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Some generalized double lacunary Zweier convergent sequence spaces

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Summary. We introduce generalized double lacunary Zweier convergent sequence spaces over n-normed spaces via a sequence of Orlicz functions. We also make an effort to study some topological properties and inclusion relations between these spaces. Furthermore, we study the concept of double lacunary statistical Zweier convergence over n-normed spaces.

Keywords
Orlicz function;
paranorm;
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1. Introduction and Preliminaries

In [15], Hardy introduced the concept of a regular convergence for double sequences. Some important work on double sequences was also done by Bromwich [2]. By convergence of a double sequence we mean the convergence in the Pringsheim sense, i.e., a double sequence $x = (x_{ij})$ has a Pringsheim limit L provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{ij} - L| < \epsilon$ whenever $i, j > n$ [21] (this is denoted by $P - \lim x = L$). In case $L = 0$, we say that the double sequence $x = (x_{ij})$ is a Pringsheim null sequence.

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The double sequence $x = (x_{ij})$ is bounded if there exists a positive integer K such that $|x_{ij}| < K$ for all i and j. We denote by l_∞^2 the space of all bounded double sequences.

1.1. Definition.[25] The double sequence $I_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing integer sequences (k_r) and (l_s) such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad \text{as } r \to \infty$$

and

$$l_0 = 0, \quad \overline{h_s} = l_s - l_{s-1} \to \infty \quad \text{as } s \to \infty.$$

Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h_s}$, and let $\theta_{r,s}$ be determined by

$$I_{r,s} = \Big\{(k,l) : k_{r-1} < k \leqslant k_r \text{ and } l_{s-1} < l \leqslant l_s\Big\},$$
$$q_r = \frac{k_r}{k_{r-1}}, \quad \overline{q_s} = \frac{l_s}{l_{s-1}} \quad \text{and} \quad q_{r,s} = q_r \overline{q_s}.$$

1.2. Definition. An Orlicz function $M\colon [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. If convexity of an Orlicz function is replaced by $M(x+y) \leqslant M(x) + M(y)$, then this function is called a modulus function.

Lindenstrauss and Tzafriri [17] used the idea of Orlicz to define the sequence space,

$$\ell_M = \left\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0\right\}$$

which is known as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leqslant 1\right\}.$$

It was shown in [17] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geqslant 1)$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η, known as the kernel of M, is right differentiable for $t \geqslant 0$, non-decreasing, $\eta(0) = 0$, $\eta(t) > 0$, and $\eta(t) \to \infty$ as $t \to \infty$.

The notion of difference sequence spaces was introduced by Kızmaz [16] who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [8] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w

denote the set of all real or complex sequences and let m, n be non-negative integers. For the sequence spaces $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta_m^n) = \left\{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\right\},$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Çolak [8]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [16]. Similarly, we can define difference operators on double sequence spaces as:

$$\begin{aligned}\Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1})\\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1},\\ \Delta^n x_{k,l} &= \Delta^{n-1} x_{k,l} - \Delta^{n-1} x_{k,l+1} - \Delta^{n-1} x_{k+1,l} + \Delta^{n-1} x_{k+1,l+1}\end{aligned}$$

and

$$\Delta_m^n x_{k,l} = \Delta_m^{n-1} x_{k,l} - \Delta_m^{n-1} x_{k,l+1} - \Delta_m^{n-1} x_{k+1,l} + \Delta_m^{n-1} x_{k+1,l+1}.$$

For more details about sequence spaces, see [1, 22, 23] and references therein.

1.3. Definition. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be a Musielak–Orlicz function (see [18, 20]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\left\{|v|u - M_k(u) : u \geqslant 0\right\}, \quad k \in \mathbb{N}$$

is called a function complementary to the Musielak–Orlicz function (M_k). For a given Musielak–Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_\mathcal{M}$ and its subspace $h_\mathcal{M}$ are defined as follows

$$\begin{aligned}t_\mathcal{M} &= \left\{x \in w : I_\mathcal{M}(cx) < \infty \text{ for some } c > 0\right\},\\ h_\mathcal{M} &= \left\{x \in w : I_\mathcal{M}(cx) < \infty \text{ for all } c > 0\right\},\end{aligned}$$

where $I_\mathcal{M}$ is a convex modular defined by

$$I_\mathcal{M}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in w.$$

We consider $t_\mathcal{M}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{k > 0 : I_\mathcal{M}\left(\frac{x}{k}\right) \leqslant 1\right\}$$

or with the Orlicz norm

$$\|x\|^0 = \inf\Big\{\tfrac{1}{k}\big(1 + I_{\mathcal{M}}(kx)\big) : k > 0\Big\}.$$

A Musielak–Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy the Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leqslant KM_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$ such that $M_k(u) \leqslant a$.

1.4. Definition. Let X be a linear metric space. A function $p\colon X \to \mathbb{R}$ is called a paranorm if

- (i) $p(x) \geqslant 0$ for all $x \in X$;
- (ii) $p(-x) = p(x)$ for all $x \in X$;
- (iii) $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$;
- (iv) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called a total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [29, Theorem 10.4.2, pp. 183]).

1.5. Definition. A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| < 1$.

1.6. Definition. A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of $\mathbb{N}$.

1.7. Definition. A sequence space E is said to be sequence algebra if $(x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

1.8. Definition. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

1.9. Definition. Let $K = \{k_1 < k_2 < \dots\} \subset \mathbb{N}$ and let E be a sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (x_k) \in E\}$.

