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WORTH property, García-Falset coefficient and Opial property of infinite sums

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Summary We prove some results concerning the WORTH property and the García-Falset coefficient of absolute sums of infinitely many Banach spaces. Also, the Opial property/uniform Opial property of infinite ℓ^p-sums is studied, and some properties analogous to the Opial property/uniform Opial property are discussed for Lebesgue–Bochner spaces $L^p(\mu, X)$.

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1. Introduction

For a real Banach space X, X^* denotes its dual space, B_X its closed unit ball, and S_X its unit sphere.

We begin by recalling the important notion of the fixed point property: the space X is said to have the fixed point property (resp. weak fixed point property) if for every closed and bounded (resp. weakly compact) convex subset $C \subseteq X$, every nonexpansive mapping $F: C \to C$ has a fixed point (where F is called nonexpansive if $\|F(x) - F(y)\| \leq \|x - y\|$ for all $x, y \in C$, or, in other words, if F is 1-Lipschitz continuous).

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A closed, bounded, convex subset $C \subseteq X$ is said to have normal structure if, for each subset $B \subseteq C$ which contains at least two elements, there exists a point $x \in B$ such that

$$\sup_{y \in B} \|x - y\| < \operatorname{diam} B,$$

where $\operatorname{diam} B$ denotes the diameter of B. The space X itself is said to have normal structure, if every closed, bounded, convex subset of X has normal structure. It is well known that if C is weakly compact and has normal structure, then every nonexpansive mapping $F\colon C \to C$ has a fixed point (see, for example, [13, Theorem 2.1]), so spaces with normal structure have the weak fixed point property.

By δ_X we denote the modulus of convexity of X, i. e., for $0 < \epsilon \leqslant 2$,

$$\delta_X(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X \text{ with } \|x - y\| \geqslant \epsilon\right\}.$$

X is called uniformly rotund iff $\delta_X(\epsilon) > 0$ for each $0 < \epsilon \leqslant 2$. It is well known that all spaces $L^p(\mu)$, for any measure μ and any $1 < p < \infty$ (in particular the spaces $\ell^p(I)$ for any index set I), are uniformly rotund.

Every uniformly rotund space has normal structure (see [2, Theorem 4.1]), and hence (since uniformly rotund spaces are also reflexive) enjoys the fixed point property. An example of a Banach space which fails to possess the weak fixed point property is $L^1[0,1]$ (see [1]).

The space X is said to have the Opial property provided that

$$\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\|$$

for every weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X and every $x \in X \setminus \{0\}$ (one could as well use lim inf instead of lim sup, or from the beginning assume that both limits exist). This property was first considered by Opial [24] (starting from Hilbert spaces as the canonical example) to prove a result on iterative approximations of fixed points of nonexpansive mappings. In [24] it was shown that the spaces ℓ^p, for $1 \leqslant p < \infty$, enjoy the Opial property, while the spaces $L^p[0,1]$, for $1 < p < \infty, p \neq 2$, fail to have it. Note that every Banach space with the Schur property (i. e., weak and norm convergence of sequences coincide) trivially possesses the Opial property. Moreover, X is said to have the nonstrict Opial property if it satisfies the definition of the Opial property with "$\leqslant$" instead of "$<$" ([30]; in [10], it is called the weak Opial property). It is known that every weakly compact convex subset of a Banach space with the Opial property has normal structure (see, for instance, [26, Theorem 5.4]), and hence the Opial property implies the weak fixed point property.

Prus introduced the notion of the uniform Opial property [25]: a Banach space X has the uniform Opial property if for every $c > 0$ there is some $r > 0$ such that

$$1 + r \leqslant \liminf_{n\to\infty} \|x_n - x\|$$

holds for every $x \in X$ with $\|x\| \geqslant c$ and every weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\liminf \|x_n\| \geqslant 1$. In [25] it was proved that a Banach space is reflexive and has the uniform Opial property if and only if it has the so called (L) property (see [25] for the definition), and that X has the fixed point property whenever X^* enjoys the said (L) property.

A modulus corresponding to the uniform Opial property was defined in [21]:

$$r_X(c) := \inf\Big\{\liminf_{n\to\infty} \|x_n - x\| - 1\Big\} \quad \forall c > 0,$$

where the infimum is taken over all $x \in X$ with $\|x\| \geqslant c$ and all weakly null sequences $(x_n)_{n\in\mathbb{N}}$ in X with $\liminf \|x_n\| \geqslant 1$ (if X has the Schur property we agree to set $r_X(c) := 1$ for all $c > 0$). Consequently, X has the uniform Opial property iff $r_X(c) > 0$ for every $c > 0$.

In this paper, we will mostly use the following equivalent formulation of the uniform Opial property ([17, Definition 3.1]): X has the uniform Opial property iff for every $\epsilon > 0$ and every $R > 0$ there is some $\eta > 0$ such that

$$\eta + \liminf_{n\to\infty} \|x_n\| \leqslant \liminf_{n\to\infty} \|x_n - x\|$$

holds for all $x \in X$ with $\|x\| \geqslant \epsilon$ and every weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\limsup \|x_n\| \leqslant R$.

We can also associate a modulus to this formulation in the following way:

$$\eta_X(\epsilon, R) := \inf\Big\{\liminf_{n\to\infty} \|x_n - x\| - \liminf_{n\to\infty} \|x_n\|\Big\} \quad \forall \epsilon, R > 0,$$

where the infimum is taken over all $x \in X$ with $\|x\| \geqslant \epsilon$ and all weakly null sequences $(x_n)_{n\in\mathbb{N}}$ in X with $\limsup \|x_n\| \leqslant R$. So X has the uniform Opial property iff $\eta_X(\epsilon, R) > 0$ for all $\epsilon, R > 0$. Actually, it is enough that for every $\epsilon > 0$ there exist some $R > 2$ with $\eta_X(\epsilon, R) > 0$. More precisely, we have the following connection between the moduli r_X and η_X:

1.1. Lemma. *Let X be a Banach space which does not possess the Schur property.*

(i) *For every $c > 0$ and every $R > 2$ we have*

$$\min\Big\{\eta_X(c, R), \frac{R}{2} - 1\Big\} \leqslant r_X(c).$$

(ii) *For all $\epsilon, R > 0$ with $r_X(\epsilon/R) > 0$, we have*

$$\frac{\epsilon r_X(\epsilon/R)}{2 + r_X(\epsilon/R)} = \max_{\beta\in[0,\epsilon/2]} \min\Big\{\beta r_X\Big(\frac{\epsilon}{R}\Big), \epsilon - 2\beta\Big\} \leqslant \eta_X(\epsilon, R).$$

Proof. (i) Let $c > 0$ and $R > 2$. Put $\tau := \min\big\{\eta_X(c, R), \frac{R}{2} - 1\big\}$. Let $(x_n)_{n\in\mathbb{N}}$ be any weakly null sequence in X with $\liminf \|x_n\| \geqslant 1$ and let $x \in X$ with $\|x\| \geqslant c$. By passing to a subsequence we may assume that $\lim_{n\to\infty} \|x_n - x\|$ and $s := \lim_{n\to\infty} \|x_n\|$ exist. If $s \leqslant R$ then $1 + \tau \leqslant s + \eta_X(c, R) \leqslant \lim_{n\to\infty} \|x_n - x\|$.

If $s > R$ and $\|x\| > R/2$, then $\lim_{n\to\infty}\|x_n - x\| \geqslant \|x\| > R/2 \geqslant 1+\tau$ by the weak lower semicontinuity of the norm. Finally, if $s > R$ and $\|x\| \leqslant R/2$, then $\lim_{n\to\infty}\|x_n - x\| \geqslant s - \|x\| > R/2 \geqslant 1+\tau$.

(ii) The first equality is easily verified. Now choose any $\beta \in (0, \epsilon/2)$ and put $v := \min\left\{\beta r_X(\frac{\epsilon}{R}), \epsilon - 2\beta\right\}$. Let $(x_n)_{n\in\mathbb{N}}$ be a weakly null sequence in X with $\limsup\|x_n\| \leqslant R$ and let $x \in X$ with $\|x\| \geqslant \epsilon$. Again we may assume that $\lim_{n\to\infty}\|x_n - x\|$ and $s := \lim_{n\to\infty}\|x_n\|$ exist. By the definition of r_X we get $s(1+r_X(\epsilon/R)) \leqslant \lim_{n\to\infty}\|x_n - x\|$, which implies $s+v \leqslant \lim_{n\to\infty}\|x_n - x\|$ if $s > \beta$. But if $s \leqslant \beta$ then $\lim_{n\to\infty}\|x_n - x\| \geqslant \|x\| - s \geqslant \epsilon - \beta \geqslant v + \beta \geqslant v + s$, and the proof is finished. □

In [10], J. García-Falset introduced the following coefficient of a Banach space X:

$$R(X) := \sup\left\{\liminf_{n\to\infty}\|x_n + x\| : x \in B_X, (x_n)_{n\in\mathbb{N}} \in \mathrm{WN}(B_X)\right\},$$

where by $\mathrm{WN}(B_X)$ we denote the set of all weakly null sequences in B_X. Obviously, $1 \leqslant R(X) \leqslant 2$ and $R(X) = 1$ if X has the Schur property (in particular if X is finite-dimensional or $X = \ell^1$). One has $R(c_0) = 1$ and $R(\ell^p) = 2^{1/p}$ for $1 < p < \infty$ (see [10, Corollary 3.2]). In [11, Theorem 3] it was proved that the condition $R(X) < 2$ implies that X has the weak fixed point property. The reflexive spaces with $R(X) < 2$ are precisely the so called weakly nearly uniformly smooth spaces ([10, Corollary 4.4]) which were introduced in [18] and include in particular all uniformly smooth spaces.

