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Insertion and extension theorems for completely distributive lattice valued functions *

Dedicated to Professor Julian Musielak on the occasion of his 85th birthday

Abstract. Insertion and extension theorems are presented for lattice-valued functions where the lattice is an appropriately based completely distributive lattice endowed with its interval topology.

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1. Introduction. Topological Insertion Theorem due to Blair [2] and Lane [9, 10] provides a necessary and sufficient condition for inserting a continuous real-valued function between two arbitrary comparable real-valued functions. It tells us: *If X is a topological space, $g, h\colon X \to \mathbb{R}$ are two arbitrary (not necessarily continuous) functions, then there exists a continuous function $f\colon X \to \mathbb{R}$ with $g \leq f \leq h$ if and only if, given $s < t$ in $\mathbb{R}$, the sets $g^{-1}[t, \infty)$ and $h^{-1}(-\infty, s]$ are completely separated.* Here we recall that two subsets A and B of X are completely separated in X if there is a continuous $k\colon X \to [0, 1]$ such that $k = 0$ on A and $k = 1$ on B. On the other hand, Topological Extension Theorem of Mrówka [12] provides a necessary and sufficient condition in order that a bounded continuous real-valued function on an arbitrary subspace be continuously extendable to the whole space. Namely: *If X is a topological space, Y its arbitrary subspace, and $f\colon Y \to [0, 1]$ a continuous function, then f has a continuous extension to the whole of X if and only if the sets $f^{-1}[0, s]$ and $f^{-1}[t, 1]$ are completely separated in X for all $s < t$ in*

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$[0,1]$. We note that the famous Urysohn Extension Theorem of Gillman and Jerison [6] is a simple consequence of the latter.

In this paper we show that all those theorems continue to hold for functions taking values in a completely distributive lattice endowed with its interval topology and in which there is a countable basis which is free of completely join irreducible elements. Examples of such lattices include, but are not limited to, any Hilbert cube with its componentwise ordering.

2. Terminology and notation. Our references for lattices are [1] and [5] for which we refer for concepts not defined therein. A lattice L is called *completely distributive* if it is complete and

$$\bigwedge_{j\in J}\bigvee_{k\in K_j} a_{jk} = \bigvee_{\varphi\in\Pi_{j\in J}K_j}\bigwedge_{j\in J} a_{j\varphi(j)}$$

for any family $\{a_{jk} : j \in J,\ k \in K_j\} \subseteq L$. The bounds of L are denoted 0 and 1. Following Raney [14], given $a, b \in L$ we let

$$b \lhd a$$

if and only if, whenever $A \subseteq L$ and $a \leq \bigvee A$, there is a $c \in A$ with $b \leq c$. The relation $\lhd$ has the following properties: (1) $b \lhd a$ implies $b \leq a$, (2) $d \leq b \lhd a \leq c$ implies $d \lhd c$. Raney showed that L is completely distributive if and only if each $a \in L$ has the following *approximation property*:

$$a = \bigvee\{b \in L : b \lhd a\},$$

in which case $\lhd$ has the *insertion property*: $b \lhd a$ implies $b \lhd c \lhd a$ for some $c \in L$.

A subset $S \subseteq L$ is a *basis* of L if each element of L is a join of elements from S. An $S \subseteq L$ is a basis of L if and only if S is $\lhd$-dense in L, i.e., given $a \lhd b$ in L, there is an $s \in S$ such that $a \lhd s \lhd b$. Because of the approximation property of $\lhd$ and the $\lhd$-density of S, a set $S \subseteq L$ is a basis of L if and only if

$$a = \bigvee\{s \in S : s \lhd a\}$$

for each $a \in L$ (see [7] for details).

An element $a \in L$ is said to be *supercompact* or *completely join irreducible* if $a \lhd a$. A completely distributive lattice L is called $\lhd$-*separable* in [7] (or *strictly* $\bigvee$-*generating* in [11]) if it has a countable basis which is free of supercompact elements.

Examples Among examples of $\lhd$-separable completely distributive lattices are the following (cf. [7] and [4]):

(1) The real unit interval $[0,1]$ in which $S = [0,1]\cap\mathbb{Q}$ is a countable basis without supercompact elements, and $b \lhd a$ if and only if $b < a$.