1.10. Definition. The canonical preimage of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in w$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

The canonical preimage of a step space λ_K^E is the set of canonical preimages of all the elements in λ_K^E, that is, y is in the canonical preimage of λ_K^E if and only if y is a canonical preimage of some $x \in \lambda_K^E$.

1.11. Definition. A sequence space E is said to be monotone if it contains the canonical preimages of its step spaces.

The concept of 2-normed spaces was initially developed by Gähler [10] in mid-1960s, while that of n-normed spaces one can trace back to Misiak [19]. Since then many other authors have studied this concept and obtained various results, see Gunawan [11, 12] and Gunawan and Mashadi [13]. Let $n \in \mathbb{N}$ and let X be a real linear space of dimension d, where $d \geqslant n \geqslant 2$. A real-valued function $\|\cdot, \cdots, \cdot\|$ on X^n satisfying the following four conditions:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X,

(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation,

(iii) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$,

(iv) $\|x + x', x_2, \ldots, x_n\| \leqslant \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

is called an n-norm on X and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an n-normed space over the field $\mathbb{R}$.

For example, we may take $X = \mathbb{R}^n$ equipped with the n-norm $\|x_1, x_2, \ldots, x_n\|_E$, i.e. the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \ldots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space of dimension $d \geqslant n \geqslant 2$ and $\{a_1, a_2, \ldots, a_n\}$ a linearly independent set in X. Then the function $\|\cdot, \ldots, \cdot\|_\infty$ on X^{n-1} given by

$$\|x_1, x_2, \ldots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \ldots, x_{n-1}, a_i\| : i = 1, 2, \ldots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \ldots, z_{n-1}\| = 0 \quad \text{for every } z_1, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \ldots, z_{n-1}\| = 0 \quad \text{for every } z_1, \ldots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. A complete n-normed space is said to be an n-Banach space. For more details on n-normed spaces, see [14, 24] and references therein.

2. Lacunary strongly Zweier convergent sequence spaces

Zweier sequence spaces for single sequences were defined and studied by Şengönül [28], Esi and Sapsızoğlu [6], Khan et. al [3, 4]. Esi and Acikgoz [7] define the double Zweier sequence spaces $[W^2, Z]$, $[N_{\theta_{r,s}}, Z]_0$, $[N_{\theta_{r,s}}, Z]$ and $[N_{\theta_{r,s}}, Z]_\infty$ as the set of all double sequences such that their $Z-$transforms are in $[W^2]$, $[N_{\theta_{r,s}}]_0$, $[N_{\theta_{r,s}}]$ and $[N_{\theta_{r,s}}]_\infty$, which were introduced by Savaş in [27], Savaş and Patterson in [26].

We define the double sequence $v = (v_{ij})$ and $w = (w_{ij})$ which will be used throughout the paper, as $Z-$transform of $x = (x_{ij})$ and $y = (y_{ij})$, respectively, i.e.,

$$v_{ij} = \tfrac{1}{2}(x_{ij} + x_{ij-1}) \quad \text{and} \quad w_{ij} = \tfrac{1}{2}(y_{ij} + y_{ij-1}), \qquad i, j \in \mathbb{N}. \tag{1}$$

Let $A = (a_{rsij})$ be a nonnegative bounded regular matrix of complex numbers. We write $Ax = (A_{ij(x)}) = \sum_{r,s=1}^{\infty} a_{rsij}(x_{ij})$ converges for each i, j.

Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and let $W(n - X)$ denote the space of X-valued sequences. Let $\mathcal{M} = (M_{ij})$ be a Musielak-Orlicz function, $A = (a_{klij})$ a nonnegative four-dimensional bounded regular matrix, $p = (p_{ij})$ a bounded double sequence of positive real numbers, and $u = (u_{ij})$ a double sequence of strictly positive real numbers. In the present paper we introduce new double Zweier sequence spaces as follows:

$$\begin{aligned}
&\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|\big]_0 = \\
&\quad \Big\{x = (x_{ij}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} = 0 \\
&\qquad \text{for some } \rho > 0\Big\},
\end{aligned}$$

$$\begin{aligned}
&\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|\big] = \\
&\quad \Big\{x = (x_{ij}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij} - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} = 0 \\
&\qquad \text{for some } L \text{ and } \rho > 0\Big\},
\end{aligned}$$

$$\begin{aligned}
&\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|\big]_\infty = \\
&\quad \Big\{x = (x_{ij}) : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty \\
&\qquad \text{for some } \rho > 0\Big\},
\end{aligned}$$

and

$$\big[W^2, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|\big] =$$

$$\Big\{x = (x_{ij}) \,:\, P - \lim_{m,n} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij} - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} = 0$$

$$\text{for some } L \text{ and } \rho > 0\Big\}.$$

2.1. Remark. Let us consider a few special cases of the above sequence spaces:

(i) If $M_{ij}(x) = x$ for all $i, j \in \mathbb{N}$, then the above sequence spaces reduce to

$$\big[N_{\theta_{r,s}}, Z, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, Z, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, Z, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|\big]_\infty \quad \text{and} \quad \big[W^2, Z, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|\big].$$

(ii) By taking $(p_{ij}) = 1$ for all $i, j \in \mathbb{N}$, the above spaces become

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, u, \|\cdot, \dots, \cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, u, \|\cdot, \dots, \cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, u, \|\cdot, \dots, \cdot\|\big]_\infty \quad \text{and} \quad \big[W^2, Z, \mathcal{M}, A, \Delta_m^n, u, \|\cdot, \dots, \cdot\|\big].$$

