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# Mean values for vector valued functions and corresponding functional equations

*Dedicated to Professor Julian Musielak on his 85th birthday*

**Abstract.** Although, in general, a straightforward generalization of the Lagrange mean value theorem for vector valued mappings fails to hold we will look for what can be salvaged in that situation. In particular, we deal with Sanderson's and McLeod's type results of that kind (see [9] and [7], respectively). Moreover, we examine mappings with a prescribed intermediate point in the spirit of the celebrated Aczél's theorem characterizing polynomials of degree at most 2 (cf. [1]).

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It is well known that mean value theorems offered by the classical one-dimensional analysis do not carry over to vector valued mappings. Nevertheless, some substitutes are known. Let us mention a few of them.

Theorem of D. E. Sanderson ([9], see also P.K. Sahoo and T. Riedel [8, p. 162]). *Let $f : [a, b] \longrightarrow \mathbb{R}^n$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If $f(a)$ and $f(b)$ are orthogonal to a non-zero vector $v \in \mathbb{R}^n$, then there exists a point $\xi \in (a, b)$ such that $f'(\xi)$ is orthogonal to $v$.*

Theorem of R. M. McLeod (1964, cf. [7]). *Let $f : [a, b] \longrightarrow \mathbb{R}^n$ be continuous on $[a, b]$ and continuously differentiable on $(a, b)$. Then there are $n$ points $\xi_1, \xi_2, ..., \xi_n \in (a, b)$ and $n$ nonnegative numbers $\lambda_1, \lambda_2, ..., \lambda_n$ such that*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$$

*and*

$$f(b) - f(a) = (b - a) \sum_{k=1}^{n} \lambda_k f'(\xi_k).$$

THEOREM (*folklore*). *Let $(X, \|\cdot\|)$ be a real Banach space and let $f : [a,b] \longrightarrow X$ be continuous on $[a,b]$ and continuously differentiable on $(a,b)$. Then*

$$\frac{f(b)-f(a)}{b-a} \in \operatorname{cl}\operatorname{conv} f'((a,b)) .$$

To get Sanderson's result it suffices to apply the following simple theorem for the case where $(X, (\cdot|\cdot))$ yields and inner product space and $x^*(x) := (x|v)$, $x \in X$.

THEOREM 1. *Let $(X, \|\cdot\|)$ be a real normed linear space and let $f : [a,b] \longrightarrow X$ be continuous on $[a,b]$ and differentiable on $(a,b)$. Fix a nonzero member $x^*$ of the space $X^*$ dual to $X$. If $f(a)$ and $f(b)$ are both in the kernel of $x^*$, then there exists a point $\xi \in (a,b)$ such that $f'(\xi)$ is in the kernel of $x^*$.*

PROOF. It suffices to observe that a map $\varphi : [a,b] \longrightarrow \mathbb{R}$ given by the formula $\varphi := x^* \circ f$ satisfies the assumptions of the classical theorem of Rolle. Therefore, there exists a point $\xi \in (a,b)$ such that $0 = \varphi'(\xi) = x^*(f'(\xi))$.

Going back to McLeod's theorem, assume that $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is continuously differentiable. Then, for arbitrarily fixed real numbers $x < y$ there are $n$ points $\xi_1(x,y), \xi_2(x,y), ..., \xi_n(x,y) \in (x,y)$ and $n$ nonnegative numbers $\lambda_1(x,y), \lambda_2(x,y), ..., \lambda_n(x,y)$ such that

$$\lambda_1(x,y) + \lambda_2(x,y) + \cdots + \lambda_n(x,y) = 1$$

and

$$\frac{f(y)-f(x)}{y-x} = \sum_{k=1}^{n} \lambda_k(x,y) f'(\xi_k(x,y)).$$

In particular, for $n = 2$ we are faced to the following mean value type result:

THEOREM. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be continuously differentiable. Then there exist two means $m_i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1,2$ and a function $\lambda : \mathbb{R} \times \mathbb{R} \longrightarrow [0,1]$ such that*

$$(1) \qquad f(y) - f(x) = (y-x)\left[\lambda(x,y) f'(m_1(x,y)) + (1-\lambda(x,y)) f'(m_2(x,y))\right]$$

for all $x, y \in \mathbb{R}$.

