2^n\varphi_1\big((n+1)^2t_{n+1}\big)}\right) \\ &\leqslant \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} + \sum_{n=1}^{\infty}\frac{1}{2^n} = 3. \end{aligned}$$

Consequently $x \in RV^*_{\varphi_1}[0,1]$. By the definition of the function x, inequality (5) and Lemma 2.1, for all $\lambda > 0$ we get the following estimate:

$$\begin{aligned} v^R_{\varphi_2}\big(\lambda x;[0,1]\big) &= \sum_{n=1}^{\infty} \varphi_2(\lambda nt_n)(a_{n-1}-a_n) \geqslant \sum_{n\geqslant\frac{1}{\lambda}}^{\infty} \varphi_2(t_n)(a_{n-1}-a_n) \\ &= \sum_{n\geqslant\frac{1}{\lambda}}^{\infty} \frac{\varphi_2(t_n)}{2^{n-1}}\left(\frac{1}{\varphi_1(n^2t_n)} - \frac{1}{2\varphi_1((n+1)^2t_{n+1})}\right) \\ &\geqslant \sum_{n\geqslant\frac{1}{\lambda}}^{\infty} \frac{\varphi_1(n^2 2^n t_n)}{2^{n-1}}\left(\frac{1}{\varphi_1(n^2t_n)} - \frac{1}{2\varphi_1((n+1)^2t_{n+1})}\right) \\ &\geqslant \sum_{n\geqslant\frac{1}{\lambda}}^{\infty} \frac{2^n\varphi_1(n^2t_n)}{2^{n-1}}\left(\frac{1}{\varphi_1(n^2t_n)} - \frac{1}{2\varphi_1((n+1)^2t_{n+1})}\right) \\ &\geqslant \sum_{n\geqslant\frac{1}{\lambda}}^{\infty}\left(2 - \frac{\varphi_1(n^2t_n)}{\varphi_1((n+1)^2t_{n+1})}\right) = \infty. \end{aligned}$$

Thus $x \in RV^*_{\varphi_1}[0,1]$ and $x \notin RV^*_{\varphi_2}[0,1]$, which is a contradiction. □

From W. Orlicz's classical result [5, p. 110], we obtain the following corollary.

3.4. Corollary. *Let φ_1, φ_2 be convex Young functions which satisfy condition (∞_1). Then $RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$ if and only if $L^*_{\varphi_1}[a,b] \subset L^*_{\varphi_2}[a,b]$ where $L^*_{\varphi}[a,b]$ is the Orlicz space generated by the class $L_\varphi[a,b]$.*

Now we present equivalent conditions for inequality (4).

3.5. Theorem. *Let φ_1, φ_2 be convex Young functions which satisfy the condition (∞_1). Then the following conditions are equivalent:*

(i) *there exist positive constants C and T such that*

$$\varphi_1(t) \leqslant \varphi_2(Ct), \quad t \geqslant T,$$

(ii) *there exist positive constants C and T such that*

$$\varphi_1(Ct) \leqslant \varphi_2(t), \quad t \geqslant T,$$

(iii) *there exist positive constants C_1, C_2 and T such that*

$$\varphi_1(t) \leqslant C_1\varphi_2(C_2 t), \quad t \geqslant T,$$

(iv) *there exist positive constants $n \in \mathbb{N}$ and T such that*

$$\varphi_1(t) \leqslant \varphi_2(nt), \quad t \geqslant T,$$

(v) *there exist positive constants $n, m \in \mathbb{N}$ and T such that*

$$\varphi_1(t) \leqslant m\varphi_2(nt), \quad t \geqslant T,$$

(vi) *there exist positive constants C and r such that*

$$\limsup_{t\to\infty} \frac{\varphi_1(t)}{\varphi_2(Ct)} = r < \infty,$$

(vii) *there exist positive constants C and r such that*

$$\liminf_{t\to\infty} \frac{\varphi_2(Ct)}{\varphi_1(t)} = r > 0,$$

(viii) *there exist a constant $n \in \mathbb{N}$ such that*

$$nRV^*_{\varphi_2}[a,b] \subset RV^*_{\varphi_1}[a,b].$$

From this theorem we obtain the following corollary.

3.6. Corollary.

(i) *If there exists a positive constant C such that*

$$\lim_{t\to\infty} \frac{\varphi_2(Ct)}{\varphi_1(t)} > 0, \quad \textit{then} \quad RV^*_{\varphi_2}[a,b] \subset RV^*_{\varphi_1}[a,b],$$

(ii) *If there exists a positive constant C such that*

$$0 < \lim_{t\to\infty} \frac{\varphi_2(Ct)}{\varphi_1(t)} < \infty, \quad \textit{then} \quad RV^*_{\varphi_2}[a,b] = RV^*_{\varphi_1}[a,b],$$

(iii) $RV_{\varphi_2}[a,b] = RV_{\varphi_1}[a,b]$ *if and only if there exists a positive constant C such that*

$$0 < \lim_{t\to\infty} \frac{\varphi_1(t)}{t^p} < \infty \quad \textit{if and only if} \quad RV^*_{\varphi_1}[a,b] = RV_p[a,b].$$

Now we present a necessary and sufficient condition on the Young function φ so that the class $RV_\varphi[a,b]$ is a vector space.

3.7. Theorem. *Let φ be a convex Young function which satisfies the condition (∞_1). The class $RV_\varphi[a,b]$ is a vector space if and only if the function φ satisfies the condition Δ_2 for large t, i.e. if there exist positive constants k and t_0 such that*

$$\varphi(2t) \leqslant k\varphi(t) \quad \textit{for all } t \geqslant t_0.$$

Proof. Similar to the proof of Theorem 1.12, p. 14 in [8]. □

3.8. Remark.

(i) Let φ_1, φ_2 be convex Young functions which satisfy the condition (∞_1). If the inequality (1) holds for all $t \geqslant 0$, then
   - $Lip[a,b] \subset RV_{\varphi_1}[a,b] \subset RV_{\varphi_2}[a,b]$,
   - $BV_{\varphi_1}[a,b] \subset BV_{\varphi_2}[a,b]$. Moreover, if additionally φ_1 satisfies the following condition: there exists a positive constant C_1 such that $\varphi_1(t) \leqslant C_1 t$ for all $t \geqslant 0$, then
   - $Lip[a,b] \subset RV_{\varphi_1}[a,b] \subset RV_{\varphi_2}[a,b] \subset BV[a,b] \subset BV_{\varphi_1}[a,b] \subset BV_{\varphi_2}[a,b]$.

(ii) Let φ_1, φ_2 be convex Young functions which satisfy the condition (∞_1). If the inequality (4) holds for all $t \geqslant 0$, then
   - $Lip[a,b] \subset RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b]$,
   - $BV^*_{\varphi_1}[a,b] \subset BV^*_{\varphi_2}[a,b]$. Moreover, if additionally φ_1 satisfies the following condition: there exists a positive constant C_1 such that $\varphi_1(t) \leqslant C_1 t$ for all $t \geqslant 0$, then
   - $Lip[a,b] \subset RV^*_{\varphi_1}[a,b] \subset RV^*_{\varphi_2}[a,b] \subset BV[a,b] \subset BV^*_{\varphi_1}[a,b] \subset BV^*_{\varphi_2}[a,b]$.

(iii) If the φ-function φ satisfies the condition Δ_2 for all $t \geqslant 0$, then the classes $RV_\varphi[a,b]$ and $BV_\varphi[a,b]$ are vector spaces.

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