(iii) By taking $(u_{ij}) = 1$ for all $i, j \in \mathbb{N}$, we get

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\big]_\infty \quad \text{and} \quad \big[W^2, Z, \mathcal{M}, A, \Delta_m^n, p, \|\cdot, \dots, \cdot\|\big].$$

(iv) If we take $M_{ij}(x) = x$, $(p_{ij}) = 1$, $(u_{ij}) = 1$ for all $i, j \in \mathbb{N}$, then the above sequence spaces reduce to

$$\big[N_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \dots, \cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \dots, \cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \dots, \cdot\|\big]_\infty \quad \text{and} \quad \big[W^2, Z, A, \Delta_m^n, \|\cdot, \dots, \cdot\|\big].$$

(v) If we take $A = (C, 1, 1)$, $M_{ij}(x) = x$, $(p_{ij}) = 1$, $(u_{ij}) = 1$ for all $i, j \in \mathbb{N}$ and $n = 0$, then the above spaces reduce to

$$\big[N_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, Z, \|\cdot, \dots, \cdot\|\big]_\infty \quad \text{and} \quad \big[W^2, Z, \|\cdot, \dots, \cdot\|\big].$$

(vi) Finally, if we take $(p_{ij}) = 1$, $(u_{ij}) = 1$ for all $i, j \in \mathbb{N}$, then the above spaces reduce to

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, \|\cdot, \dots, \cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, \|\cdot, \dots, \cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, \|\cdot, \dots, \cdot\|\big]_\infty \quad \text{and} \quad \big[W^2, Z, \mathcal{M}, A, \Delta_m^n, \|\cdot, \dots, \cdot\|\big].$$

The following inequality will be used throughout the paper. If $0 \leqslant p_{ij} \leqslant \sup p_{ij} = G$, $D = \max(1, 2^{G-1})$, then

$$|a_{ij} + b_{ij}|^{p_{ij}} \leqslant D\big(|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}}\big) \tag{2}$$

for all $i, j \in \mathbb{N}$ and $a_{ij}, b_{ij} \in \mathbb{C}$. Also $|a|^{p_{ij}} \leqslant \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to introduce double lacunary Zweier strongly convergent sequence spaces over n-normed spaces and study various properties of these spaces, like linearity, existence of a paranorm, solidity, monotonicity, etc. Some inclusion relations between these spaces are also established. Finally, we study the concept of the double lacunary Zweier statistical convergence over n-normed spaces.

3. Main Results

3.1. Theorem. *Let $\mathcal{M} = (M_{ij})$ be a sequence of Orlicz functions, $p = (p_{ij})$ any bounded double sequence of positive real numbers and $u = (u_{ij})$ a double sequence of strictly positive real numbers. Then the double Zweier sequence spaces*

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot,\dots,\cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot,\dots,\cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot,\dots,\cdot\|\big]_\infty \quad \textit{and} \quad \big[W^2, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot,\dots,\cdot\|\big]$$

are linear spaces over the field $\mathbb{R}$ of real numbers.

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot,\dots,\cdot\|]_\infty$, and $\alpha, \beta \in \mathbb{R}$. Then there exist positive real numbers ρ_1, ρ_2 such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho_1}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty$$
$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n w_{ij}}{\rho_2}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_{ij}'s are non-decreasing and convex, by inequality (2), we have

$$\begin{aligned}
&\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n(\alpha v_{ij} + \beta w_{ij})}{\rho_3}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad \leqslant \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n \alpha v_{ij}}{\rho_3}, z_1, \dots, z_{n-1}\Big\|\Big) \\
&\qquad + u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n \beta w_{ij}}{\rho_3}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant D\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}\frac{1}{2^{p_{ij}}}M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho_1},z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\quad+D\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}\frac{1}{2^{p_{ij}}}M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n w_{ij}}{\rho_2},z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\leqslant D\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho_1},z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\quad+D\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n w_{ij}}{\rho_2},z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}<\infty.
\end{aligned}
$$

Thus $\alpha x+\beta y\in\big[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,p,u,\|\cdot,\dots,\cdot\|\big]_\infty$. This proves that

$$\big[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,p,u,\|\cdot,\dots,\cdot\|\big]_\infty$$

is a linear space. Similarly we can prove that

$$\big[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,p,u,\|\cdot,\dots,\cdot\|\big]_0,\quad \big[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,p,u,\|\cdot,\dots,\cdot\|\big]$$
$$\text{and }\big[W^2,Z,\mathcal{M},A,\Delta_m^n,p,u,\|\cdot,\dots,\cdot\|\big]$$

are linear spaces. □

3.2. Theorem. *Let $\mathcal{M}=(M_{ij})$ be a sequence of Orlicz functions, $p=(p_{ij})$ a bounded double sequence of positive real numbers, and $u=(u_{ij})$ a double sequence of strictly positive real numbers. Then the double Zweier sequence space $[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,p,u,\|\cdot,\dots,\cdot\|]_\infty$ is a paranormed space with the paranorm defined by*

$$
\begin{aligned}
g(x)=\inf\Big\{(\rho)^{\frac{p_{ij}}{H}}\ :\ &\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}}\leqslant 1,\\
&\textit{for some } \rho>0\Big\},
\end{aligned}
$$

where $0<p_{ij}\leqslant\sup p_{ij}=G$ *and* $H=\max(1,G)$.