In what follows, we shall study equation (1) with the coefficient function $\lambda(x,y) \equiv \frac{1}{2}$ and, to get rid of the differentiability assumption, with the derivatives $\frac{1}{2}f'$ in (2) replaced by another unknown function $g : \mathbb{R} \longrightarrow \mathbb{R}^2$. That is, we shall examine a Pexider type functional equation

$$\frac{f(y)-f(x)}{y-x} = g(m_1(x,y)) + g(m_2(x,y)) \tag{2}$$

with some means $m_1$ and $m_2$. Expecting to have quadratic "polynomials" $f(x) = ax^2 + bx + c,\ x \in \mathbb{R}$, with some fixed vectors $a, b, c \in \mathbb{R}^2$ as potential solutions, we shall be looking for solutions of equation (1) with the coefficient function $\lambda(x,y) \equiv \frac{1}{2}$ which, in turn, forces the existence of a mean $m$ such that

$$m_1(x,y) = m(x,y) \quad \text{and} \quad m_2(x,y) = x + y - m(x,y)\,.$$

This transforms equation (1) applied for $\lambda(x,y) \equiv \frac{1}{2}$ to the equation

$$f(y) - f(x) = (y-x)\left[\frac{1}{2}f'(m(x,y)) + \frac{1}{2}f'(x+y-m(x,y))\right]. \tag{3}$$

and, accordingly, equation (2) to the equation

$$\frac{f(y)-f(x)}{y-x} = g(m(x,y)) + g(x+y-m(x,y)). \tag{4}$$

On the other hand, keeping in mind that we wish to characterize quadratic "polynomials", by means of Theorem 2 below, we may force $f$ to satisfy the functional equation

$$\frac{f(y)-f(x)}{y-x} = h(x+y)$$

with some new unknown function $h$ which jointly with (2) leads to another Pexider type functional equation

$$h(x+y) = g(m_1(x,y)) + g(m_2(x,y)), \tag{$E_{1,2}$}$$

which will prove to be very interesting for its own.

To proceed, let us first note that, actually, quadratic "polynomialsśatisfy equation (3) for quite arbitrary (!) mean $m$. Therefore, it is seems natural to ask what about the converse, that is to look for possibly small families $\Lambda$ of means in order that equation

$$f(y) - f(x) = (y-x)\left[g(m(x,y)) + g(x+y-m(x,y))\right], \tag{E}$$

assumed to be valid for all members $m \in \Lambda$, forces $f$ to be a quadratic "polynomial". Once we are lucky enough a singleton $\Lambda$ may do the job. That is, for instance, the case where $\Lambda = \{A\}$ with $A$ being the arithmetic mean.

Theorem 2. *Let $X$ be a real linear space and let functions $f : \mathbb{R} \longrightarrow X$ and $g : \mathbb{R} \longrightarrow X$ satisfy the equation*

$$\frac{f(y)-f(x)}{y-x} = h(x+y)$$

*for all pairs* $(x,y)\in \mathbb{R}^2$ *such that* $x\neq y$, *then there exist points* $a,b,c\in X$ *such that*

$$f(x)=ax^2+bx+c,\quad x\in\mathbb{R},$$

*and*

$$h(x)=ax+b,\quad x\in\mathbb{R}.$$

Proof. Fix arbitrarily a real linear functional $\ell$ from the algebraic dual $X'$ of the space $X$ and put $\varphi:=\ell\circ f_0$ with $f_0:=f-f(0)$. Let $\psi:=\ell\circ h$; then

