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Off-diagonal multilinear interpolation between adjoint operators

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Summary We extend a theorem by Grafakos and Tao on multilinear interpolation between adjoint operators [*Multilinear interpolation between adjoint operators*, J. Funct. Anal. 199 (2003), 379–385] to an off-diagonal situation. We provide an application.

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1. Introduction and the main result

Multilinear interpolation has proved to be a powerful and indispensable tool in analysis. The two main linear interpolation theorems, the Marcinkiewicz and the Riesz–Thorin theorem, have well-established multilinear analogs. The works [3, 4, 6, 10] provide multilinear extensions of the Marcinkiewicz interpolation theorem. The Riesz–Thorin theorem is adapted in the multilinear case in [12, 21, Chapter XII, (3.3)] and [1, Theorem 4.4.2]; related versions appeared in [8, 9, 11].

A different type of interpolation is that between adjoint operators. In the linear case, a typical result would be as follows: If an operator and its adjoint are of weak type $(1, 1)$, then it is L^p-bounded for all $p \in (1, \infty)$. A multilinear version of this result was obtained in [5]. It says that, under the initial condition similar to (1) below, if an m-linear operator and all of its m adjoints are of restricted weak type $(1, 1, \dots, 1, 1/m)$ then the operator is bounded

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from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p, for all $1 < p_1, \dots, p_m < \infty$ and all $1/m < p < \infty$. In this context, an m-linear operator is called of restricted weak type $(p_1, \dots, p_m, p)$ if it maps $L^{p_1} \times \cdots \times L^{p_m}$ to $L^{p,\infty}$ when restricted to the characteristic functions of sets of finite measure. The jth adjoint of an m-linear operator T (acting on products of simple functions on measure spaces (X_j, μ_j), $j \in \{1, \dots, m\}$, and taking values in another measure space (X_0, μ_0)) is defined as the operator T^{*j} such that

$$\int_{A_0} T(\chi_{A_1}, \dots, \chi_{A_{j-1}}, \chi_{A_j}, \chi_{A_{j+1}}, \dots, \chi_{A_m})\, d\mu_0 = \int_{A_j} T^{*j}(\chi_{A_1}, \dots, \chi_{A_{j-1}}, \chi_{A_0}, \chi_{A_{j+1}}, \dots, \chi_{A_m})\, d\mu_j$$

for all measurable subsets A_i of X_i with nonzero finite measure. When we write T^{*0}, we understand it to be T itself. For $0 < q < \infty$, q' denotes the number $q/(q-1)$ or ∞ if $q = 1$.

In this note, we prove the following off-diagonal version of the main result of [5], where the diagonal case $t = 1$ and $s = 1/m$ was considered.

1.1. Theorem. *Let $1 \leqslant t < \infty$, $0 < s \leqslant 1$, $1 < p < t'$ and $t < p_1, \dots, p_m < \infty$ be such that $1/p_1 + \cdots + 1/p_m - 1/p = m/t - 1/s$. Let $(X_0, \mu_0), (X_1, \mu_1), \dots, (X_m, \mu_m)$ be σ-finite measure spaces. Suppose that an m-linear operator T is defined on the space of simple functions on $X_1 \times \cdots \times X_m$ and takes values in the space of measurable functions on $(X_0\ \mu_0)$. Assume that T satisfies*

$$\sup_{A_0, A_1, \dots, A_m} \frac{1}{\mu_0(A_0)^{\frac{1}{p'}} \mu_1(A_1)^{\frac{1}{p_1}} \cdots \mu_m(A_m)^{\frac{1}{p_m}}} \left| \int_{A_0} T(\chi_{A_1}, \dots, \chi_{A_m})\, d\mu_0 \right| < \infty, \tag{1}$$

where the supremum is taken over all measurable subsets A_i of X_i with nonzero finite measure. Suppose further that for each $j \in \{0, 1, \dots, m\}$, T^{*j} is of restricted weak type $(t, t, \dots, t, s)$ with norm B_j. Then there is a constant $C = C(p_1, \dots, p_m, p, t, s)$ such that T is of restricted weak type $(p_1, \dots, p_m, p)$ with norm at most

$$C B_0^{\theta\left(\frac{1}{t} - \frac{1}{p'}\right)} B_1^{\theta\left(\frac{1}{t} - \frac{1}{p_1}\right)} \cdots B_m^{\theta\left(\frac{1}{t} - \frac{1}{p_m}\right)}, \tag{2}$$

where $\theta = (1/t + 1/s - 1)^{-1}$.