Proof. (i) Clearly, $g(x)\geqslant 0$ for $x=(x_{ij})\in[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,p,u,\|\cdot,\dots,\cdot\|]_\infty$. Since $M_{ij}(0)=0$, we have $g(0)=0$.

(ii) $g(-x)=g(x)$.

(iii) Let $x=(x_{ij})$ and $y=(y_{ij})\in[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,p,u,\|\cdot,\dots,\cdot\|]_\infty$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho_1},z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\leqslant 1$$

and

$$\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n w_{ij}}{\rho_2}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \leqslant 1.$$

Let $\rho = \rho_1 + \rho_2$. By Minkowski's inequality, we have

$$\begin{aligned}
&\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n (v_{ij}+w_{ij})}{\rho}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\quad= \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n (v_{ij}+w_{ij})}{\rho_1+\rho_2}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\quad\leqslant \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho_1+\rho_2}, z_1,\dots,z_{n-1}\Big\|\Big)\\
&\qquad+ u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n w_{ij}}{\rho_1+\rho_2}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\quad\leqslant \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho_1}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\qquad+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n w_{ij}}{\rho_2}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \leqslant 1
\end{aligned}$$

and thus

$$\begin{aligned}
&g(x+y)\\
&\quad= \inf\Big\{(\rho)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n (v_{ij}+w_{ij})}{\rho_1+\rho_2}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}} \leqslant 1\Big\}\\
&\quad\leqslant \inf\Big\{(\rho_1)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho_1}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}} \leqslant 1\Big\}\\
&\qquad+ \inf\Big\{(\rho_2)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n w_{ij}}{\rho_2}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}} \leqslant 1\Big\}.
\end{aligned}$$

Therefore, $g(x+y) \leqslant g(x) + g(y)$.

(iv) Finally, we prove that scalar multiplication is continuous. Let λ be any complex number. By definition,

$$\begin{aligned}
g(\lambda x) &= \inf\Big\{(\rho)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n \lambda v_{ij}}{\rho}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}} \leqslant 1\Big\}\\
&= \inf\Big\{(|\lambda| t)^{\frac{p_{ij}}{H}} : \sup_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{t}, z_1,\dots,z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}} \leqslant 1\Big\}.
\end{aligned}$$

where $t = \frac{\rho}{|\lambda|} > 0$. Since $|\lambda|^{p_{ij}} \leqslant \max(1, |\lambda|^{\sup p_{ij}})$, we have

$$g(\lambda x) \leqslant \max(1, |\lambda|^{\sup p_{ij}}) \\ \times \inf\left\{ t^{\frac{p_{ij}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{t}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{\frac{p_{ij}}{H}} \leqslant 1 \right\}.$$

The fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem.

□

3.3. Theorem. *If* $0 < p_{ij} < q_{ij} < \infty$ *for each i and j, then*

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_\infty \subset \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, q, u, \|\cdot, \ldots, \cdot\|\big]_\infty.$$

Proof. Let $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$. Then there exists $\rho > 0$ such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty.$$

This implies that $M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < 1$ for sufficiently large values of i and j. Since M_{ij}'s are non-decreasing, we get

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{q_{ij}} \\ \leqslant \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty.$$

Thus, $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, q, u, \|\cdot, \ldots, \cdot\|]_\infty$. This completes the proof. □

3.4. Theorem. *Suppose* $\mathcal{M} = (M_{ij})$ *is a sequence of Orlicz functions,* $p = (p_{ij})$ *a bounded double sequence of positive real numbers, and* $u = (u_{ij})$ *a double sequence of strictly positive real numbers. Then*

(i) *If* $0 < \inf p_{ij} < p_{ij} \leqslant 1$, *then*

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_\infty \subset \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, u, \|\cdot, \ldots, \cdot\|\big]_\infty.$$

(ii) *If* $1 \leqslant p_{ij} \leqslant \sup p_{ij} < \infty$, *then*

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, u, \|\cdot, \ldots, \cdot\|\big]_\infty \subset \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_\infty.$$

Proof. (i). Let $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty$. Since $0 < \inf p_{ij} \leqslant 1$, we obtain the following

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big] \leqslant \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty,$$

and hence $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, u, \|\cdot, \dots, \cdot\|]_\infty$.

(ii). Let $p_{ij} \geqslant 1$ for each i and j, and $\sup p_{ij} < \infty$. Let

$$x = (x_{ij}) \in \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, u, \|\cdot, \dots, \cdot\|\big]_\infty.$$

Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big] \leqslant \epsilon < 1 \quad \text{for all } r, s \geqslant N.$$

This implies that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \leqslant \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big] < \infty.$$

Therefore, $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty$. This completes the proof. □

3.5. Theorem. *Let $\mathcal{M}' = (M'_{ij})$ and $\mathcal{M}'' = (M''_{ij})$ be two sequences of Orlicz functions. Then we have*

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}', A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|\big]_\infty \cap \big[N_{\theta_{r,s}}, Z, \mathcal{M}'', A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|\big]_\infty \subset \big[N_{\theta_{r,s}}, Z, \mathcal{M}' + \mathcal{M}'', A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|\big]_\infty.$$

Proof. Let

$$x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}', A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty \cap [N_{\theta_{r,s}}, Z, \mathcal{M}'', A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty.$$

Then

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M'_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho_1}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty \quad \text{for some } \rho_1 > 0$$

and

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M''_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho_2}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty \quad \text{for some } \rho_2 > 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. We have

$$\begin{aligned}
&\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (M'_{ij} + M''_{ij})\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&= \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M'_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad + \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M''_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\leqslant D \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M'_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho_1}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad + D \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M''_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho_2}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty.
\end{aligned}$$

Therefore, $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}' + \mathcal{M}'', A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty$. This completes the proof. □

3.6. Theorem. *Let $\mathcal{M} = (M_{ij})$ be a sequence of Orlicz functions, $p = (p_{ij})$ a bounded double sequence of positive real numbers, and $u = (u_{ij})$ a double sequence of strictly positive real numbers. Then*

(i) $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_0 \subset [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty$.