$$\frac{\varphi(y)-\varphi(x)}{y-x}=\psi(x+y)$$

for all pairs $(x,y)\in\mathbb{R}^2$ such that $x\neq y$. By means of Aczél's characterization theorem from [1], there exist real numbers $a(\ell)$ and $b(\ell)$ such that

$$\varphi(x)=a(\ell)x^2+b(\ell)x\quad\text{and}\quad\psi(x)=a(\ell)x+b(\ell)$$

for all $x\in\mathbb{R}$. Since $\varphi(1)=a(\ell)+b(\ell)$ and $\psi(0)=b(\ell)$ we infer that

$$\begin{aligned}\ell(f_0(x))&=(\varphi(1)-\psi(0))x^2+\psi(0)x\\&=(\ell(f_0(1))-\ell(h(0))\,x^2+\ell(h(0))x=\ell((f_0(1)-h(0))x^2+h(0)x)\end{aligned}$$

for every $x\in\mathbb{R}$. Due to the unrestricted choice of the functional $\ell\in X'$ and the fact that $X'$ separates points, we conclude that

$$f_0(x)=ax^2+bx,\ x\in\mathbb{R},\quad\text{where}\quad a:=f_0(1)-h(0)\ \text{ and }\ b:=h(0).$$

Consequently,

$$f(x)=ax^2+bx+c,\ x\in\mathbb{R},\quad\text{where}\quad c:=f(0).$$

Moreover, we have also,

$$h(x)=\frac{f(x)-c}{x}=\frac{ax^2+bx}{x}=ax+b\ \text{ for all }\ x\in\mathbb{R}\setminus\{0\},$$

which completes the proof because for every $x\in\mathbb{R}\setminus\{0\}$ one has

$$h(0)=\frac{f(x)-f(-x)}{2x}=b.$$

In general, i.e. in the case where the means $m(x,y)$ and $x+y-m(x,y)$ do not coincide, the situation is more complicated. To get a characterization of quadratic "polynomials"via equation (E) we need more means to be involved. In what follows, we shall present several results of that kind. We begin with the family $\Lambda\subset\{m_\lambda:\lambda\in[0,1]\}$ of weighted arithmetic means

$$m_\lambda(x):=\lambda x+(1-\lambda)y,\ x,y\in\mathbb{R},\ \lambda\in[0,1]\ ,$$

consisting of two elements: $\Lambda = \{m_\lambda, m_{1-\lambda}\}$. As we shall see, the corresponding functional equation will characterize quadratic "polynomials"for all but two particular irrational $\lambda$'s. This fact may be viewed as somewhat surprising one but, as in everyday life, exceptions happen.

Theorem 3. *Let $X$ be a real linear space and let functions $f, g : \mathbb{R} \longrightarrow X$ satisfy the equation*

$$(\mathrm{E}_\lambda) \qquad f(y) - f(x) = (y - x)\left[g(\lambda x + (1-\lambda)y) + g(\lambda y + (1-\lambda)x)\right],$$

*for all pairs $(x, y) \in \mathbb{R}^2$ and a coefficient $\lambda \in [0, 1]$. Then there exist points $a, b, c \in X$ such that*

$$\begin{cases} f(x) = ax^2 + bx + c, & x \in \mathbb{R}, \\ g(x) = ax + b, & x \in \mathbb{R}, \end{cases}$$

*if $\lambda \notin \{\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\}$, and, in the case where $\lambda \in \{\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\}$, there exist points $a, b, c, d \in X$ such that*

$$\begin{cases} f(x) = \frac{2}{3}dx^3 + ax^2 + bx + c, & x \in \mathbb{R}, \\ g(x) = dx^2 + ax + b, & x \in \mathbb{R}. \end{cases}$$

*Conversely, if $(f, g)$ are given by one of the above formulae, then the pair satisfies* $(\mathrm{E}_\lambda)$.