In [29] Sims introduced the notion of the WORTH (weak orthogonality) property: X is said to have the WORTH property provided that for all weakly null sequences $(x_n)_{n\in\mathbb{N}}$ in X and every $x \in X$ one has $\|x_n + x\| - \|x_n - x\| \to 0$.

Again spaces with the Schur property obviously enjoy the WORTH property. Hilbert spaces are easily seen to have the WORTH property as well. Also, the class of spaces with the WORTH property includes all so called weakly orthogonal Banach lattices (the notion introduced earlier by Borwein and Sims in [4]), which in turn include in particular all spaces $\ell^p(I)$ for $1 \leqslant p < \infty$, and $c_0(I)$. However, the spaces $L^p[0,1]$ with $1 \leqslant p \leqslant \infty, p \neq 2$, do not have the WORTH property (see the remark at the end of [30]). In [29], it was proved that the WORTH property implies the nonstrict Opial property, and in [30] it was shown that a space with the WORTH property, which is ϵ-inquadrate in every direction for some $0 < \epsilon < 2$ (see [30] for the definition), has the weak fixed point property (even more, every weakly compact convex subset of such a space has normal structure). By [10, Proposition 3.6], a uniformly non-square Banach space X with the WORTH property satisfies $R(X) < 2$. Recall that X is said to be uniformly non-square if there is some $\delta > 0$ such that whenever $x, y \in B_X$ one has $\|x + y\| < 2(1-\delta)$ or $\|x - y\| < 2(1-\delta)$.

The degree $w(X)$ of WORTHness of X was also introduced in [30] as the supremum of all $r \geqslant 0$ such that

$$r \liminf_{n\to\infty}\|x_n + x\| \leqslant \liminf_{n\to\infty}\|x_n - x\|$$

holds for all $x \in X$ and all weakly null sequences $(x_n)_{n\in\mathbb{N}}$ in X. Then $1/3 \leqslant w(X) \leqslant 1$, and X has the WORTH property if and only if $w(X) = 1$.

In this paper, we study the WORTH property and the García-Falset coefficient of infinite absolute sums, as well as various Opial properties, specifically of infinite ℓ^p-sums of Banach spaces (for normal structure in finite or infinite direct sums of Banach spaces, see [7] and references therein; for more information on the fixed point property and normal structure in general, see [13]). Also some Opial-type results for Lebesgue–Bochner spaces will be obtained. The next section contains the necessary preliminaries on absolute sums.

2. Preliminaries on absolute sums

Throughout this paper, if not stated otherwise, we denote by I a (usually infinite) index set and E a subspace of the space of all real-valued functions on I containing all functions with finite support and endowed with an absolute, normalised norm $\|\cdot\|_E$. The latter means that $\|\cdot\|_E$ is a complete norm on E such that the following conditions are satisfied:

(i) If $(a_i)_{i\in I} \in E$ and $(b_i)_{i\in I} \in \mathbb{R}^I$ such that $|a_i| = |b_i|$ for all $i \in I$, then $(b_i)_{i\in I} \in E$ and $\|(b_i)_{i\in I}\|_E = \|(a_i)_{i\in I}\|_E$.

(ii) $\|e_i\|_E = 1$ for all $i \in I$, where $e_i = (e_{ij})_{j\in I}$ with $e_{ij} = 0$ for $j \neq i$ and $e_{ii} = 1$.

It is important to note that such norms are automatically monotone, i. e., we actually have

$$(a_i)_{i\in I} \in E, (b_i)_{i\in I} \in \mathbb{R}^I \text{ with } |b_i| \leqslant |a_i| \;\forall i \in I \Rightarrow (b_i)_{i\in I} \in E \text{ and } \|(b_i)_{i\in I}\|_E \leqslant \|(a_i)_{i\in I}\|_E.$$

For a proof, see, for instance, [20, Remark 2.1]. The standard examples of spaces with absolute, normalised norm are of course the spaces $\ell^p(I)$ (with $1 \leqslant p \leqslant \infty$) and $c_0(I)$.

If we put

$$E' := \Big\{(a_i)_{i\in I} \in \mathbb{R}^I : \|(a_i)_{i\in I}\|_{E'} := \sup_{(b_i)_{i\in I}\in B_E} \sum_{i\in I} |a_i b_i| < \infty\Big\},$$

then $(E', \|\cdot\|_{E'})$ is again a space with absolute, normalised norm and the map $T\colon E' \to E^*$ defined by

$$T((a_i)_{i\in I})((b_i)_{i\in I}) := \sum_{i\in I} a_i b_i \quad \forall (a_i)_{i\in I} \in E',\; \forall (b_i)_{i\in I} \in E$$

is an isometric embedding. T is onto if $\operatorname{span}\{e_i : i \in I\}$ is dense in E, so in this case $E^* = E'$.

Now, given a family $(X_i)_{i\in I}$ of Banach spaces, the absolute sum of $(X_i)_{i\in I}$ with respect to E is defined as the space

$$\Big[\bigoplus_{i\in I} X_i\Big]_E := \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i : (\|x_i\|)_{i\in I} \in E\Big\}$$

and is endowed with the norm $\|(x_i)_{i\in I}\|_E := \|(\|x_i\|)_{i\in I}\|_E$. It is not hard to see that this sum is indeed a Banach space. For $E = \ell^p(I)$ one obtains the usual p-sum.

As for the dual space of an absolute sum, the map

$$S\colon \Big[\bigoplus_{i\in I} X_i^*\Big]_{E'} \to \Big[\bigoplus_{i\in I} X_i\Big]_E^*$$
$$S\big((x_i^*)_{i\in I}\big)\big((x_i)_{i\in I}\big) := \sum_{i\in I} x_i^*(x_i)$$

is an isometric embedding, and it is onto if $\operatorname{span}\{e_i : i \in I\}$ is dense in E.

3. Results on absolute sums

WORTH property of absolute sums. By [15, Theorem 4.7], $w(X \oplus_E Y) = \min\{w(X), w(Y)\}$ holds for all Banach spaces X and Y and every absolute, normalised norm $\|\cdot\|_E$ on $\mathbb{R}^2$ (actually, in [15] the notion of ψ-direct sums is used, but this is an equivalent formulation; see Section 2 in [15]). In particular, $X \oplus_E Y$ has the WORTH property if and only if X and Y have the WORTH property (for this, see also [15, Theorem 4.2]). It is possible to generalise [15, Theorem 4.7] to sums of arbitrarily many Banach spaces under a mild condition on E:

3.1. Proposition. *If* $\operatorname{span}\{e_i : i \in I\}$ *is dense in* E *and* $(X_i)_{i\in I}$ *is any family of Banach spaces, then*

$$w\left(\Big[\bigoplus_{i\in I} X_i\Big]_E\right) = \inf\{w(X_i) : i \in I\}.$$

In particular, $[\bigoplus_{i\in I} X_i]_E$ *has the WORTH property if and only if* X_i *has the WORTH property for every* $i \in I$.

Proof. Let us write $X = [\bigoplus_{i\in I} X_i]_E$ and $s = \inf\{w(X_i) : i \in I\}$. We clearly have $w(X) \leqslant s$. Now let $x_n = (x_{n,i})_{i\in I} \in X$ for every $n \in \mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}}$ converges weakly to zero and let $x = (x_i)_{i\in I} \in X$. Without loss of generality we may assume that the limits $a := \lim_{n\to\infty}\|x_n + x\|_E$ and $b := \lim_{n\to\infty}\|x_n - x\|_E$ exist.

