The Schur and Steinhaus Theorems for 4-Dimensional Infinite Matrices

Abstract. This paper is a sequel to [2]. Throughout this paper, entries of double sequences, double series and 4-dimensional infinite matrices are real or complex numbers. We prove the Schur and Steinhaus theorems for 4-dimensional infinite matrices.

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1. Introduction and Preliminaries. This paper is a sequel to [2]. Throughout the present paper, entries of double sequences, double series and 4-dimensional infinite matrices are real or complex numbers. If $A = (a_{m,n,k,\ell})$ is a 4-dimensional infinite matrix, and $m,n,k,\ell = 0, 1, 2, \ldots$, by the $A$-transform of a double sequence $x = \{x_{k,\ell}\}$, $k,\ell = 0, 1, 2, \ldots$, we mean the double sequence $A(x) = \{(Ax)_{m,n}\}$,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell}x_{k,\ell}, \quad m,n = 0, 1, 2, \ldots,$$

where we suppose that the double series on the right converges. The double sequence $x = \{x_{k,\ell}\}$ is said to be summable $A$ or $A$-summable to $s$ if

$$\lim_{m+n\to\infty} (Ax)_{m,n} = s.$$

Let $c_{ds}, \ell^\infty_{ds}$ denote the spaces of convergent double sequences and bounded double sequences, respectively. If $A = (a_{m,n,k,\ell})$ is such that $\{(Ax)_{m,n}\} \in c_{ds}$ whenever $x = \{x_{k,\ell}\} \in c_{ds}$, $A$ is said to be convergence-preserving. If, further, $\lim_{m+n\to\infty} (Ax)_{m,n} = \lim_{k+\ell\to\infty} x_{k,\ell}$, we say that $A$ is regular.

Natarajan proved the following result.
Theorem 1.1 ([2]) \( A = (a_{m,n,k,\ell}) \) is regular if and only if

1. \( \lim_{m+n \to \infty} a_{m,n,k,\ell} = 0, \ k, \ell = 0, 1, 2, \ldots; \)

2. \( \lim_{m+n \to \infty} \sum_{k,\ell=0}^{\infty} a_{m,n,k,\ell} = 1; \)

3. \( \lim_{m+n \to \infty} \sum_{k=0}^{\infty} |a_{m,n,k,\ell}| = 0, \ \ell = 0, 1, 2, \ldots; \)

4. \( \lim_{m+n \to \infty} \sum_{\ell=0}^{\infty} |a_{m,n,k,\ell}| = 0, \ k = 0, 1, 2, \ldots; \)

and

5. \( \sup_{m,n} \sum_{k,\ell=0}^{\infty} |a_{m,n,k,\ell}| < \infty. \)

A is called a Schur matrix if \( \{(Ax)_{m,n}\} \in c_{ds} \) whenever \( x = \{x_{k,\ell}\} \in \ell^\infty_{ds}. \) The main object of this paper is to get necessary and sufficient conditions for \( A \) to be a Schur matrix and then deduce Steinhaus theorem.

Definition 1.2 The double sequence \( \{x_{m,n}\} \) is called a Cauchy sequence if for every \( \epsilon > 0, \) there exists \( N \in \mathbb{N} \) (the set of all non-negative integers) such that the set

\( \{(m,n),(k,\ell) \in \mathbb{N}^2 : |x_{m,n} - x_{k,\ell}| \geq \epsilon, \ m,n,k,\ell \geq N\} \)

if finite.

It is now easy to prove the following result.

Theorem 1.3 The double sequence \( \{x_{m,n}\} \) is Cauchy if and only if

6. \( \lim_{m+n \to \infty} |x_{m+u,n} - x_{m,n}| = 0, \ u = 0, 1, 2, \ldots; \)

and

7. \( \lim_{m+n \to \infty} |x_{m,n+v} - x_{m,n}| = 0, \ v = 0, 1, 2, \ldots. \)

Definition 1.4 If every Cauchy double sequence of a normed linear space \( X \) converges to an element of \( X, \) \( X \) is said to be double sequence complete or \( ds \)-complete.
Note that \( \mathbb{R} \) (the set of all real numbers) and \( \mathbb{C} \) (the set of all complex numbers) are \( ds \)-complete.

For \( x = \{x_{m,n}\} \in \ell^\infty_{ds} \), define \( \|x\| = \sup_{m,n}|x_{m,n}|. \) One can easily prove that \( \ell^\infty_{ds} \) is a normed linear space which is \( ds \)-complete. With the same definition of norm for elements of \( c_{ds} \), \( c_{ds} \) is a closed subspace of \( \ell^\infty_{ds} \).

2. Main Results. Schur’s theorem and Steinhaus theorem for 2-dimensional infinite matrices are very well-known (see, for instance, [1]). In this section, we obtain these results for 4-dimensional infinite matrices.