(ii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|] \subset [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty$.

Proof. The proof is easy, hence it will be omitted. □

3.7. Theorem. *The double Zweier sequence space $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty$ is solid.*

Proof. Suppose $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \dots, \cdot\|]_\infty$. Then

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty \quad \text{for some } \rho > 0.$$

Let (α_{ij}) be a double sequence of scalars such that $|\alpha_{ij}| \leqslant 1$ for all $i, j \in \mathbb{N}$. Then we get

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m \alpha_{ij} v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}$$

$$\leqslant \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} < \infty.$$

This completes the proof. □

3.8. Theorem. *The double Zweier sequence space* $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$ *is monotone.*

Proof. The proof is trivial and will be omitted. □

3.9. Theorem. *The double Zweier sequence spaces*

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_\infty \quad \textit{and} \quad \big[W^2, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]$$

are linearly isomorphic to the double sequence spaces

$$\big[N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_0, \quad \big[N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big],$$
$$\big[N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_\infty \quad \textit{and} \quad \big[W^2, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big],$$

respectively, i.e.,

(i) $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]_0 \approx [N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]_0$,

(ii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|] \approx [N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]$,

(iii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty \approx [N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]_\infty$,

(iv) $[W^2, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|] \approx [W^2, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]$.

Proof. We consider only $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|]_0$. We should show the existence of a linear bijection between the double sequence spaces

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_0 \quad \text{and} \quad \big[N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_0.$$

Consider the transformation Z

$$\big[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_0 \to \big[N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \ldots, \cdot\|\big]_0$$

given by

$$x \mapsto Zx = v, \quad v = (v_{ij})$$

and

$$v_{ij} = \tfrac{1}{2}(x_{ij} + x_{ij-1}) \quad (i, j \in \mathbb{N}).$$

The linearity of Z is clear. Further, it is trivial that $x = 0$ whenever $Zx = 0$, hence Z is injective. Let $v = (v_{ij}) \in [N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]_0$ and define the sequence $x = (x_{ij})$ by

$$x_{ij} = 2 \sum_{k=0}^{j} (-1)^{j-k} v_{ik}, \quad i \in \mathbb{N}.$$

Then

$$\begin{aligned}
&\|x\|_{[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]_0} \\
&\quad = \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n \frac{1}{2}(x_{ij} + x_{ij-1})}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad = \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\Big(A_{ij}\Delta_m^n \frac{1}{2}(2 \sum_{k=0}^{j} (-1)^{j-k} v_{ik} \\
&\qquad + 2 \sum_{k=0}^{j-1} (-1)^{(j-1)-k} v_{ik})\Big)(\rho)^{-1}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad = \sup_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}
\end{aligned}$$

which implies that $x = (x_{ij}) \in [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]_0$. Additionally, we observe that $\|x\|_{[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]_\infty} = \|v\|_{[N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]_\infty}$. Thus the transformation Z is surjective. Hence Z is a linear bijection which means that the double sequence spaces

$$[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]_0 \quad \text{and} \quad [N_{\theta_{r,s}}, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]_0$$

are linearly isomorphic. The other linear isomorphisms can be proved similarly. This completes the proof. □

3.10. Theorem. *Let $\theta_{r,s}$ be a double lacunary sequence. Then*

(i) $[W^2, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|] \subset [N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]$ *if* $\liminf q_r > 1$ *and* $\liminf \overline{q_s} > 1$;

(ii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|] \subset [W^2, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]$ *if* $\limsup q_r < \infty$ *and* $\limsup \overline{q_s} < \infty$;

(iii) $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|] = [W^2, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot, \dots, \cdot\|]$ *if* $1 < \liminf q_r < \infty$ *and* $1 < \limsup \overline{q_s} < \infty$.

Proof. (i). Suppose that $\liminf q_r > 1$ and $\liminf \overline{q_s} > 1$. Then there exists $\delta > 0$ such that both $q_r > 1 + \delta$ and $\overline{q_s} > 1 + \delta$. This implies that $\frac{h_r}{k_r} \geqslant \frac{\delta}{1+\delta}$ and $\frac{\overline{h_s}}{l_s} \geqslant \frac{\delta}{1+\delta}$. If $x = (x_{ij}) \in [W^2, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \ldots, \cdot\|]$, then we obtain the following:

$$\begin{aligned}
B_{rs} &= \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&= \frac{1}{h_{r,s}} \sum_{i=1}^{k_r} \sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad - \frac{1}{h_{r,s}} \sum_{i=1}^{k_{r-1}} \sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad - \frac{1}{h_{r,s}} \sum_{i=k_{r-1}+1}^{k_r} \sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad - \frac{1}{h_{r,s}} \sum_{j=l_{s-1}+1}^{l_s} \sum_{i=1}^{k_{r-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&= \frac{k_r l_s}{h_{r,s}}\Big(\frac{1}{k_r l_s} \sum_{i=1}^{k_r} \sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big) \\
&\quad - \frac{k_{r-1} l_{s-1}}{h_{r,s}}\Big(\frac{1}{k_{r-1} l_{s-1}} \sum_{i=1}^{k_{r-1}} \sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big) \\
&\quad - \frac{1}{h_r} \sum_{i=k_{r-1}+1}^{k_r} \frac{l_{s-1}}{\overline{h_s}} \frac{1}{l_{s-1}} \sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}} \\
&\quad - \frac{1}{\overline{h_s}} \sum_{j=l_{s-1}+1}^{l_s} \frac{k_{r-1}}{h_r} \frac{1}{k_{r-1}} \sum_{i=1}^{k_{r-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}.
\end{aligned}$$

Since $x = (x_{ij}) \in [W^2, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \ldots, \cdot\|]$ the last two terms tend to zero in the Pringsheim sense. Thus

$$\begin{aligned}
B_{rs} &= \frac{k_r l_s}{h_{r,s}}\Big(\frac{1}{k_r l_s} \sum_{i=1}^{k_r} \sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big) \\
&\quad - \frac{k_{r-1} l_{s-1}}{h_{r,s}}\Big(\frac{1}{k_{r-1} l_{s-1}} \sum_{i=1}^{k_{r-1}} \sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta^n_m v_{ij} - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big) + 0(1).
\end{aligned}$$

Since $h_{r,s} = k_r l_s - k_r l_{s-1} - k_{r-1} l_s + k_{r-1} l_{s-1}$ we obtain the following:

$$\frac{k_r l_s}{h_{r,s}} \leqslant \Big(\frac{1+\delta}{\delta}\Big)^2 \quad \text{and} \quad \frac{k_{r-1} l_{s-1}}{h_{r,s}} \leqslant \frac{1}{\delta}.$$

Hence the terms

$$\frac{k_r l_s}{h_{r,s}}\Big(\frac{1}{k_r l_s}\sum_{i=1}^{k_r}\sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big)$$

and

$$\frac{k_{r-1} l_{s-1}}{h_{r,s}}\Big(\frac{1}{k_{r-1} l_{s-1}}\sum_{i=1}^{k_{r-1}}\sum_{j=1}^{l_{s-1}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big)$$

are both Pringsheim null sequences. Thus $B_{r,s}$ is a Pringsheim null sequence. Therefore, $x=(x_{ij})\in[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot,\ldots,\cdot\|]$.

(ii). Suppose that $\limsup q_r<\infty$ and $\limsup \overline{q_s}<\infty$, then there exists $K>0$ such that $q_r\leqslant K$, $\overline{q_s}\leqslant K$ for all r and s. Let $x=(x_{ij})\in[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta_m^n, p, u, \|\cdot,\ldots,\cdot\|]$ and $\epsilon>0$. Also there exist $r_0>0$ and $s_0>0$ such that for every $k\geqslant r_0$ and $l\geqslant s_0$

$$B'_{k,l}=\frac{1}{h_{k,l}}\sum_{(i,j)\in I_{k,l}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}<\epsilon.$$

Let $N=\max\{B'_{k,l}:1\leqslant k\leqslant r_0 \text{ and } 1\leqslant l\leqslant s_0\}$ and p and q be such that $k_{r-1}<p\leqslant k_r$ and $l_{s-1}<q\leqslant l_s$. Then we obtain the following

$$\begin{aligned}
&\frac{1}{pq}\sum_{i,j=1,1}^{p,q} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\quad\leqslant \frac{1}{k_{r-1}l_{s-1}}\sum_{i=1}^{k_r}\sum_{j=1}^{l_s} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\\
&\quad\leqslant \frac{1}{k_{r-1}l_{s-1}}\sum_{p,q=1,1}^{r,s}\Big(\sum_{(i,j)\in I_{p,q}} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}\Big)\\
&\quad= \frac{1}{k_{r-1}l_{s-1}}\sum_{p,q=1,1}^{r_0,s_0} h_{p,q}B'_{p,q}+\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\leqslant r)\cup(s_0<q\leqslant s)} h_{p,q}B'_{p,q}\\
&\quad\leqslant \frac{N}{k_{r-1}l_{s-1}}\sum_{p,q=1,1}^{r_0,s_0} h_{p,q}+\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\leqslant r)\cup(s_0<q\leqslant s)} h_{p,q}B'_{p,q}\\
&\quad\leqslant \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+\Big(\sup_{(p\geqslant r_0)\cup(q\geqslant s_0)} B'_{p,q}\Big)\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\leqslant r)\cup(s_0<q\leqslant s)} h_{p,q}\\
&\quad\leqslant \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+\epsilon\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\leqslant r)\cup(s_0<q\leqslant s)} h_{p,q}\\
&\quad\leqslant \frac{Nk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}}+\epsilon K^2.
\end{aligned}$$

Since k_r and l_s both approach infinity as both r and s approach infinity, it follows that

$$P-\lim_{p,q}\frac{1}{pq}\sum_{i,j=1,1}^{p,q} M_{ij}\Big[u_{ij}\Big(\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho}, z_1,\ldots,z_{n-1}\Big\|\Big)\Big]^{p_{ij}}=0.$$

Therefore, $x = (x_{ij}) \in [W^2, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \ldots, \cdot\|]$.