The following well-known characterization of the weak L^p space will be used in the proof; see, for instance, Proposition 7.2.12 in [2].

1.2. Proposition. *Let $0 < p < \infty$, $A, B > 0$ and let f be a measurable function on a σ-finite measure space (X, μ).*

(i) *Suppose that $||f||_{L^{p,\infty}} \leqslant A$. Then for every measurable set E of finite measure there exists a measurable subset E' of E with $\mu(E') \geqslant \mu(E)/2$ such that f is bounded on E' and*

$$\left| \int_{E'} f\, d\mu \right| \leqslant 2^{\frac{1}{p}} A\, \mu(E)^{1 - \frac{1}{p}}.$$

(ii) *Suppose that a measurable function f on X has the property that for any measurable subset E of X with $\mu(E) < \infty$ there is a measurable subset E' of E with $\mu(E') \geqslant \mu(E)/2$ such that f is integrable on E' and*

$$\left|\int_{E'} f\,d\mu\right| \leqslant B\,\mu(E)^{1-\frac{1}{p}}.$$

Then

$$\|f\|_{L^{p,\infty}} \leqslant B\,2^{\frac{2}{p}+\frac{3}{2}}.$$

2. The proof of Theorem 1.1

Proof. First consider the case where

$$\frac{\mu_0(A_0)}{B_0^\theta} \geqslant \max\left(\frac{\mu_1(A_1)}{B_1^\theta}, \dots, \frac{\mu_m(A_m)}{B_m^\theta}\right). \tag{3}$$

Let M be the supremum given in (1). It is enough to show that M is bounded above by the constant in (2), from which, by Proposition 1.2(ii), the desired boundedness will follow.

Since T is of restricted weak type $(t, \dots, t, s)$, by Proposition 1.2(i) there is a subset A_0' of A_0 with measure $\mu_0(A_0') \geqslant \mu_0(A_0)/2$ such that

$$\left|\int_{A_0'} T(\chi_{A_1}, \dots, \chi_{A_m})\,d\mu_0\right| \leqslant KB_0\,\mu_1(A_1)^{\frac{1}{t}} \dots \mu_m(A_m)^{\frac{1}{t}}\,\mu_0(A_0)^{1-\frac{1}{s}}$$

for some constant $K = K(s)$. It follows that

$$\begin{aligned}\left|\int_{A_0} T(\chi_{A_1}, \dots, \chi_{A_m})\,d\mu_0\right| &\leqslant \left|\int_{A_0'} T(\chi_{A_1}, \dots, \chi_{A_m})\,d\mu_0\right| + \left|\int_{A_0\setminus A_0'} T(\chi_{A_1}, \dots, \chi_{A_m})\,d\mu_0\right| \\ &=: I + II.\end{aligned}$$

Using (3) we obtain

$$\begin{aligned} I &\leqslant KB_0\mu_1(A_1)^{\frac{1}{t}} \dots \mu_m(A_m)^{\frac{1}{t}}\,\mu_0(A_0)^{1-\frac{1}{s}} \\ &\leqslant \mu_0(A_0)^{\frac{1}{p'}}\,\mu_1(A_1)^{\frac{1}{p_1}} \dots \mu_m(A_m)^{\frac{1}{p_m}}\left(K\,B_0\left(\frac{B_1^\theta}{B_0^\theta}\right)^{\frac{1}{t}-\frac{1}{p_1}} \dots \left(\frac{B_m^\theta}{B_0^\theta}\right)^{\frac{1}{t}-\frac{1}{p_m}}\right)\end{aligned}$$