Since $\operatorname{span}\{e_i : i \in I\}$ is dense in E, it is not hard to see that we actually have $(\|x_i\|)_{i\in I} = \sum_{i\in I}\|x_i\|e_i$. So if $\epsilon > 0$ is given, we find a finite set $J \subseteq I$ such that

$$\Big\|\sum_{i\in I\setminus J}\|x_i\|e_i\Big\|_E = \Big\|(\|x_i\|)_{i\in I} - \sum_{i\in J}\|x_i\|e_i\Big\|_E \leqslant \epsilon. \tag{1}$$

By passing to an appropriate subsequence we may assume that the limits $a_i := \lim_{n\to\infty}\|x_{n,i} + x_i\|$ and $b_i := \lim_{n\to\infty}\|x_{n,i} - x_i\|$ exist for each $i \in J$. Since $(x_{n,i})_{n\in\mathbb{N}}$ is weakly convergent to zero in X_i for every $i \in I$ it follows that $sa_i \leqslant b_i \leqslant s^{-1}a_i$ and consequently

$$|a_i - b_i| \leqslant \frac{1-s}{s}b_i \quad \forall i \in J. \tag{2}$$

For every $n \in \mathbb{N}$ we have, because of (1),

$$\begin{aligned}\big|\|x_n + x\|_E - \|x_n - x\|_E\big| &\leqslant \big\|(\|x_{n,i} + x_i\| - \|x_{n,i} - x_i\|)_{i\in I}\big\|_E \\ &\leqslant \Big\|\sum_{i\in J}(\|x_{n,i} + x_i\| - \|x_{n,i} - x_i\|)e_i\Big\|_E + 2\Big\|\sum_{i\in I\setminus J}\|x_i\|e_i\Big\|_E \\ &\leqslant \Big\|\sum_{i\in J}(\|x_{n,i} + x_i\| - \|x_{n,i} - x_i\|)e_i\Big\|_E + 2\epsilon.\end{aligned}$$

So, for $n \to \infty$, we obtain

$$|a - b| \leqslant \Big\|\sum_{i\in J}(a_i - b_i)e_i\Big\|_E + 2\epsilon.$$

Taking (2) into account we arrive at

$$|a - b| \leqslant \frac{1-s}{s}\Big\|\sum_{i\in J} b_i e_i\Big\|_E + 2\epsilon.$$

But $\|\sum_{i\in J}\|x_{n,i} - x_i\|e_i\|_E \leqslant \|x_n - x\|_E$ for each n, thus $\|\sum_{i\in J} b_i e_i\|_E \leqslant b$ and hence

$$|a - b| \leqslant \frac{1-s}{s}b + 2\epsilon.$$

Letting $\epsilon \to 0$ leaves us with $|a - b| \leqslant (1 - s)b/s$ which implies $sa \leqslant b$, and we are done. □

García-Falset coefficient of absolute sums. In [6, Theorem 7] it was proved that $R((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_E) < 2$ whenever $R(X_i) < 2$ for $i = 1, \ldots, n$ and $\|\cdot\|_E$ is any strictly convex, absolute, normalised norm on $\mathbb{R}^n$. For absolute sums of two Banach spaces a stronger result was obtained in [15, Theorem 3.6] (in the equivalent formulation of ψ-direct sums): $R(X \oplus_E Y) < 2$ provided that $R(X), R(Y) < 2$ and $\|\cdot\|_E$ is any absolute, normalised norm on $\mathbb{R}^2$ with $\|\cdot\|_E \neq \|\cdot\|_1$. A complete characterisation of ψ-direct sums of finitely many spaces having García-Falset coefficient less than 2 is given in [16, Theorem 6.2].

For infinite sums we have the following partial result (for $J \subseteq I$ we denote by $\big[\bigoplus_{i\in J} X_i\big]_E$ the sum of the family whose i-th member is X_i for $i \in J$ and $\{0\}$ for $i \in I \setminus J$).

3.2. Theorem. *If I is an infinite index set, E a subspace of $\mathbb{R}^I$ with absolute, normalised norm such that* span$\{e_i : i \in I\}$ *is dense in E, and $(X_i)_{i\in I}$ is a family of Banach spaces with*

$$\alpha := \sup\Big\{R\Big(\Big[\bigoplus_{i\in J} X_i\Big]_E\Big) : J \subseteq I \text{ finite}\Big\} < 2$$

and $\delta_E((1 - \alpha/2)^2) > 0$, then $R\big(\big[\bigoplus_{i\in I} X_i\big]_E\big) < 2$.

Proof. Let us write $X = \big[\bigoplus_{i\in I} X_i\big]_E$ for short. It is well known that δ_E is continuous on $(0, 2)$ (see, for example, [12, Lemma 5.1]), so we can find $0 < \tau < (1 - \alpha/2)^2$ with $\delta_E(\tau) > 0$. Let $\gamma := \sqrt{\tau}$ and choose $0 < \eta < \min\{\delta_E(\tau), 1/2 - \gamma\}$.

Suppose that $R(X) = 2$. Then there would exist a weakly null sequence $(x_n)_{n\in\mathbb{N}} = ((x_{n,i})_{i\in I})_{n\in\mathbb{N}}$ in B_X and an element $x = (x_i)_{i\in I} \in B_X$ such that $\lim_{n\to\infty}\|x_n + x\|_E > 2 - \eta$. We may assume that $\|x_n + x\|_E > 2 - 2\eta$ for all $n \in \mathbb{N}$. Since $\|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E \geq \|x_n + x\|_E$ and $\eta < \delta_E(\tau)$ it follows that

$$\big\|(|\|x_{n,i}\| - \|x_i\||)_{i\in I}\big\|_E < \tau \quad \forall n \in \mathbb{N}. \tag{3}$$

Similarly,

$$4(1-\eta) < 2\|x_n + x\|_E \leq \big\|(\|x_{n,i}\| + \|x_i\| + \|x_{n,i} + x_i\|)_{i\in I}\big\|_E \leq 4,$$

and hence

$$\big\|(\|x_{n,i}\| + \|x_i\| - \|x_{n,i} + x_i\|)_{i\in I}\big\|_E < 2\tau \quad \forall n \in \mathbb{N}. \tag{4}$$

We further have $\|x\|_E \geq \|x_n + x\|_E - 1 > 1 - 2\eta > 2\gamma$. Since $(\|x_i\|)_{i\in I} = \sum_{i\in I}\|x_i\|e_i$ we can find a finite set $J \subseteq I$ such that

$$\Big\|\sum_{i\in J}\|x_i\|e_i\Big\|_E > 2\gamma. \tag{5}$$

Put $y := (x_i)_{i\in J}$, $y_n = (x_{n,i})_{i\in J} \in [\bigoplus_{i\in J} X_i]_E$ as well as $a := \sum_{i\in J}\|x_i\|e_i$ and $a_n := \sum_{i\in J}\|x_{n,i}\|e_i$. By (3) we have

$$\|a_n - a\|_E \leq \big\|(|\|x_{n,i}\| - \|x_i\||)_{i\in I}\big\|_E < \tau \quad \forall n \in \mathbb{N}, \tag{6}$$

which implies in particular that $\big|\|y\|_E - \|y_n\|_E\big| = |\|a\|_E - \|a_n\|_E| < \tau$, hence $\|y_n\|_E > \|y\|_E - \tau > 2\gamma - \tau > 0$ by (5).

Furthermore, for every $n \in \mathbb{N}$,

$$\big|\|a_n + a\|_E - \|y_n + y\|_E\big| \leq \Big\|\sum_{i\in J}(\|x_{n,i}\| + \|x_i\| - \|x_{n,i} + x_i\|)e_i\Big\|_E,$$

so, because of (4), it follows that

$$\big|\|a_n + a\|_E - \|y_n + y\|_E\big| < 2\tau \quad \forall n \in \mathbb{N}.$$

Also, by (6), we have $|\|a_n + a\|_E - 2\|y\|_E| = |\|a_n + a\|_E - 2\|a\|_E| \leq \|a_n - a\| < \tau$ for each n. Consequently,

$$\big|\|y_n + y\|_E - 2\|y\|_E\big| < 3\tau \quad \forall n \in \mathbb{N}.$$

Since $\|y_n/\|y_n\|_E - y_n/\|y\|_E\|_E = |1 - \|y_n\|_E/\|y\|_E| < \tau/\|y\|_E$ we get

$$\left|2 - \left\|\frac{y}{\|y\|_E} + \frac{y_n}{\|y_n\|_E}\right\|_E\right| < \frac{4\tau}{\|y\|_E} < \frac{2\tau}{\gamma} \quad \forall n \in \mathbb{N}, \tag{7}$$

where the last inequality holds because of (5). Note that $(x_{n,i})_{n\in\mathbb{N}}$ converges weakly to zero in X_i for each $i \in I$ and thus, by the representation of the dual of $[\bigoplus_{i\in J} X_i]_E$ as $[\bigoplus_{i\in J} X_i^*]_{E'}$ and finiteness of J, the sequence $(y_n/\|y_n\|_E)_{n\in\mathbb{N}}$ is also a weakly null sequence (as noted above, $(\|y_n\|)_{n\in\mathbb{N}}$ is bounded away from zero).