(iii). Combining (i) and (ii), we can easily prove (iii). □

3.11. Example. Suppose $\liminf_r q_r = 1$ or $\liminf_s \overline{q_s} = 1$, and assume without loss of generality that $\liminf_r q_r = 1$ [9]; then there exists an ordinary subsequence $\{k_{\alpha_j}\}$ of the lacunary sequence θ_r such that $\frac{k_{\alpha_j}}{k_{\alpha_j} - 1} < 1 + \frac{1}{j}$ and $\frac{k_{\alpha_j} - 1}{k_{\alpha_j} - 1} > j$, where $\alpha_j \geqslant \alpha_{j-1} + 2$. Let us define x as follows:

$$x_{ij} = \begin{cases} 1, & \text{if } i \in I_{\alpha_j} \text{ and } j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly the rows are not in $[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \ldots, \cdot\|]$ but each row is such that x is in $|\sigma_{1,1}|$ where $|\sigma_{1,1}| = \{x : P - \lim_{m,n} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} |x_{i,j} - L| = 0 \text{ for some } L\}$. Therefore, each row is in $[W^2, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \ldots, \cdot\|]$. Since the double lacunary sequence $\theta_{r,s}$ is factorable, we have

$$\left[W^2, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \ldots, \cdot\|\right] \not\subset \left[N_{\theta_{r,s}}, Z, \mathcal{M}, A, \Delta^n_m, p, u, \|\cdot, \ldots, \cdot\|\right].$$

4. Lacunary Zweier statistical convergence sequence space

The following definition was presented by Mursaleen and Edely in [5]:

4.1. Definition. A real double sequence $x = (x_{ij})$ is said to be statistically convergent to L, provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \left|\left\{(i,j) : i \leqslant m, j \leqslant n, \text{ and } |x_{ij} - L| \geqslant \epsilon\right\}\right| = 0$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $st_2 - \lim_{ij} x_{ij} = L$ and we denote the set of all $P-$ statistically convergent double sequences by st_2.

4.2. Remark.

(i) If x is a convergent double sequence, then x is also statistically convergent to the same number. Since there is only a finite number of bounded (unbounded) rows and/or columns,

$$K(m,n) \leqslant s_1 m + s_2 n,$$

where s_1 and s_2 are finite numbers, from which one concludes that x is statistically convergent.

(ii) If x is statistically convergent to the number L, then L is determined uniquely.

(iii) If x is statistically convergent, then x need not be convergent. Also x is not necessarily bounded. For example, let $x = (x_{ij})$ be defined as

$$x_{ij} = \begin{cases} ij, & \text{if } i \text{ and } j \text{ are squares} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $st_2 - \lim x_{ij} = 1$, since the cardinality of the set $\{(i,j) : |x_{i,j} - 1| \geqslant \epsilon\} \leqslant \sqrt{i}\sqrt{j}$ for every $\epsilon > 0$ but x is neither convergent nor bounded.

Recently, in [27], Savaş defined double lacunary statistical convergence as follows:

4.3. Definition. A real double sequence $x = (x_{ij})$ is said to be $S_{\theta_{r,s}}$-convergent to L, provided that for each $\epsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(i,j) \in I_{r,s} : |x_{ij} - L| \geqslant \epsilon\right\}\right| = 0.$$

4.4. Definition. A real double sequence $x = (x_{ij})$ is said to be double lacunary Zweier statistical convergent to L provided that for each $\epsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(i,j) \in I_{r,s} : |v_{ij} - L| \geqslant \epsilon\right\}\right| = 0$$

where v_{ij} is of the form (1).

4.5. Theorem. *Let $\theta_{r,s}$ be a double lacunary sequence. If*

$$x_{ij} \to L([N_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|]),$$

then $x_{ij} \to L([S_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|])$.

Proof. If $\epsilon > 0$ and $x_{ij} \to L([N_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|])$ then we can write, where $S(\varepsilon) = \{(i,j) : \|A_{ij}\Delta_m^n \frac{1}{2}(x_{ij} + x_{ij-1}) - L, z_1, \ldots, z_{n-1}\| \geqslant \epsilon\}$

$$\begin{aligned}
&\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \|A_{ij}\Delta_m^n v_{ij} - L, z_1, \ldots, z_{n-1}\| \\
&\quad \geqslant \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}\cap S(\varepsilon)} \|A_{ij}\Delta_m^n v_{ij} - L, z_1, \ldots, z_{n-1}\| \\
&\quad \geqslant \frac{1}{h_{r,s}} \left|\left\{(i,j) \in I_{r,s} : \|A_{ij}\Delta_m^n v_{ij} - L, z_1, \ldots, z_{n-1}\| \geqslant \epsilon\right\}\right|.
\end{aligned}$$

It follows that $x_{ij} \to L([S_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|])$, that is, $[N_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|] \subset [S_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|]$ and the inclusion is strict. To show the latter, we establish an example as follows.