(2) Let L be the set of closed intervals $a = [a_1, a_2]$ of the unit interval ordered by componentwise order, i.e. $b \leq a$ if and only if $b_1 \leq a_1$ and $b_2 \leq a_2$. Then $b \lhd a$ if and only if either $b_1 = 0$ and $b_2 < a_2$ or $b_2 < a_1$ and hence

$$\bigvee\{b \in L : b \lhd a\} = \bigvee\{b \in L \mid \big(b_1 = 0 \text{ and } b_2 < a_2\big) \text{ or } b_2 < a_1\} = a$$

for each $a \in L$. It follows that L is completely distributive. Moreover,

$$a = \bigvee\{b \in L \mid \big(b_1 = 0 \text{ and } b_2 \in \mathbb{Q} \cap (0, a_2)\big) \text{ or } b_1 = b_2 \in \mathbb{Q} \cap (0, a_1)\}$$

for each $a \in L$ and so

$$S = \{a \in L \mid \big(a_1 = 0 \text{ and } a_2 \in \mathbb{Q} \cap (0, 1)\big) \text{ or } a_1 = a_2 \in \mathbb{Q} \cap (0, 1)\}$$

is a countable basis without supercompact elements.

(3) The class of $\lhd$-separable completely distributive lattices is closed under the formation of countable products with componentwise ordering [7, Prop. 3.5]. In particular, the Hilbert cube $[0, 1]^\omega$ is a $\lhd$-separable completely distributive lattice.

The set L^X of all maps from a set X into L is ordered pointwise, i.e. $f \leq g$ in L^X if and only if $f(x) \leq g(x)$ for all $x \in X$. Given $f \in L^X$ and $a \in L$, we let

$$[f \geq a] = \{x \in X : a \leq f(x)\},$$
$$[f \rhd a] = \{x \in X : a \lhd f(x)\},$$
$$[f \not\rhd a] = \{x \in X : a \not\lhd f(x)\}.$$

The *interval topology* on L is one which is generated by the sets

$$L \backslash {\downarrow} a = \{b \in L : b \nleq a\} \quad \text{and} \quad L \backslash {\uparrow} a = \{b \in L : a \nleq b\}$$

for all $a \in L$. We quote from [13]:

Fact *A completely distributive lattice is a compact Hausdorff topological lattice for the interval topology.*

Standing assumption In what follows, L is endowed with its interval topology.

For L a completely distributive lattice and X a topological space, if $f\colon X \to L$ is continuous, then $[f \rhd a]$ is open for all $a \in L$ [7, Prop. 5.2]. In particular, by continuity of identity map $\mathrm{id}_L\colon L \to L$, the set

$$\Uparrow a = \{b \in L : a \lhd b\}$$

is open in the interval topology for all $a \in L$. A fuller account of these matters can be found in [7, Sect. 4].

3. Inserting lattice-valued functions. The right generalization of the concept of complete separation to the lattice-valued setting is as follows:

Definition 3.1 Two subsets C and D of a topological space X are called *completely L-separated* if there is a continuous function $f\colon X \to L$ such that

$$C \subseteq [f = 1] \quad \text{and} \quad D \subseteq [f \not\rhd 0].$$

The following technical lemma is obvious when $L = [0,1]$.

LEMMA 3.2 *For X a topological space and L a $\lhd$-separable completely distributive lattice the following hold: $C, D \subseteq X$ are completely L-separated if and only if there are $s \lhd t$ in L and a continuous $h\colon X \to L$ such that $C \subseteq [h \geq t]$ and $D \subseteq [h \not\rhd s]$*

PROOF The 'only if' part is evident, for $s = 0 \lhd 1 = t$ and $C \subseteq [f = 1]$ and $D \subseteq [f \not\rhd 0]$. For the 'if' part, we assume $C \subseteq [h \geq t]$ and $D \subseteq [h \not\rhd s]$ for some continuous h and $s \lhd t$. Let $K = (L \setminus \Uparrow s) \cup (\uparrow t)$, a union of two disjoint closed subsets of L. Define $\varphi\colon K \to L$ by $\varphi = 0$ on $L \setminus \Uparrow s$ and $\varphi = 1$ on $\uparrow t$. Since L is compact and Hausdorff, φ has a continuous extension $\widehat{\varphi}$ to the whole of X by [7, Thm. 6.5] (= Tietze's extension theorem for L-valued functions). Then $f = \widehat{\varphi} \circ h$ is continuous from X into L. Also, $[h \geq t] \subseteq [f = 1]$ and $[h \not\rhd s] \subseteq [f \not\rhd 0]$. Thus C and D are completely separated L-separated by f. ■