**Theorem 2.1 (Schur)** The necessary and sufficient conditions for a 4-dimensional infinite matrix \( A = (a_{m,n,k,\ell}) \) to be a Schur matrix, i.e., \( \{(Ax)_{m,n}\} \in c_{ds} \) whenever \( x = \{x_{k,\ell}\} \in \ell^\infty_{ds} \) are:

\[
\sum_{k,\ell=0}^\infty |a_{m,n,k,\ell}| < \infty, \quad m, n = 0, 1, 2, \ldots; \tag{8}
\]

\[
\lim_{m+n \to \infty} \sum_{k,\ell=0}^\infty |a_{m+u,n,k,\ell} - a_{m,n,k,\ell}| = 0, \quad u = 0, 1, 2, \ldots; \tag{9}
\]

and

\[
\lim_{m+n \to \infty} \sum_{k,\ell=0}^\infty |a_{m,n+v,k,\ell} - a_{m,n,k,\ell}| = 0, \quad v = 0, 1, 2, \ldots. \tag{10}
\]

**Proof** Sufficiency part. Let (8), (9), (10) hold and \( x = \{x_{k,\ell}\} \in \ell^\infty_{ds} \). First, we note that in view of (8),

\[
(Ax)_{m,n} = \sum_{k,\ell=0}^\infty a_{m,n,k,\ell}x_{k,\ell}, \quad m, n = 0, 1, 2, \ldots
\]

is defined, the double series on the right being convergent. Now, for \( u = 0, 1, 2, \ldots, \)

\[
|(Ax)_{m+u,n} - (Ax)_{m,n}| = \left| \sum_{k,\ell=0}^\infty (a_{m+u,n,k,\ell} - a_{m,n,k,\ell})x_{k,\ell} \right|
\]

\[
\leq M \sum_{k,\ell=0}^\infty |a_{m+u,n,k,\ell} - a_{m,n,k,\ell}|
\]

\[
\to 0, \quad m + n \to \infty, \quad \text{using (9)},
\]

where \( |x_{k,\ell}| \leq M, \quad k, \ell = 0, 1, 2, \ldots, \quad M > 0. \) Similarly it follows that

\[
|(Ax)_{m,n+v} - (Ax)_{m,n}| \to 0, \quad m + n \to \infty, \quad v = 0, 1, 2, \ldots,
\]
using (10). Thus \( \{(Ax)_{m,n}\} \) is a Cauchy double sequence. Since \( \mathbb{R} \) (or \( \mathbb{C} \)) is \( \ell^\infty \)-complete, \( \{(Ax)_{m,n}\} \) converges, i.e., \( \{(Ax)_{m,n}\} \in \ell^\infty \), completing the sufficiency part of the proof.

Necessity part. Let \( A \) be a Schur matrix. For \( m, n = 0, 1, 2, \ldots \), consider the double sequence \( \{x_{k,\ell}\} \), where \( x_{k,\ell} = \text{sgn } a_{m,n,k,\ell}, k, \ell = 0, 1, 2, \ldots \). Then \( \{x_{k,\ell}\} \in \ell^\infty \) so that, by hypothesis,

\[
(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty} |a_{m,n,k,\ell}|, \quad m, n = 0, 1, 2, \ldots
\]

is defined. Since the series on the right converges, (8) holds. Suppose (9) does not hold. Then there exist \( \ell_0, u_0 \in \mathbb{N} \) such that

\[
\lim_{m+n \to \infty} \sum_{k=0}^{\infty} |a_{m+u_0,n,k,\ell_0} - a_{m,n,k,\ell_0}| = 0
\]

does not hold. So there exists \( \epsilon > 0 \) such that the set

\[
\left\{(m, n) \in \mathbb{N}^2 : \sum_{k=0}^{\infty} |a_{m+u_0,n,k,\ell_0} - a_{m,n,k,\ell_0}| > 2\epsilon \right\}
\]

is infinite. Thus we can choose pairs of integers \( m_p, n_p \in \mathbb{N} \) such that \( m_1 + n_1 < m_2 + n_2 < \cdots < m_p + n_p < \cdots \) and

\[
\sum_{k=0}^{\infty} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| > 2\epsilon, \quad p = 1, 2, \ldots.
\]

Using (8), we have,

\[
\sum_{k=0}^{\infty} |a_{m_1+u_0,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| < \infty.
\]

Consequently there exists \( r_1 \in \mathbb{N} \) such that

\[
\sum_{k=r_1}^{\infty} |a_{m_1+u_0,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| < \frac{\epsilon}{4}.
\]

In view of (11) and (12), we have,

\[
\sum_{k=0}^{r_1-1} |a_{m_1+u_0,n_1,k,\ell_0} - a_{m_1,n_1,k,\ell_0}| < \frac{7\epsilon}{4} > \epsilon.
\]