So, from (7) and the definition of α it follows that $\alpha \geq 2(1 - \tau/\gamma)$. But $\gamma = \sqrt{\tau}$ and $\tau < (1 - \alpha/2)^2$, thus $2(1 - \tau/\gamma) > \alpha$ and with this contradiction the proof is complete. □

The above theorem reduces the case of infinite sums to the one of finite sums. The condition $\alpha < 2$ is clearly necessary for $R(X) < 2$. Unfortunately, the author does not know whether the simpler condition $\beta := \sup_{i\in I} R(X_i) < 2$ would already be enough to ensure that $\alpha < 2$. The proofs of [15, Theorem 3.6] and [16, Theorem 6.2] do not give quantitative bounds for the García-Falset coefficient of finite sums. The proof of [6, Theorem 7] shows that when $\beta < 2$, for every finite subset $J \subseteq I$ with $|J| = N$ one has that $R([\bigoplus_{i\in J} X_i]_E) \leqslant 2 - \delta$, where first $\epsilon > 0$ is chosen so that $\beta(1+N\epsilon) < 2$ and then $0 < \delta < \min\{2\delta_E(\epsilon), 2 - \beta(1 + N\epsilon)\}$, hence it might still be that $R([\bigoplus_{i\in J} X_i]_E)$ tends to 2 for $N \to \infty$.

Next we will discuss some applications of Theorem 3.2. First, since the Schur property is inherited by finite sums, we get the following corollary.

3.3. Corollary. *If $(X_i)_{i\in I}$ is a family of Banach spaces with the Schur property (in particular, a family of finite-dimensional Banach spaces) and* $\operatorname{span}\{e_i : i \in I\}$ *is dense in E with $\delta_E(1/4) > 0$, then $R([\bigoplus_{i\in I} X_i]_E) < 2$. In particular, $R\Big([\bigoplus_{i\in I} X_i]_p\Big) < 2$ for all $1 < p < \infty$.*

For another application of Theorem 3.2 consider the following example.

3.4. Example. If $N \geqslant 2$ and $I_1, \ldots, I_N$ are non-empty sets at least one of which is infinite, then

$$R\Big(\Big[\bigoplus_{k=1}^{N} c_0(I_k)\Big]_p\Big) = 2^{1/p} \tag{8}$$

for every $1 \leqslant p < \infty$. Consequently, by Theorem 3.2, if $(I_k)_{k\in I}$ is any family of non-empty sets we have

$$R\Big(\Big[\bigoplus_{k\in I} c_0(I_k)\Big]_p\Big) < 2 \quad \text{for } 1 < p < \infty.$$

Proof. To prove (8) put $X := \big[\bigoplus_{k=1}^{N} c_0(I_k)\big]_p$ and suppose without loss of generality that I_1 is infinite. Fix a sequence $(i_n)_{n\in\mathbb{N}}$ of distinct elements of I_1 and $j \in I_2$, and put $x_n := (e_{i_n}, 0, \ldots, 0) \in S_X$ as well as $x := (0, e_j, 0, \ldots, 0) \in S_X$. Then $x_n \to 0$ weakly in X and $\|x_n + x\|_p = 2^{1/p}$ for each n, thus $2^{1/p} \leqslant R(X)$.

To prove the reverse inequality, for each $n \in \mathbb{N}$ let $x_n = (x_{n,1}, \ldots, x_{n,N}) \in B_X$ be such that $x_n \to 0$ weakly and let $x = (x_1, \ldots, x_N) \in B_X$. Without loss of generality we may suppose that $\lim_{n\to\infty} \|x_n + x\|_p$ and also $a_k := \lim_{n\to\infty} \|x_{n,k}\|_\infty$, for each $k \in \{1, \ldots, N\}$, exist.

Take an arbitrary $\epsilon > 0$. Then for each $k \in \{1, \ldots, N\}$ the set $J_k := \{i \in I_k : |x_k(i)| > \epsilon\}$ is finite. Since $x_n \to 0$ weakly we have $x_{n,k}(i) \to 0$ for all $k \in \{1, \ldots, N\}$ and all $i \in I_k$. It follows that there exists $n_0 \in \mathbb{N}$ such that $|x_{n,k}(i)| \leqslant \epsilon$ for all $k \in \{1, \ldots, N\}$, all $i \in J_k$ and all $n \geqslant n_0$.

But then $|x_{n,k}(i) + x_k(i)| \leqslant |x_{n,k}(i)| + |x_k(i)| \leqslant \max\{|x_{n,k}(i)|, |x_k(i)|\} + \epsilon$ for all $k \in \{1, \ldots, N\}$, all $i \in I_k$ and all $n \geqslant n_0$. From this we conclude that

$$\|x_n + x\|_p^p = \sum_{k=1}^{N} \|x_{n,k} + x_k\|_\infty^p \leqslant \sum_{k=1}^{N} \big(\max\{\|x_{n,k}\|_\infty, \|x_k\|_\infty\} + \epsilon\big)^p \quad \forall n \geqslant n_0.$$

For $n \to \infty$ it follows that

$$\lim_{n\to\infty} \|x_n + x\|_p^p \leqslant \sum_{k=1}^{N} (\max\{a_k, \|x_k\|_\infty\} + \epsilon)^p.$$

Letting $\epsilon \to 0$ we obtain

$$\lim_{n\to\infty} \|x_n + x\|_p^p \leqslant \sum_{k=1}^{N} \max\{a_k^p, \|x_k\|_\infty^p\} \leqslant \sum_{k=1}^{N} (a_k^p + \|x_k\|_\infty^p) = \lim_{n\to\infty} \|x_n\|_p^p + \|x\|_p^p \leqslant 2.$$

Hence $\lim_{n\to\infty} \|x_n + x\|_p \leqslant 2^{1/p}$, and we are done. □

A Banach space X is said to be a U-space if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X and every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} the conditions $\|x_n + y_n\| \to 2$ and $x_n^*(x_n) = 1$ for each $n \in \mathbb{N}$ imply $x_n^*(y_n) \to 1$. U-spaces were introduced by Lau in [19]. Uniformly rotund and uniformly smooth spaces are examples of U-spaces. Every U-space has normal structure ([8, Theorem 3.2]). Gao [9] defined the modulus of u-convexity of X by

$$u_X(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X \ \exists x^* \in S_{X^*} \ x^*(x) = 1, x^*(y) \leqslant 1 - \epsilon\right\},$$

for $0 < \epsilon \leqslant 2$. Then X is a U-space if and only if $u_X(\epsilon) > 0$ for all $0 < \epsilon \leqslant 2$. Obviously, $u_X \geqslant \delta_X$. It is proved in [9, Theorem 3] that even the condition $u_X(\epsilon) > 0$ for some $\epsilon \in \left(0, \frac{1}{2}\right)$ is enough to ensure that X has normal structure. In [27, Proposition 3.3] an even stronger result was obtained: if $u_X(1) > 0$, then X and X^* have normal structure.

By [23, Theorem 5], $R(X) < 2$ if $u_X(\epsilon) > 0$ for some $0 < \epsilon < 1$.

Putting several results together, it is now possible to obtain the following corollary.

3.5. Corollary. *Let $(X_i)_{i\in I}$ be a family of Banach spaces and $1 < p < \infty$. Suppose that there exist four pairwise disjoint (possibly empty) subsets $I_1, I_2, I_3, I_4 \subseteq I$ such that*

(i) *X_i has the Schur property for each $i \in I_1$,*
(ii) *$\inf_{i\in I_2} u_{X_i}(\epsilon) > 0$ for all $0 < \epsilon \leqslant 2$,*
(iii) *for each $i \in I_3$ there is a set J_i with $X_i = c_0(J_i)$.*
(iv) *I_4 is finite and $R(X_i) < 2$ for all $i \in I_4$.*

Then $R([\bigoplus_{i\in I} X_i]_p) < 2$.

Proof. Let us put $X := [\bigoplus_{i\in I} X_i]_p$ and $X_k := [\bigoplus_{i\in I_k} X_i]_p$ for $k = 1, 2, 3, 4$ (or $X_k = \{0\}$ if $I_k = \emptyset$). By Corollary 3.3 we have $R(X_1) < 2$ and by Example 3.4 we have $R(X_3) < 2$. Also, by [14, Corollary 3.17] and the remarks after [14, Definition 1.5], X_2 is again a U-space, so $R(X_2) < 2$. From the aforementioned result [6, Theorem 7] it follows that $R(X_4) < 2$ and, since $X \cong X_1 \oplus_p X_2 \oplus_p X_3 \oplus_p X_4$, [6, Theorem 7] implies that $R(X) < 2$. □

The case of c_0-sums is not covered by the above results. However, it is easy to prove the following proposition directly.