Let $\mathcal{P}(X)$ be the powerset of a set X. A binary relation $\Subset$ on $\mathcal{P}(X)$ is called a *subordination* if it has the following properties for all $A, B, C \subseteq X$:

($\Subset_1$) $A \Subset B$ implies $A \subseteq B$,

($\Subset_2$) $C \subseteq A \Subset B \subseteq D$ implies $C \Subset D$,

($\Subset_3$) $A, B \Subset C$ implies $A \cup B \Subset C$,

($\Subset_4$) $A \Subset B, C$ implies $A \Subset B \cap C$,

($\Subset_5$) $A \Subset B$ implies $A \Subset E \Subset B$ for some $E \subseteq X$ (*insertion property*).

The following is a special case of Lemma 6.3 in [7]:

LEMMA 3.3 (INSERTION LEMMA) *Let X be a set and let $\Subset$ be an arbitrary subordination on $\mathcal{P}(X)$. Let S be an arbitrary countable set endowed with a transitive and irreflexive relation $\prec$. Let $\{A_s : s \in S\} \subseteq \mathcal{P}(X)$ and $\{B_s : s \in S\} \subseteq \mathcal{P}(X)$ be such that*

$$A_t \subseteq A_s,\ A_t \Subset B_s \text{ and } B_t \subseteq A_s \text{ whenever } s \prec t.$$

Then there exists $\{C_s : s \in S\} \subseteq \mathcal{P}(X)$ such that

$$A_t \Subset C_s,\ C_t \Subset C_s \text{ and } C_t \Subset B_s \text{ whenever } s \prec t.$$

Given $A, B \subseteq X$ we write

$$A \sqsubset B$$

if and only if A and $X \setminus B$ are completely L-separated. Therefore

$$A \sqsubset B \quad \text{if and only if} \quad A \subseteq [h \geq t] \subseteq [h \rhd s] \subseteq B$$

for a continuous $h\colon X \to L$ and $s \lhd t$ in L. For convenience, we may write $A \sqsubset_h B$ to indicate the function involved in the separation. We note that $A \sqsubset B$ implies $\overline{A} \subseteq \operatorname{Int} B$.

FACT 3.4 *The relation $\sqsubset$ is a subordination in the powerset $\mathcal{P}(X)$.*

Proof Axioms ($\Subset_1$) and ($\Subset_2$) are evident. For ($\Subset_3$), let $A \sqsubset_h C$ and $B \sqsubset_k C$. Since L is a topological lattice for the interval topology, it follows that $A \cup B \sqsubset_{h\vee k} C$. A similar argument applies to ($\Subset_4$). For ($\Subset_5$), let $A \sqsubset_h B$. With $s \lhd r \lhd t$ we get $A \subseteq [h \geq t] \subseteq [h \rhd r] = C \subseteq [h \geq r] \subseteq [h \rhd s] \subseteq B$, so that $A \sqsubset C \sqsubset B$. ∎

Theorem 3.5 (Topological Insertion Theorem) *Let L be a $\lhd$-separable completely distributive lattice and let $S \subseteq L$ be its basis. Let X be a topological space and let $g, h\colon X \to L$ be arbitrary functions. The following are equivalent*:

(1) *There exists a continuous function $f\colon X \to L$ such that $g \leq f \leq h$.*

(2) *If $s \lhd t$ in S, then $[g \geq t]$ and $[h \not\rhd s]$ are completely L-separated.*

Proof (1) $\Rightarrow$ (2): This is obvious for if $g \leq f \leq h$ and $s \lhd t$ in S, then $[g \geq t] \subseteq [f \geq t] \subseteq [f \rhd s] \subseteq [h \rhd s]$, i.e. $[g \geq t]$ and $[h \not\rhd s]$ are completely separated by f.