Proof. We will use a result from A. Lisak's and M. Sablik's paper [6, Lemma 1]. Let $G$ and $H$ be commutative groups. Then $SA^i(G; H)$ denotes the group of all $i$-additive, symmetric mappings from $G^i$ into $H$ for $i \geq 2$, while $SA^0(G; H)$ denotes the family of constant functions from $G$ to $H$ and $SA^1(G; H) = \mathrm{Hom}(G; H)$. We also denote by $\mathcal{I}$ the subset of $\mathrm{Hom}(G; G) \times \mathrm{Hom}(G; G)$ containing all pairs $(\alpha, \beta)$ for which $\mathrm{Ran}(\alpha) \subset \mathrm{Ran}(\beta)$. Furthermore, we adopt a convention that a sum over empty set of indices equals 0. Then the authors of [6] have proved the following Lemma 1.

*Fix $N, M, K \in \mathbb{N}$ and let $I_0, ..., I_{M+K}$ be finite subsets of $\mathcal{I}$. Suppose further that $H$ is uniquely divisible by $N!$ and let functions $\varphi_i : G \to SA^i(G; H)$, $i \in \{0, ..., N\}$ and functions $\psi_{i,(\alpha,\beta)} : G \to SA^i(G; H)$, $(\alpha, \beta) \in I_i$, $i \in \{0, ..., M + K\}$ satisfy*

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) =$$
$$\sum_{i=0}^{M} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x) + \beta(y))(y^i) +$$
$$\sum_{i=M+1}^{M+K} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x) + \beta(y))(x^i)$$

*for every $x, y \in G$. Then $\varphi_N$ is a generalized polynomial of degree at most equal to*

$$\sum_{i=0}^{M+K} \operatorname{card}(\bigcup_{s=i}^{M+K} I_s) - 1.$$

Now, let us look at the equation ($E_\lambda$) and suppose that $\lambda = 0$ (let us note that changing $\lambda$ into $1 - \lambda$ does not change the right-hand side, thus the case $\lambda = 1$ is analogous).

The equation ($E_0$) reads as follows

$$f(y) - f(x) = (y - x)\left[g(y) + g(x)\right],$$

or, after introducing the new variable $z = y - x$ and reordering,

$$g(x)z + f(x) = g(x + z)z + f(x + z).$$

From the above Lemma we get the following: $g$ is a polynomial function and

$$\deg g \leq 1.$$

Hence $g(x) = A(x) + b$, where $A$ is an additive function and $b$ is a constant. Setting $x = 0$ into ($E_0$), we get $f(y) = f(0) + 2by + A(y)y$. Let us insert the forms of $f$ and $g$ into ($E_0$), and replace $y$ by $x + z$. We get

$$A(x)z + A(z)x + A(z)z + 2bz = 2A(x)z + A(z)z + 2bz,$$

or

$$A(z)x = A(x)z,$$

for all $x, z \in \mathbb{R}$, whence it follows that for some constant $a$ we have $A(x) = ax$. Consequently, $g(x) = ax + b$, and, denoting $c = f(0)$, we have $f(x) = ax^2 + bx + c$.

Assume now that $\lambda \in (0, 1)$. Write ($E_\lambda$) in the form

$$f(x + z) - f(x) = z[g(x + \lambda z) + g(x + (1 - \lambda)z)].$$

Using again the Lemma 1 from [6] one can easily show that $g$ and $f$ are polynomial functions of degrees at most 3, and 4, respectively. Let us write

$$g(x) = b_0 + b_1(x) + b_2^*(x) + b_3^*(x),$$

$$f(x) = a_0 + a_1(x) + a_2^*(x) + a_3^*(x) + a_4^*(x),$$

where $a_0$ and $b_0$ are constants, $a_1$ and $b_1$ are additive, $a_i^*$, $i \in \{2, 3, 4\}$, as well as $b_j^*$,
$j \in \{2, 3\}$ are diagonalizations of $i$-additive, resp. $j$-additive and symmetric functions.