By hypothesis, (1) holds so that we can suppose that

\[
\sum_{k=0}^{r_1-1} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| < \frac{\epsilon}{4}.
\]
Using (11), we have,

\[
\sum_{k=0}^{\infty} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| > 2\epsilon.
\]

Using (8),

\[
\sum_{k=0}^{\infty} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| < \infty
\]

so that there exists \( r_2 \in \mathbb{N}, r_2 > r_1 \), such that

\[
\sum_{k=r_2}^{\infty} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| < \frac{\epsilon}{4}.
\]

From (14), (15), (16), we have,

\[
\sum_{k=r_1}^{r_2-1} |a_{m_2+u_0,n_2,k,\ell_0} - a_{m_2,n_2,k,\ell_0}| > \frac{3\epsilon}{2} > \epsilon.
\]

Inductively, we can choose a strictly increasing sequence \( \{r_p\} \) such that

\[
\sum_{k=0}^{r_p-1} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| < \frac{\epsilon}{4};
\]

\[
\sum_{k=r_p}^{\infty} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| < \frac{\epsilon}{4};
\]

and

\[
\sum_{k=r_p-1}^{r_p-1} |a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}| > \epsilon.
\]

Now, define \( \{x_{k,\ell}\} \in \ell^\infty_{\Delta_3} \), where

\[
x_{k,\ell} = \begin{cases} 
\text{sgn} (a_{m_p+u_0,n_p,k,\ell_0} - a_{m_p,n_p,k,\ell_0}), & \text{if } \ell = \ell_0, r_p-1 \leq k < r_p; \\
0, & \text{otherwise}.
\end{cases}
\]
Then,

\[(Ax)_{m+p,u_0,n_p} - (Ax)_{m_p,n_p} = \sum_{k,\ell=0}^{\infty} (a_{m_p+u_0,n_p,k,\ell} - a_{m_p,n_p,k,\ell}) x_k,\ell \]

\[= \sum_{k=0}^{\infty} (a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0}) x_k,0 \]

\[= \sum_{k=0}^{r_p-1} (a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0}) x_k,0 \]

\[+ \sum_{k=r_p}^{r_p-1} (a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0}) x_k,0 \]

\[+ \sum_{k=r_p}^{\infty} (a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0}) x_k,0 \]

\[= \sum_{k=0}^{r_p-1} (a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0}) x_k,0 \]

\[+ \sum_{k=r_p}^{r_p-1} \left| a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0} \right| \]

\[+ \sum_{k=r_p}^{\infty} (a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0}) x_k,0 \]

so that

\[\sum_{k=r_p}^{r_p-1} \left| a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0} \right| = \{ (Ax)_{m_p+u_0,n_p} - (Ax)_{m_p,n_p} \} \]

\[- \sum_{k=0}^{r_p-1} (a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0}) x_k,0 \]

\[- \sum_{k=r_p}^{\infty} (a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0}) x_k,0 \]

In view of (17), (18), (19), we have,

\[\epsilon < \sum_{k=r_p}^{r_p-1} \left| a_{m_p+u_0,n_p,k,0} - a_{m_p,n_p,k,0} \right| \]

\[\leq \left| (Ax)_{m_p+u_0,n_p} - (Ax)_{m_p,n_p} \right| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \]

from which it follows that

\[\left| (Ax)_{m_p+u_0,n_p} - (Ax)_{m_p,n_p} \right| > \frac{\epsilon}{2}, \quad p = 1, 2, \ldots \]
Consequently \( (Ax)_{m,n} \not\in \text{cds} \), a contradiction. Thus (9) holds. Similarly (10) holds too. This completes the proof of the theorem.

We now deduce the following result.

**Theorem 2.2 (Steinhaus)** A 4-dimensional infinite matrix \( A = (a_{m,n,k,\ell}) \) cannot be both a regular and a Schur matrix, i.e., given a 4-dimensional regular matrix \( A \), there exists a bounded divergent double sequence which is not \( A \)-summable.

**Proof** If \( A \) is regular, then (1) and (2) hold. If \( A \) is a Schur matrix too, then, \( \{a_{m,n,k,\ell}\}_{m,n=0}^\infty \) is uniformly Cauchy with respect to \( k,\ell = 0,1,2,\ldots \). Since \( \mathbb{R} \) (or \( \mathbb{C} \)) is \( \text{ds} \)-complete, \( \{a_{m,n,k,\ell}\}_{m,n=0}^\infty \) converges uniformly to 0 with respect to \( k,\ell = 0,1,2,\ldots \). Consequently, we have,

\[
\lim_{m+n \to \infty} \sum_{k,\ell=0}^\infty a_{m,n,k,\ell} = 0,
\]

a contradiction of (2), completing the proof.

**References**