(2) $\Rightarrow$ (1): The families $\{[g \geq s]\}_{s\in S}$ and $\{[h \rhd s]\}_{s\in S}$ satisfy the assumptions of Lemma 3.3 with the relation $\sqsubset$. Hence there is a family $\{F_s\}_{s\in S}$ of subsets of X such that if $s \lhd p \lhd q \lhd t$, then $[g \geq t] \sqsubset F_q \sqsubset F_p \sqsubset [h \rhd s]$. Let $f\colon X \to L$ be defined by

$$f(x) = \bigvee\{s \in S : x \in F_s\}$$

for each $x \in X$. Then f is continuous. Indeed, for each $a \in L$ one has

$$[f \geq a] = \bigcap_{s\lhd a} \overline{F_s} \quad \text{and} \quad [f \rhd a] = \bigcup_{a\lhd s} \operatorname{Int} F_s.$$

Given $x \in X$ we get

$$g(x) = \bigvee\{s \in S : x \in [g \geq s]\} \leq \bigvee\{s \in S : x \in F_s\} = f(x)$$

and

$$f(x) = \bigvee\{s \in S : x \in F_s\} \leq \bigvee\{s \in S : x \in [h \rhd s]\} = h(x)$$

(see [7, Sect. 4] for details). Hence $f \leq h \leq g$. ∎

Remark 3.6 Special cases of Theorem 3.5 include the L-valued version of the Katětov-Tong insertion theorem of [7] and [11]. It should however be remarked that the latter theorem has essentially been used in the proof of Lemma 3.2.

4. Extending lattice-valued functions. The first important corollary is the following L-valued variant of the extension theorem of Mrówka [12] concerning extendability of a single function:

Theorem 4.1 *Let L be a $\lhd$-separable completely distributive lattice and let $S \subseteq L$ be its basis. Let X be a topological space and let $Y \subseteq X$ be its arbitrary subspace. For $f\colon Y \to L$ a continuous function the following are equivalent*:

(1) *The function f has a continuous extension to the whole of X.*

(2) *If $s \lhd t$ in S, then $[f \geq t]$ and $[f \not\rhd s]$ of Y are completely L-separated in X.*

PROOF (1) $\Rightarrow$ (2): Let $\widehat{f}$ be a continuous extension of f to all of X. Then $[f \geq t] \subseteq [\widehat{f} \geq t]$ and $[f \not\trianglerighteq s] \subseteq [\widehat{f} \not\trianglerighteq s]$.

(2) $\Rightarrow$ (1): Let us standartly define $g, h\colon X \to L$ by $g = f = h$ on Y and $g = 0$ and $h = 1$ on $X \setminus Y$. Then $[g \geq t] = [f \geq t]$ and $[h \not\trianglerighteq s] = [f \not\trianglerighteq s]$ and there exists a continuous $\widehat{f}\colon X \to L$ such that $g \leq \widehat{f} \leq h$ which extends f over X. ∎

As another corollary we get an L-valued variant of the well-known Urysohn Extension Theorem of Gillman and Jerison [6] which characterizes simultaneous extendability of every function on a subspace.

THEOREM 4.2 (URYSOHN EXTENSION THEOREM) *Let L be a $\lhd$-separable completely distributive lattice. For X a topological space and $Y \subseteq X$, the following are equivalent*:

(1) *Each continuous $f\colon Y \to L$ has a continuous extension to the whole of X.*

(2) *Any two completely L-separated sets in Y are completely L-separated in X.*

PROOF (1) $\Rightarrow$ (2): This is obvious: if $A, B \subseteq Y$ are completely L-separated by a continuous $f\colon Y \to L$, then extension of f completely L-separates A and B in X.

(2) $\Rightarrow$ (1): Let S be a basis of L and let $f\colon Y \to L$ be continuous. Given $s \lhd t$ in S, the sets $[f \geq t]$ and $[f \not\trianglerighteq s]$ are completely L-separated in Y by f itself, hence they are completely L-separated in X. By Theorem 4.1, f has a continuous extension to the whole of X. ∎

REMARK 4.3 There are counterparts of the presented results for hedgehog valued functions (cf. [3] and [8]) These are hoped to appear thereafter.

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