3.6. Proposition. *Let $(X_i)_{i\in I}$ be any family of Banach spaces and $X := \left[\bigoplus_{i\in I} X_i\right]_{c_0(I)}$. Then*

$$R(X) = \sup_{i\in I} R(X_i).$$

Proof. We clearly have $\alpha := \sup_{i\in I} R(X_i) \leqslant R(X)$. To prove the reverse inequality, fix a weakly null sequence $(x_n)_{n\in\mathbb{N}} = ((x_{n,i})_{i\in I})_{n\in\mathbb{N}}$ in B_X and $x = (x_i)_{i\in I} \in B_X$. Without loss of generality we may assume that $\lim_{n\to\infty}\|x_n + x\|_\infty$ exists.

Let $\epsilon > 0$ be arbitrary. Then $J := \{i \in I : \|x_i\| \geqslant \epsilon\}$ is finite, so by passing to an appropriate subsequence once more, we may also assume that $\lim_{n\to\infty}\|x_{n,i} + x_i\|$ exists for all $i \in J$.

Since $x_{n,i} \to 0$ weakly for all $i \in I$ it follows that $\lim_{n\to\infty}\|x_{n,i} + x_i\| \leqslant R(X_i) \leqslant \alpha$ for all $i \in J$, so $\|x_{n,i} + x_i\| \leqslant \alpha + \epsilon$ for all $i \in J$ and all sufficiently large n. But for $i \in I \setminus J$ we have $\|x_{n,i} + x_i\| \leqslant \|x_{n,i}\| + \|x_i\| \leqslant 1 + \epsilon \leqslant \alpha + \epsilon$. Consequently, $\|x_n + x\|_\infty \leqslant \alpha + \epsilon$ for all sufficiently large n, hence $\lim_{n\to\infty}\|x_n + x\|_\infty \leqslant \alpha + \epsilon$. Since $\epsilon > 0$ was arbitrary, we are done. □

Concerning ℓ^1-sums it was already proved in [15, Theorem 3.13] that $R(X \oplus_1 Y) < 2$ if and only if both X and Y have the Schur property (in [16, Proposition 6.7] this is generalised to finite ℓ^1-sums together with various other equivalent conditions). The proof of the "only if" part from [15, Theorem 3.13] directly generalises to sums of infinitely many spaces and, since it was proved in [31] that the ℓ^1-sum of any family of Banach spaces has the Schur property if and only if each summand has the Schur property, we obtain the following characterisation.

3.7. Proposition. *Let I be any index set with at least two elements. Let $(X_i)_{i\in I}$ be a family of Banach spaces and $X := \left[\bigoplus_{i\in I} X_i\right]_1$. Then the following assertions are equivalent*:

(i) $R(X) < 2$.
(ii) *X_i has the Schur property for each $i \in I$.*
(iii) *X has the Schur property.*
(iv) $R(X) = 1$.

Opial properties of finite absolute sums. In this subsection we briefly consider the Opial properties of finite sums. They are surely well known, but we will include the results and some of the proofs here as the author was not able to find them explicitly in the literature.

Recall that an absolute, normalised norm $\|\cdot\|_E$ on $\mathbb{R}^m$ is said to be strictly monotone if for all $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ we have

$$\|a\|_E = \|b\|_E \quad \text{and} \quad |a_i| \leqslant |b_i| \quad \forall i = 1, \dots, m \;\Rightarrow\; |a_i| = |b_i| \quad \forall i = 1, \dots, m.$$

It is easy to see that strictly convex, absolute, normalised norms are strictly monotone.

3.8. Proposition. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^m$ and $X_1, \dots, X_m$ Banach spaces with the nonstrict Opial property. Then $\left[\bigoplus_{i=1}^m X_i\right]_E$ has the nonstrict Opial property.*

If, in addition, $\|\cdot\|_E$ is strictly monotone and each X_i has the Opial property, then $\left[\bigoplus_{i=1}^m X_i\right]_E$ also has the Opial property.

The proof is straightforward and will be omitted.

As is well known, every strictly monotone, absolute, normalised norm on $\mathbb{R}^m$ is actually uniformly monotone in the following sense (the proof consists in an easy compactness argument).

3.9. Lemma. *Let $\|\cdot\|_E$ be a strictly monotone, absolute, normalised norm on $\mathbb{R}^m$. Let $\epsilon, R > 0$. Then there exists $\delta > 0$ such that for all $a = (a_1, \dots, a_m), b = (b_1, \dots, b_m) \in \mathbb{R}^m$ with $\|b\|_E \leqslant R$ and $|a_i| \leqslant |b_i|$ for $i = 1 \dots, m$, we have*

$$\|b\|_E - \|a\|_E < \delta \;\Rightarrow\; |b_i| - |a_i| < \epsilon \;\forall i = 1, \dots, m.$$

Utilizing this fact, we can obtain the following

3.10. Proposition. *Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^m$ which is strictly monotone and let $X_1, \dots, X_m$ be Banach spaces with the uniform Opial property. Then $X := \left[\bigoplus_{i=1}^m X_i\right]_E$ also has the uniform Opial property.*

Proof. Let $\epsilon, R > 0$ and put $\eta := \min\{\eta_{X_i}(\epsilon/m, R) : i = 1, \dots, m\}$. Choose $0 < \delta \leqslant 1$, according to Lemma 3.9, corresponding to the values η and $3R + 1$.

Now consider a weakly null sequence $(x_n)_{n\in\mathbb{N}} = \left((x_{n,1}, \dots, x_{n,m})\right)_{n\in\mathbb{N}}$ in X satisfying the condition $\limsup \|x_n\|_E \leqslant R$ and $y = (y_1, \dots, y_m) \in X$ with $\|y\|_E \geqslant \epsilon$. Since $\|y\|_E \leqslant \sum_{i=1}^m \|y_i\|$, there is some $i_0 \in \{1, \dots, m\}$ with $\|y_{i_0}\| \geqslant \epsilon/m$. There is no loss of generality in assuming that all the limits in the subsequent calculations exist. From the definition of η we get

$$\lim_{n\to\infty} \|x_{n,i_0}\| + \eta \leqslant \lim_{n\to\infty} \|x_{n,i_0} - y_{i_0}\|.$$

Since each X_i has in particular the nonstrict Opial property, we also have

$$\lim_{n\to\infty} \|x_{n,i}\| \leqslant \lim_{n\to\infty} \|x_{n,i} - y_i\| \quad \forall i \in \{1, \dots, m\} \setminus \{i_0\}.$$

If $\|y\|_E \leqslant 2R + 1$, then $\lim_{n\to\infty} \|x_n - y\|_E \leqslant \lim_{n\to\infty} \|x_n\|_E + 2R + 1 \leqslant 3R + 1$ and the choice of δ implies $\lim_{n\to\infty} \|x_n\|_E + \delta \leqslant \lim_{n\to\infty} \|x_n - y\|_E$.

If on the other hand $\|y\|_E > 2R+1$, then $\lim_{n\to\infty} \|x_n - y\|_E \geqslant \|y\|_E - \lim_{n\to\infty} \|x_n\|_E \geqslant R+1 \geqslant \lim_{n\to\infty} \|x_n\|_E + \delta$. So X has the uniform Opial property. □

Opial properties of some infinite sums. We will first show that the Opial and the nonstrict Opial property are preserved under infinite ℓ^p-sums.

3.11. Proposition. *If $1 \leqslant p < \infty$, I is any index set and $(X_i)_{i\in I}$ is a family of Banach spaces with the Opial property (nonstrict Opial property), then $X := \left[\bigoplus_{i\in I} X_i\right]_p$ also has the Opial property (nonstrict Opial property).*

Proof. We will only prove the strict case, the nonstrict case is treated analogously. For every $n \in \mathbb{N}$ let $x_n = (x_{n,i})_{i\in I} \in X$ be such that $(x_n)_{n\in\mathbb{N}}$ converges weakly to zero and let $x = (x_i)_{i\in I} \in X \setminus \{0\}$. Fix $i_0 \in I$ for which $x_{i_0} \neq 0$. We may assume that $\lim_{n\to\infty}\|x_n\|_p$ and $\lim_{n\to\infty}\|x_n - x\|_p$ as well as $a := \lim_{n\to\infty}\|x_{n,i_0}\|$ and $b := \lim_{n\to\infty}\|x_{n,i_0} - x_{i_0}\|$ exist. Note that $(x_{n,i})_{n\in\mathbb{N}}$ is a weakly null sequence in X_i for each $i \in I$. So, since X_{i_0} has the Opial property, it follows that $\delta := b^p - a^p > 0$. Put $K := \sup_{n\in\mathbb{N}}\|x_n\|_p$ and let $0 < \epsilon \leqslant 1$. We can find a finite set $J \subseteq I$ with $i_0 \in J$ such that

$$\left\|(\|x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p \leqslant \epsilon, \tag{9}$$

where $\chi_{I\setminus J}$ denotes the characteristic function of $I \setminus J$. By passing to a further subsequence, we may assume that $\lim_{n\to\infty}\|x_{n,i}\|$ and $\lim_{n\to\infty}\|x_{n,i} - x_i\|$ exist for all $i \in J$. Then, using the Opial property of each of the summands X_i, the definition of δ, and (9), we obtain

$$\begin{aligned}
\lim_{n\to\infty}\|x_n\|_p^p &= \sum_{i\in J\setminus\{i_0\}} \lim_{n\to\infty}\|x_{n,i}\|^p + a^p + \lim_{n\to\infty}\left\|(\|x_{n,i}\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p \\
&\leqslant \lim_{n\to\infty} \sum_{i\in J\setminus\{i_0\}} \|x_{n,i} - x_i\|^p + b^p - \delta + \lim_{n\to\infty}\left\|(\|x_{n,i}\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p \\
&\leqslant \lim_{n\to\infty} \sum_{i\in J}\|x_{n,i} - x_i\|^p - \delta + \lim_{n\to\infty}\left(\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p + \epsilon\right)^p.
\end{aligned}$$