and using (1) we obtain

$$\begin{aligned} II &\leqslant M\mu_1(A_1)^{\frac{1}{p_1}} \dots \mu_m(A_m)^{\frac{1}{p_m}}\left(\tfrac{1}{2}\mu_0(A_0)\right)^{\frac{1}{p'}} \\ &= \mu_0(A_0)^{\frac{1}{p'}}\mu_1(A_1)^{\frac{1}{p_1}} \dots \mu_m(A_m)^{\frac{1}{p_m}}\left(M\,2^{-\frac{1}{p'}}\right).\end{aligned}$$

Consequently,

$$M \leqslant KB_0\left(\frac{B_1^\theta}{B_0^\theta}\right)^{\frac{1}{t}-\frac{1}{p_1}} \dots \left(\frac{B_m^\theta}{B_0^\theta}\right)^{\frac{1}{t}-\frac{1}{p_m}} + M\,2^{-\frac{1}{p'}},$$

and using (1) it follows

$$M \leqslant \frac{K}{1-2^{-\frac{1}{p'}}} B_0^{\theta\left(\frac{1}{t}-\frac{1}{p'}\right)} B_1^{\theta\left(\frac{1}{t}-\frac{1}{p_1}\right)} \dots B_m^{\theta\left(\frac{1}{t}-\frac{1}{p_m}\right)}.$$

The last inequality is a consequence of the fact that

$$1-\theta\Big(\frac{m}{t}-\frac{1}{p_1}-\cdots-\frac{1}{p_m}\Big) = 1-\theta\Big(\frac{1}{s}-\frac{1}{p}\Big) = \theta\Big(\frac{1}{\theta}-\frac{1}{s}+\frac{1}{p}\Big) = \theta\Big(\frac{1}{t}-\frac{1}{p'}\Big).$$

There are m more cases left, in each of which $\mu_j(A_j)/B_j^\theta$ is interchanged with $\mu_0(A_0)/B_0^\theta$ in (3), for $j \in \{1,\dots,m\}$. Fix j. Recall that, by assumption, the jth adjoint T^{*j} of T is also of restricted weak type $(t,\dots,t,s)$. Setting $p_0 = p'$ observe that (1) can be written as

$$\sup_{A_0,A_1,\dots,A_m} \frac{1}{\mu_j(A_j)^{\frac{1}{(p_j')'}}\,\mu_0(A_0)^{\frac{1}{p_0}} \prod_{i\neq j} \mu_i(A_i)^{\frac{1}{p_i}}} \left| \int_{A_j} T^{*j}(\chi_{A_1},\dots,\chi_{A_0},\dots,\chi_{A_m})\, d\mu_j \right| < \infty,$$

where the $(m+1)$-tuple $(p_1,\dots,p_{j-1},p_0,p_{j+1},p_j')$ replaces $(p_1,\dots,p_m,p)$ and the identity $(\frac{1}{p_0} + \sum_{i\neq j}\frac{1}{p_i}) - \frac{1}{p_j'} = \frac{m}{t} - \frac{1}{s}$ replaces $\frac{1}{p_1}+\cdots+\frac{1}{p_m}-\frac{1}{p} = \frac{m}{t}-\frac{1}{s}$. The argument in this case follows by a repetition of the argument in the preceding case under the above change of notation. This concludes the proof in all cases. □

2.1. Remark. One may wonder whether hypothesis (1) indeed weakens the statement of the theorem. As in [5], it is an essential element of the proof; however, in most applications it does not present any significant restriction. In fact, in most cases one may work with truncated versions of the operator T, for which (1) holds with constants depending on the truncation. Then boundedness is obtained for truncated operators, with bounds independent of the truncation, and a limiting argument implies the same conclusion for the original operator T.