Inserting the above forms into ($E_\lambda$) and comparing the respective terms we easily get that

- $a_0$ may be arbitrary;

- $a_1(x) = b_0 x$.

Looking for the form of $a_2^*$, which is a diagonalization of a biadditive and symmetric function $A_2$, we get the following relations

$$A_2(x,z) = xb_1(z) = zb_1(x),$$

for all $x, z \in \mathbb{R}$. It follows that there exists a constant, say $\beta$, such that $b_1(x) = \beta x$, and hence

$$A_2(x,z) = \beta xz.$$

Let us look now for the terms $a_3^*$ and $b_2^*$, being diagonalizations of $A_3$ and $B_2$, respectively. After inserting them into $(\mathrm{E}_\lambda)$, and performing the obvious reductions we compare the terms of the same degree and eventually obtain the following system of equations.

$$\begin{cases} 3A_3(x,x,y) = 2xB_2(x,y) \\ 3A_3(x,y,y) = 2xB_2(y,y) \\ A_3(x,x,x) = x[B_2(\lambda x, \lambda x) + B_2((1-\lambda)x, (1-\lambda)x)]. \end{cases}$$

The second equation implies

$$3A_3(x,x,y) = 2yB_2(x,x),$$

and together with the first one, after dividing both sides by $x^2 y$ for $x, y \neq 0$

$$\frac{B_2(x,x)}{x^2} = \frac{B_2(x,y)}{xy},$$

whence it follows (after changing $x$ and $y$ on the right-hand side) that there exists a constant $\gamma$ such that

$$B_2(x,x) = \gamma x^2.$$

It is clear that also

$$B_2(x,y) = \gamma xy,$$

and *a fortiori*

$$A_3(x,y,z) = \frac{2}{3}\gamma xyz.$$

Inserting these to the third equation of the system above, we obtain

$$\frac{2}{3}\gamma x^3 = \gamma\left(2\lambda^2 - 2\lambda + 1\right)x^3.$$

Thus either $\gamma = 0$ or $2\lambda^2 - 2\lambda + \frac{1}{3} = 0$, which yields $\lambda \in \{\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\}$.
Finally, let us investigate the terms $a_4^*$ and $b_3^*$ which are diagonalizations of $A_4$ and $B_3$, respectively. Inserting the forms into $(\mathrm{E}_\lambda)$ and comparing the terms we obtain the following system of equations.

$$2A_4(x,x,x,z) = zB_3(x,x,x). \tag{5}$$

$$2A_4(x,x,z,z) = zB_3(x,x,z).$$

(6) $$4A_4(x,z,z,z) = z\left[6B_3(x,\lambda z,\lambda z) + 3B_3(x,z,z) - 6B_3(x,z,\lambda z)\right].$$

(7)
$$A_4(z,z,z,z) = z\left[B_3(z,z,z) - 3B_3(z,z,\lambda z) + 3B_3(z,\lambda z,\lambda z) - B_3(\lambda z,\lambda z,\lambda z)\right].$$

Putting $x = z$ in (5) and inserting the obtained equality into (6) we obtain, after dividing both sides by $z \neq 0$,

$$2B_3(z,z,z) = 6B_3(z,\lambda z,\lambda z) + 3B_3(z,z,z) - 6B_3(z,z,\lambda z),$$

whence it follows that

(8) $$0 = 6B_3(z,\lambda z,\lambda z) - 6B_3(z,z,\lambda z) + B_3(z,z,z),$$

for all $z \neq 0$. Now, putting $x = z$ into (6) and comparing with (7) we obtain, after dividing by $z \neq 0$,

$$\frac{3}{2}B_3(z,\lambda z,\lambda z) - \frac{3}{2}B_3(z,z,\lambda z) + \frac{3}{4}B_3(z,z,z) = \\ B_3(z,z,z) - 3B_3(z,z,\lambda z) + 3B_3(z,\lambda z,\lambda z) - B_3(\lambda z,\lambda z,\lambda z),$$

whence, after rearrangement, we obtain

$$0 = 6B_3(z,\lambda z,\lambda z) - 6B_3(z,z,\lambda z) + B_3(z,z,z) - 4B_3(\lambda z,\lambda z,\lambda z).$$