But, since $\left|s^p - t^p\right| \leqslant pA^{p-1}|s - t|$ for all $0 \leqslant s, t \leqslant A$, we also have

$$\begin{aligned}
\lim_{n\to\infty}\left(\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p + \epsilon\right)^p &\leqslant \lim_{n\to\infty}\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p \\
&\quad + \lim_{n\to\infty}\Big|\left(\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p + \epsilon\right)^p \\
&\quad - \left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p\Big| \\
&\leqslant \lim_{n\to\infty}\left\|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p^p + p(K + \|x\|_p + 1)^{p-1}\epsilon.
\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|x_n\|_p^p \leqslant \lim_{n\to\infty}\|x_n - x\|_p^p - \delta + p(K + \|x\|_p + 1)^{p-1}\epsilon.$$

Since $\epsilon \in (0, 1]$ was arbitrary and δ independent of ϵ, we conclude that

$$\lim_{n\to\infty}\|x_n\|_p^p \leqslant \lim_{n\to\infty}\|x_n - x\|_p^p - \delta < \lim_{n\to\infty}\|x_n - x\|_p^p,$$

and the proof is complete. □

c_0 is a typical example of a Banach space which has the nonstrict Opial property but not the usual (strict) Opial property. We will see that c_0-sums preserve the nonstrict Opial property.

3.12. Proposition. *Let I be an index set and let $(X_i)_{i\in I}$ be a family of Banach spaces with the nonstrict Opial property. Then $X := \left[\bigoplus_{i\in I} X_i\right]_{c_0(I)}$ has the nonstrict Opial property.*

Proof. For every $n \in \mathbb{N}$ let $x_n = (x_{n,i})_{i\in I} \in X$ be such that $(x_n)_{n\in\mathbb{N}}$ converges weakly to zero and let $x = (x_i)_{i\in I} \in X$. Take arbitrary $\epsilon > 0$ and choose a finite subset $J \subseteq I$ such that $\|x_i\| \leqslant \epsilon$ for every $i \in I \setminus J$. Again, there is no loss of generality in assuming that all the limits involved in the subsequent calculations exist. Since each X_i has the nonstrict Opial property we have

$$\lim_{n\to\infty}\|x_{n,i}\| \leqslant \lim_{n\to\infty}\|x_{n,i} - x_i\| \quad \forall i \in J.$$

Therefore we obtain

$$\begin{aligned}\lim_{n\to\infty}\|x_n\|_\infty &= \max\left\{\max_{i\in J}\lim_{n\to\infty}\|x_{n,i}\|, \lim_{n\to\infty}\|(\|x_{n,i}\|\chi_{I\setminus J}(i))_{i\in I}\|_\infty\right\}\\ &\leqslant \max\left\{\max_{i\in J}\lim_{n\to\infty}\|x_{n,i} - x_i\|, \lim_{n\to\infty}\|(\|x_{n,i}\|\chi_{I\setminus J}(i))_{i\in I}\|_\infty\right\}\\ &\leqslant \lim_{n\to\infty}\max\left\{\max_{i\in J}\|x_{n,i} - x_i\|, \|(\|x_{n,i} - x_i\|\chi_{I\setminus J}(i))_{i\in I}\|_\infty\right\} + \epsilon\\ &= \lim_{n\to\infty}\|x_n - x\|_\infty + \epsilon,\end{aligned}$$

and since $\epsilon > 0$ was arbitrary we are done. □

Concerning the uniform Opial property we have the following result for infinite ℓ^p--sums, resembling in structure Theorem 3.2.

3.13. Theorem. *Let* $1 \leqslant p < \infty$ *and let* I *be an infinite index set. For a family* $(X_i)_{i\in I}$ *of Banach spaces put* $X_J := \left[\bigoplus_{i\in J} X_i\right]_p$ *for every finite* $J \subseteq I$*. Suppose that*

$$\omega(\epsilon, R) := \inf\left\{\eta_{X_J}(\epsilon, R) : J \subseteq I \text{ finite}\right\} > 0 \quad \forall \epsilon, R > 0.$$

Then $X := \left[\bigoplus_{i\in I} X_i\right]_p$ *has the uniform Opial property.*

Proof. Let $0 < \epsilon \leqslant 1$ and $R > 0$. We put

$$v := \min\left\{3R + 1, \omega(\epsilon/2, R)\right\} \quad \text{and} \quad \tau := \min\left\{1, 3R + 1 - ((3R+1)^p - v^p)^{1/p}\right\}.$$

Now consider a weakly null sequence $(x_n)_{n\in\mathbb{N}} = ((x_{n,i})_{i\in I})_{n\in\mathbb{N}}$ in X with $\limsup\|x_n\|_p \leqslant R$ and let $x = (x_i)_{i\in I} \in X$ with $\|x\|_p \geqslant \epsilon$. As before we may assume that $\lim_{n\to\infty}\|x_n\|_p$ and $\lim_{n\to\infty}\|x_n - x\|_p$ exist. Let $K := \sup_{n\in\mathbb{N}}\|x_n\|_p$. For $0 < \alpha \leqslant \epsilon/2$ we can find a finite subset $J \subseteq I$ such that

$$\left\|(\|x_i\|\chi_{I\setminus J}(i))_{i\in I}\right\|_p \leqslant \alpha.$$

It follows that

$$\left\|\sum_{i\in J}\|x_i\|e_i\right\|_p \geqslant \|x\|_p - \alpha \geqslant \epsilon/2. \tag{10}$$

We may also assume that $\lim_{n\to\infty}\|x_{n,i}\|$ and $\lim_{n\to\infty}\|x_{n,i}-x_i\|$ exist for all $i \in J$. As in the proof of Proposition 3.11 we can show that

$$\lim_{n\to\infty}\|x_n\|_p^p \leqslant \lim_{n\to\infty}\sum_{i\in J}\|x_{n,i}\|^p + \lim_{n\to\infty}\|(\|x_{n,i}-x_i\|\chi_{I\setminus J}(i))_{i\in I}\|_p^p$$
$$+\,p(K+\|x\|_p+1)^{p-1}\alpha. \tag{11}$$

If we put $y_n := (x_{n,i})_{i\in J}$ for each $n \in \mathbb{N}$, and $y := (x_i)_{i\in J}$, then $(y_n)_{n\in\mathbb{N}}$ is a weakly null sequence in X_J with $\lim_{n\to\infty}\|y_n\|_p \leqslant \lim_{n\to\infty}\|x_n\|_p \leqslant R$ and $y \in X_J$ with $\|y\|_p \geqslant \epsilon/2$ (because of (10)), thus

$$\lim_{n\to\infty}\|y_n\|_p + \eta_{X_J}(\epsilon/2, R) \leqslant \lim_{n\to\infty}\|y_n - y\|_p. \tag{12}$$

Since $(a-b)^p \leqslant a^p - b^p$ for all $a \geqslant b \geqslant 0$, we can deduce from (11) and (12) that

$$\begin{aligned}\lim_{n\to\infty}\|x_n\|_p^p &\leqslant \lim_{n\to\infty}\|y_n - y\|_p^p - \eta_{X_J}(\epsilon/2, R)^p\\ &\quad + \lim_{n\to\infty}\|(\|x_{n,i}-x_i\|\chi_{I\setminus J}(i))_{i\in I}\|_p^p + p(K+\|x\|_p+1)^{p-1}\alpha\\ &\leqslant \lim_{n\to\infty}\|x_n - x\|_p^p - v^p + p(K+\|x\|_p+1)^{p-1}\alpha.\end{aligned}$$

Letting $\alpha \to 0$, we obtain

$$\lim_{n\to\infty}\|x_n\|_p^p \leqslant \lim_{n\to\infty}\|x_n - x\|_p^p - v^p. \tag{13}$$

If $\|x\|_p \geqslant 2R+1$ then $\lim_{n\to\infty}\|x_n - x\|_p \geqslant 2R+1-\lim_{n\to\infty}\|x_n\|_p \geqslant \lim_{n\to\infty}\|x_n\|_p + 1 \geqslant \lim_{n\to\infty}\|x_n\|_p + \tau$.