4.6. Example. Let v_{ij} be of the form (1) and defined as follows:

$$v_{ij} = \begin{pmatrix} 1 & 2 & 3 & \ldots & [\sqrt[3]{h_{r,s}}] & 0 & 0 & \ldots \\ 2 & 2 & 3 & \ldots & [\sqrt[3]{h_{r,s}}] & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 & [\sqrt[3]{h_{r,s}}] & [\sqrt[3]{h_{r,s}}] & \ldots & [\sqrt[3]{h_{r,s}}] & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

It is clear that $x = (x_{ij})$ is an unbounded double sequence. By taking $A = (C, 1, 1)$, $n = 0$, for $\epsilon > 0$ and for every $z_1, \ldots, z_{n-1} \in X$ we have

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ (i,j) \in I_{r,s} \,:\, \|A_{ij}\Delta_m^n v_{ij} - L, z_1, \ldots, z_{n-1}\| \geqslant \epsilon \right\} \right| = P - \lim_{r,s} \frac{1}{h_{r,s}} \frac{[\sqrt[3]{h_{r,s}}]}{h_{r,s}} = 0.$$

Therefore, $x_{ij} \to 0([S_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|])$. But

$$P\text{–}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \|A_{ij}\Delta_m^n v_{ij}, z_1, \ldots, z_{n-1}\| = P\text{–}\lim_{r,s} \frac{[\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}]([\sqrt[3]{h_{r,s}}] + 1))}{2h_{r,s}} = \frac{1}{2}.$$

Therefore $x_{ij} \nrightarrow 0([N_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|])$. This completes the proof.

□

4.7. Theorem. *Let $\theta_{r,s}$ be a double lacunary sequence. If $x = (x_{ij}) \in l_\infty^2$ and $x_{ij} \to L([S_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|])$, then $x_{ij} \to L([N_{\theta_{r,s}}, Z, A, \Delta_m^n, \|\cdot, \ldots, \cdot\|])$.*

Proof. Suppose that $x = (x_{ij}) \in l_\infty^2$. Then there exists a positive integer K such that $\|A_{ij}\Delta_m^n v_{ij} - L, z_1, \ldots, z_{n-1}\| < K$ for all $i, j \in \mathbb{N}$. For every $\epsilon > 0$ we have, where as before $S(\varepsilon) = \{(i,j) : \|A_{ij}\Delta_m^n \frac{1}{2}(x_{ij} + x_{ij-1}) - L, z_1, \ldots, z_{n-1}\| \geqslant \epsilon\}$ and $S'(\varepsilon) = \{(i,j) : \|A_{ij}\Delta_m^n \frac{1}{2}(x_{ij} + x_{ij-1}) - L, z_1, \ldots, z_{n-1}\| < \epsilon\}$

$$\begin{aligned} P - \lim_{r,s} \frac{1}{h_{r,s}} & \sum_{(i,j)\in I_{r,s}} \|A_{ij}\Delta_m^n v_{ij} - L, z_1, \ldots, z_{n-1}\| \\ &= \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}\cap S(\varepsilon)} \|A_{ij}\Delta_m^n v_{ij} - L, z_1, \ldots, z_{n-1}\| \\ &\quad + \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}\cap S'(\varepsilon)} \|A_{ij}\Delta_m^n v_{ij} - L, z_1, \ldots, z_{n-1}\| \end{aligned}$$

$$\leqslant \frac{K}{h_{r,s}}\Big|\big\{(i,j)\in I_{r,s}\,:\,\|A_{ij}\Delta_m^n v_{ij}-L,z_1,\dots,z_{n-1}\|\geqslant\epsilon\big\}\Big|+\epsilon.$$

Therefore, $x=(x_{ij})\in l_\infty^2$ and $x_{ij}\to L([S_{\theta_{r,s}},Z,A,\Delta_m^n,\|\cdot,\dots,\cdot\|])$ implies

$$x_{ij}\to L\big([N_{\theta_{r,s}},Z,A,\Delta_m^n,\|\cdot,\dots,\cdot\|]\big).$$

□

4.8. Corollary. *Let $\theta_{r,s}$ be a double lacunary sequence. Then*

$$\big[N_{\theta_{r,s}},Z,A,\Delta_m^n,\|\cdot,\dots,\cdot\|\big]\cap l_\infty^2=\big[S_{\theta_{r,s}},Z,A,\Delta_m^n,\|\cdot,\dots,\cdot\|\big]\cap l_\infty^2.$$

Proof. It follows directly from Theorem4.5 and Theorem 4.7. □

4.9. Theorem. *For any sequence of Orlicz functions $\mathcal{M}$, $[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,\|\cdot,\dots,\cdot\|]\subset[S_{\theta_{r,s}},Z,A,\Delta_m^n,\|\cdot,\dots,\cdot\|]$.*

Proof. Let $x=(x_{ij})\in[N_{\theta_{r,s}},Z,\mathcal{M},A,\Delta_m^n,\|\cdot,\dots,\cdot\|]$. Then for $\epsilon>0$ and all $z_1,\dots,z_{n-1}\in X$, where as before $S(\varepsilon)=\{(i,j):\|A_{ij}\Delta_m^n\frac{1}{2}(x_{ij}+x_{ij-1})-L,z_1,\dots,z_{n-1}\|\geqslant\epsilon\}$

$$\begin{aligned}&\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}M_{ij}\Big[\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big]\\&\geqslant\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}\cap S(\varepsilon)}M_{ij}\Big[\Big\|\frac{A_{ij}\Delta_m^n v_{ij}-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big]\\&>\frac{1}{h_{r,s}}M_{ij}\Big(\frac{\epsilon}{\rho}\Big)\Big|\big\{(i,j)\in I_{r,s}\,:\,\|A_{ij}\Delta_m^n v_{ij}-L,z_1,\dots,z_{n-1}\|\geqslant\epsilon\big\}\Big|.\end{aligned}$$

This shows that

$$x=(x_{ij})\in[S_{\theta_{r,s}},Z,A,\Delta_m^n,\|\cdot,\dots,\cdot\|].$$

□

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