2.2. Remark. It is worth noting that if (1) holds for every point $(p_1,\dots,p_m)$ with $1 < p < t'$, $t < p_1,\dots,p_m < \infty$ and $1/p_1+\cdots+1/p_m-1/p = m/t-1/s$, then one can obtain restricted weak type estimates at every point in the open convex hull H of these points and the point $(\frac{1}{t},\dots,\frac{1}{t},\frac{1}{s})$. Then by the multilinear Marcinkiewicz interpolation theorem (see, for instance, [4]), it follows that T satisfies the strong type bounds in H.

3. An application

Let $0 < \alpha < n$. Consider the bilinear fractional integral

$$I_\alpha(f,g)(x) = \int_{\mathbb{R}^n} f(x+y)g(x-y)|y|^{\alpha-n}\, dy$$

defined for positive functions f, g on $\mathbb{R}^n$. It was shown in [3] and [7] that I_α maps the product $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ whenever $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + \frac{\alpha}{n}$, and $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ lies in the open convex hull of the points $(\frac{n}{\alpha}, \infty, \infty)$, $(\infty, \frac{n}{\alpha}, \infty)$, $(1, \infty, \frac{n}{n-\alpha})$, $(\infty, 1, \frac{n}{n-\alpha})$, and $(1, 1, \frac{n}{2n-\alpha})$. The proof is achieved in two steps: (a) restricted weak type estimates are proven at the aforementioned five points first; (b) multilinear interpolation is used to obtain boundedness in the open convex hull H of these five points.

We provide a simpler proof of the boundedness of I_α in H by reducing it to a restricted weak type estimate *only* at the point $(1, 1, \frac{n}{2n-\alpha})$ and its two adjoints. We will use Theorem 1.1 with $t = 1$ and $s = \frac{n}{2n-\alpha}$ for which $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = \frac{2}{t} - \frac{1}{s}$. To satisfy condition (1) we introduce the following truncated version of I_α:

$$I_\alpha^{\varepsilon,N,M}(f,g)(x) = \chi_{|x|\leq M} \int_{\varepsilon\leq|y|\leq N} f(x+y)g(x-y)|y|^{\alpha-n} dy\,.$$

For $1 < p < \infty$, it is easy to see that

$$\begin{aligned} \|I_\alpha^{\varepsilon,N,M}(\chi_{A_1}, \chi_{A_2})\|_{L^p} &\leq C_{\varepsilon,N,M} \min(1, |A_1|)^{\frac{1}{p}} \min(1, |A_2|)^{\frac{1}{p}} \min(1, |A_1|, |A_2|)^{\frac{1}{p'}} \\ &\leq C_{\varepsilon,N,M} |A_1|^{\frac{1}{p_1}} |A_2|^{\frac{1}{p_2}}, \end{aligned}$$

and (1) follows for $I_\alpha^{\varepsilon,N,M}$ via Hölder's inequality. Here $p' = \frac{p}{p-1}$ and $0 < \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} < 1$. Analogous estimates hold for the two adjoints of $I_\alpha^{\varepsilon,N,M}$. For instance,

$$(I_\alpha^{\varepsilon,N,M})^{*1}(h,g)(x) = \int_{\varepsilon\leq|y|\leq N} h(x-y)g(x-2y)|y|^{\alpha-n} \chi_{|x-y|\leq M}\, dy$$

is bounded by

$$\chi_{|x|\leq M+N} \int_{\varepsilon\leq|y|\leq N} h(x-y)g(x-2y)\, |y|^{\alpha-n}\, dy$$

when $g, h \geq 0$, thus a similar estimate holds for it. Then Theorem 1.1 and Remark 2.2 yield boundedness for $I_\alpha^{\varepsilon,N,M}$ in H with bounds as in (2), i.e., independent of ε, N, M. Letting $\varepsilon \downarrow 0$ and $N, M \uparrow \infty$, we obtain the same conclusion for I_α via the Lebesgue monotone theorem.

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