Now consider the case $\|x\|_p < 2R+1$. Define $f(s) := s - (s^p - v^p)^{1/p}$ for all $s \geqslant v$. It is easily checked that f is decreasing. Since $\lim_{n\to\infty}\|x_n - x\|_p \leqslant \lim_{n\to\infty}\|x_n\|_p + \|x\|_p \leqslant 3R+1$ it follows from (13) that

$$\begin{aligned}\lim_{n\to\infty}\|x_n\|_p &\leqslant \lim_{n\to\infty}\|x_n - x\|_p - f(\lim_{n\to\infty}\|x_n - x\|_p)\\ &\leqslant \lim_{n\to\infty}\|x_n - x\|_p - f(3R+1) \leqslant \lim_{n\to\infty}\|x_n - x\|_p - \tau\end{aligned}$$

(where the last inequality holds by the definition of τ) and the proof is complete. □

As a corollary, we again obtain the already known result that the ℓ^p-sum of any family of Banach spaces with the Schur property has the uniform Opial property (see [26, Example 4.23 (2)] or [28, Theorem 7]).

3.14. Corollary. *Let $1 \leqslant p < \infty$ and let $(X_i)_{i\in I}$ be a family of Banach spaces with the Schur property. Then $\left[\bigoplus_{i\in I} X_i\right]_p$ has the uniform Opial property.*

Let us remark that in [22, Theorem 3.4] it is claimed that certain spaces $\ell_\Phi^A(X)$ of vector-valued sequences have the uniform Opial property, where the class of sequence spaces ℓ_Φ^A includes in particular the spaces ℓ^p for $1 \leqslant p < \infty$ but no assumptions on the Banach space

X are made. This cannot be true since, for example, the fact that $\ell^p(X) = \left[\bigoplus_{n\in\mathbb{N}} X\right]_p$ has the uniform Opial property clearly implies that X itself has the uniform Opial property. Examining the proof of [22, Theorem 3.4], one finds it implicitly assumes that X has the Schur property.

The author does not know whether the condition $\inf_{i\in I} \eta_{X_i} > 0$ is already enough to ensure that $\left[\bigoplus_{i\in I} X_i\right]_p$ has the uniform Opial property (the proof of Proposition 3.10 does not give a uniform lower bound for the moduli of finite sums).

4. Some Opial-type properties in Lebesgue–Bochner spaces

We consider a complete, finite measure space $(S, \mathcal{A}, \mu)$ and a real Banach space X. Recall that for $1 \leqslant p \leqslant \infty$ the Lebesgue–Bochner space $L^p(\mu, X)$ is defined as the space of all Bochner-measurable functions $f\colon S \to X$ (modulo equality μ-almost everywhere) such that $\|f(\cdot)\| \in L^p(\mu)$. Equipped with the norm $\|f\|_p := \|\|f(\cdot)\|\|_p$, $L^p(\mu, X)$ becomes a Banach space.

As was mentioned in the introduction, even the spaces $L^p[0,1]$, $1 < p < \infty, p \neq 2$, of scalar-valued functions do not have the Opial property. However, some results which are in a certain sense analogous to the Opial property are available. For example, in [5] it was shown that any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $L^p(\mu)$ $(0 < p < \infty)$ which converges pointwise almost everywhere to a function $f \in L^p(\mu)$ satisfies

$$\lim_{n\to\infty}\left(\|f_n\|_p^p - \|f_n - f\|_p^p\right) = \|f\|_p^p,$$

hence

$$\liminf_{n\to\infty} \|g_n - g\|_p^p = \liminf_{n\to\infty} \|g_n\|_p^p + \|g\|_p^p$$

for any bounded sequence $(g_n)_{n\in\mathbb{N}}$ in $L^p(\mu)$ which converges pointwise almost everywhere to zero, and any $g \in L^p(\mu)$.

In [3, Chapter 2, Lemma 3.3] it was shown that any sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(\mu, X)$ (where $(S, \mathcal{A}, \mu)$ is a probability space and X an arbitrary Banach space) and any $f \in L^1(\mu, X)$ such that

$$\lim_{n\to\infty} \mu(\{t \in S : \|f_n(t) - f(t)\| \geqslant \epsilon\}) = 0 \quad \forall \epsilon > 0,$$

satisfy the equality

$$\liminf_{n\to\infty} \|f_n - f\|_1 + \|f - g\|_1 = \liminf_{n\to\infty} \|f_n - g\|_1$$

for every $g \in L^1(\mu, X)$.

We are going to consider weak convergence pointwise almost everywhere in Lebesgue–Bochner spaces and prove some results analogous to the Opial property in this setting.

4.1. Theorem. *Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \leqslant p < \infty$ and X a Banach space with the nonstrict Opial property. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\mu, X)$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero for almost every $t \in S$. Suppose further that there is a function $g \in L^p(\mu)$ such that $\|f_n(t)\| \to g(t)$ for almost every $t \in S$. Then,*

$$\int_A \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p \, \mathrm{d}\mu(t) - \int_A g(t)^p \, \mathrm{d}\mu(t) \leqslant \limsup_{n\to\infty} \|f_n - f\|_p^p - \limsup_{n\to\infty} \|f_n\|_p^p$$

holds for every $f \in L^p(\mu, X)$ and every $A \in \mathcal{A}$. In particular,

$$\limsup_{n\to\infty} \|f_n\|_p \leqslant \limsup_{n\to\infty} \|f_n - f\|_p \quad \forall f \in L^p(\mu, X).$$

Proof. Without loss of generality we may assume that $\lim_{n\to\infty} \|f_n(t)\| = g(t)$ and $f_n(t) \to 0$ weakly for every $t \in S$ and also that $\lim_{n\to\infty} \|f_n\|_p$ and $\lim_{n\to\infty} \|f_n - f\|_p$ exist.

Note that from Fatou's Lemma and the boundedness of $(f_n)_{n\in\mathbb{N}}$ it follows that

$$\liminf_{n\to\infty} \|f_n - f\|^p$$

is integrable over S.

Since X has the nonstrict Opial property we have $g(t) \leqslant \liminf \|f_n(t) - f(t)\|$ for every $t \in S$. Therefore, it suffices to prove the statement for $A = S$.

Now let $0 < \epsilon < 1$. By the equi-integrability of finite subsets of $L^1(\mu)$ there exists $\delta > 0$ such that

$$B \in \mathcal{A},\ \mu(B) \leqslant \delta \ \Rightarrow \ \int_B h(t)\, \mathrm{d}\mu(t) \leqslant \epsilon \text{ for each } h \in \left\{ \|f(\cdot)\|^p, g^p, \liminf_{n\to\infty} \|f_n(\cdot) - f(\cdot)\|^p \right\}. \tag{14}$$

By Egorov's theorem, there exists $C \in \mathcal{A}$ with $\mu(S \setminus C) \leqslant \delta$ such that $\lim_{n\to\infty} \|f_n(t)\|^p = g(t)^p$ uniformly in $t \in C$, which implies

$$\lim_{n\to\infty} \int_C \|f_n(t)\|^p \, \mathrm{d}\mu(t) = \int_C g(t)^p \, \mathrm{d}\mu(t). \tag{15}$$

We can find a subsequence such that

$$\lim_{k\to\infty} \int_C \|f_{n_k}(t) - f(t)\|^p \, \mathrm{d}\mu(t)$$

exists. Now we can calculate, using (15),

$$\begin{aligned} \lim_{n\to\infty} \|f_n\|_p^p &= \int_C g(t)^p \, \mathrm{d}\mu(t) + \lim_{n\to\infty} \int_{S\setminus C} \|f_n(t)\|^p \, \mathrm{d}\mu(t) \\ &= \int_C \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p \, \mathrm{d}\mu(t) \\ &\quad + \int_C \left(g(t)^p - \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p \right) \mathrm{d}\mu(t) + \lim_{n\to\infty} \int_{S\setminus C} \|f_n(t)\|^p \, \mathrm{d}\mu(t). \end{aligned} \tag{16}$$

But
$$\left|\int_C g(t)^p \,\mathrm{d}\mu(t) - \int_S g(t)^p \,\mathrm{d}\mu(t)\right| = \int_{S\setminus C} g(t)^p \,\mathrm{d}\mu(t) \leq \epsilon \tag{17}$$
because of $\mu(S \setminus C) \leq \delta$ and (14). Analogously,
$$\int_{S\setminus C} \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p \,\mathrm{d}\mu(t) \leq \epsilon. \tag{18}$$
Putting (16), (17) and (18) together we obtain
$$\begin{aligned}
\lim_{n\to\infty} \|f_n\|_p^p &\leq \int_C \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p \,\mathrm{d}\mu(t) + \lim_{n\to\infty} \int_{S\setminus C} \|f_n(t)\|^p \,\mathrm{d}\mu(t) \\
&\quad + \int_S \left(g(t)^p - \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p\right) \mathrm{d}\mu(t) + 2\epsilon \\
&\leq \lim_{k\to\infty} \int_C \|f_{n_k}(t) - f(t)\|^p \,\mathrm{d}\mu(t) + \lim_{k\to\infty} \int_{S\setminus C} \|f_{n_k}(t)\|^p \,\mathrm{d}\mu(t) \\
&\quad + \int_S \left(g(t)^p - \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p\right) \mathrm{d}\mu(t) + 2\epsilon,
\end{aligned} \tag{19}$$
where we used Fatou's lemma in the second step.

Since $\mu(S \setminus C) \leq \delta$ we have $\int_{S\setminus C} \|f(t)\|^p \,\mathrm{d}\mu(t) \leq \epsilon$, by (14). Hence,
$$\lim_{k\to\infty} \int_{S\setminus C} \|f_{n_k}(t)\|^p \,\mathrm{d}\mu(t) \leq \lim_{k\to\infty} \left(\left(\int_{S\setminus C} \|f_{n_k}(t) - f(t)\|^p \,\mathrm{d}\mu(t)\right)^{1/p} + \epsilon^{1/p}\right)^p.$$
Since $|s^p - t^p| \leq pA^{p-1}|s-t|$ for all $0 \leq s, t \leq A$, we obtain as in the proof of Proposition 3.11 that
$$\lim_{k\to\infty} \int_{S\setminus C} \|f_{n_k}(t)\|^p \,\mathrm{d}\mu(t) \leq \lim_{k\to\infty} \int_{S\setminus C} \|f_{n_k}(t) - f(t)\|^p \,\mathrm{d}\mu(t) + pL^{p-1}\epsilon^{1/p} \tag{20}$$
where $L := \|f\|_p + 1 + \sup_{n\in\mathbb{N}} \|f_n\|_p$.

From (19) and (20) it follows that
$$\begin{aligned}
\lim_{n\to\infty} \|f_n\|_p^p &\leq \lim_{k\to\infty} \int_S \|f_{n_k}(t) - f(t)\|^p \,\mathrm{d}\mu(t) + pL^{p-1}\epsilon^{1/p} \\
&\quad + \int_S \left(g(t)^p - \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p\right) \mathrm{d}\mu(t) + 2\epsilon \\
&= \lim_{n\to\infty} \|f_n - f\|_p^p + pL^{p-1}\epsilon^{1/p} + 2\epsilon \\
&\quad + \int_S \left(g(t)^p - \liminf_{n\to\infty} \|f_n(t) - f(t)\|^p\right) \mathrm{d}\mu(t).
\end{aligned}$$
Letting $\epsilon \to 0$ yields the desired inequality. □

If X has the Opial property, we have the following corollary.

4.2. Corollary. *Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \leq p < \infty$ and X a Banach space with the Opial property. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\mu, X)$ such that $(f_n(t))_{n\in\mathbb{N}}$*

converges weakly to zero for almost every $t \in S$. Suppose further that there is a function $g \in L^p(\mu)$ such that $\|f_n(t)\| \to g(t)$ for almost every $t \in S$. Then

$$\limsup_{n\to\infty} \|f_n\|_p < \limsup_{n\to\infty} \|f_n - f\|_p \quad \forall f \in L^p(\mu, X) \setminus \{0\}.$$

Proof. Put $A := \{t \in S : f(t) \neq 0\}$ in Theorem 4.1. Then $\mu(A) > 0$ and since X has the Opial property we have $\liminf \|f_n(t) - f(t)\| < g(t)$ for every $t \in A$, so the result follows from Theorem 4.1. □

In the case when X has even the uniform Opial property, we have the following two results.

4.3. Theorem. *Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \leqslant p < \infty$ and X a Banach space with the uniform Opial property. Let $M, R > 0$ and $f \in L^p(\mu, X) \setminus \{0\}$. Then there exists $\eta > 0$ such that the following holds: Whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^p(\mu, X)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_p \leqslant R$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero and $\lim_{n\to\infty} \|f_n(t)\| \leqslant M$ for almost every $t \in S$, then*

$$\limsup_{n\to\infty} \|f_n\|_p + \eta \leqslant \limsup_{n\to\infty} \|f_n - f\|_p.$$

Proof. We define $\tau := \|f\|_p (2\mu(S))^{-1/p}$ and $A := \{t \in S : \|f(t)\| \geqslant \tau\}$. If $\mu(A) = 0$, then we would obtain $\|f\|_p^p \leqslant \mu(S \setminus A)\tau^p \leqslant \|f\|_p^p/2$, contradicting the fact that $f \in L^p(\mu, X) \setminus \{0\}$. Thus $\mu(A) > 0$.

Next we put $w := \eta_X(\tau, M)$, $\delta := \min\{(3R+1)^p, \mu(A)w^p\}$, $\omega := R + 1 - ((R+1)^p - \delta)^{1/p}$, and finally $\eta := \min\{\omega, 1\}$.

Let $(f_n)_{n\in\mathbb{N}}$ be as above. Without loss of generality we may assume that

$$g(t) := \lim_{n\to\infty} \|f_n(t)\| \leqslant M$$

and $f_n(t) \to 0$ weakly for every $t \in S$. The definition of η_X implies that

$$\liminf_{n\to\infty} \|f_n(t) - f(t)\| - g(t) \geqslant \eta_X(\tau, M) = w \quad \forall t \in A.$$

Since $(a-b)^p \leqslant a^p - b^p$ for all $a \geqslant b \geqslant 0$, it follows that

$$\liminf_{n\to\infty} \|f_n(t) - f(t)\|^p - g(t)^p \geqslant w^p \quad \forall t \in A.$$

Combinig this with Theorem 4.1 yields

$$\limsup_{n\to\infty} \|f_n - f\|_p^p - \limsup_{n\to\infty} \|f_n\|_p^p \geqslant \mu(A)w^p \geqslant \delta.$$

As in the proof of Theorem 3.13, by distinguishing the cases $\|f\|_p \geqslant 2R+1$ and $\|f\|_p < 2R+1$, we deduce that

$$\limsup_{n\to\infty} \|f_n\|_p + \eta \leqslant \limsup_{n\to\infty} \|f_n - f\|_p.$$

□

4.4. Theorem. *Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \leqslant p < \infty$ and X a Banach space with the uniform Opial property. Let $p < r \leqslant \infty$ and $\epsilon, M, R, K > 0$. Then there exists $\eta > 0$ such that the following holds: Whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^p(\mu, X)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_p \leqslant R$ such that $(f_n(t))_{n\in\mathbb{N}}$ converges weakly to zero and $\lim_{n\to\infty} \|f_n(t)\| \leqslant M$ for almost every $t \in S$, and $f \in L^r(\mu, X) \subseteq L^p(\mu, X)$ such that $\|f\|_r \leqslant K$ and $\|f\|_p \geqslant \epsilon$, then*

$$\limsup_{n\to\infty} \|f_n\|_p + \eta \leqslant \limsup_{n\to\infty} \|f_n - f\|_p.$$

Proof. We put $s := r/p \in (1, \infty]$. Let $s' \in [1, \infty)$ be such that $1/s' + 1/s = 1$. Choose $0 < \tau < \epsilon\mu(S)^{-1/p}$ and put $Q := (\epsilon^p - \mu(S)\tau^p)^{s'} K^{-ps'}$. Let $w := \eta_X(\tau, M)$ and $\delta := \min\{Qw^p, (3R+1)^p\}$, where ω and η are defined as in the preceding proof.

Now let $(f_n)_{n\in\mathbb{N}}$ and f be as above. For $A := \{t \in S : \|f(t)\| \geqslant \tau\}$ we have

$$\begin{aligned}
\epsilon^p \leqslant \|f\|_p^p &= \int_A \|f(t)\|^p \, d\mu(t) + \int_{S\setminus A} \|f(t)\|^p \, d\mu(t) \\
&\leqslant \int_A \|f(t)\|^p \, d\mu(t) + \mu(S \setminus A)\tau^p \leqslant \mu(A)^{1/s'} \|f\|_r^p + \mu(S)\tau^p \\
&\leqslant \mu(A)^{1/s'} K^p + \mu(S)\tau^p,
\end{aligned}$$

where we used Hölder's inequality in the second line. It follows that $\mu(A) \geqslant Q$. As in the preceding proof we deduce that

$$\limsup_{n\to\infty} \|f_n - f\|_p^p - \limsup_{n\to\infty} \|f_n\|_p^p \geqslant \mu(A)w^p \geqslant Qw^p \geqslant \delta,$$

and from there we come to

$$\limsup_{n\to\infty} \|f_n\|_p + \eta \leqslant \limsup_{n\to\infty} \|f_n - f\|_p$$

as in the proof of Theorem 3.13. □

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