Taking into account (8), we see that

$$B_3(\lambda z,\lambda z,\lambda z) = 0,$$

for all $z \in \mathbb{R}$. This implies obviously that $B_3 = 0$ and also $A_4 = 0$. Putting $d = \gamma$, $a = \beta$, $b = a_0$ and $c = a_0$ we get the formulae from the assertion.

It is a matter of simple calculation that the functions defined in assertion, actually satisfy the equation $(\mathrm{E}_\lambda)$. Hence the proof is finished.

Now, we shall continue our study considering the family $\Lambda$ in question that consists of two abstract quasi-arithmetic means.

THEOREM 4. *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ and $\psi : \mathbb{R}$ be continuous and strictly monotonic functions generating the quasi-arithmetic and not arithmetic means*

$$m_1(x,y) := \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) \quad \textit{and} \quad m_2(x,y) := \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right), \ x,y \in \mathbb{R}.$$

*If*

$$m_1(x,y) + m_2(x,y) = x + y \quad \textit{for all} \quad x,y \in \mathbb{R},$$

*then each pair* $(g,h)$ *of functions from* $\mathbb{R}$ *into an Abelian group* $(X,+)$ *satisfying equation* $(E_{1,2})$ *has to be of the form*

$$g(x)=\ell\left(e^{px}\right)+c \quad \text{and} \quad h(x)=\ell\left(e^{px}\right)+2c \quad \text{for all} \quad x\in\mathbb{R}, \tag{9}$$

*where* $\ell:(0,\infty)\longrightarrow X$ *is a logarithmic function, that is a solution to the functional equation*

$$\ell(st)=\ell(s)+\ell(t), \quad s,t\in(0,\infty),$$

$p\neq 0$ *is a real constant and* $c$ *stands for a fixed element of* $X$*.*

*Conversely, for any logarithmic function* $\ell:(0,\infty)\longrightarrow X$ *and every constants* $p\in\mathbb{R}\setminus\{0\}$, $c\in X$, *the functions* $g$ *and* $h$ *given by* (9) *satisfy equation* $(\mathrm{E}_{1,2})$ *with the means* $m_1,m_2$ *spoken of.*

Proof. A very deep result proved by Z. Daróczy and Zs. Páles in [2] states that whenever the equality

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=x+y$$

holds for all $x,y\in\mathbb{R}$ and these two quasi-arithmetic means are not the arithmetic ones, there exists a nonzero real constant $p$ such that

$$\varphi(x)=\alpha e^{px}+\beta \quad \text{and} \quad \psi(x)=\gamma e^{-px}+\delta,\ x\in\mathbb{R},$$

for some constants $\alpha,\gamma\in\mathbb{R}\setminus\{0\}$ and $\beta,\delta\in\mathbb{R}$. Hence, for all $x,y$ from $\mathbb{R}$ we have

$$h(x+y)=g\left(\frac{1}{p}\log\left(\frac{e^{px}+e^{py}}{2}\right)\right)+g\left(\frac{1}{p}\log\left(\frac{e^{-px}+e^{-py}}{2}\right)^{-1}\right).$$

Setting here $H:=h\circ\frac{1}{p}\log$, $G:=g\circ\frac{1}{p}\log$ and $s:=e^{px}$, $t:=e^{py}$ we obtain the equation

$$H(st)=G\left(\frac{s+t}{2}\right)+G\left(\frac{2st}{s+t}\right) \tag{B}$$

valid for all $s,t\in(0,\infty)$.

Now, fix arbitrarily $u,v\in(0,\infty)$ with $u\le v$ and put

$$s:=v-\sqrt{v^2-uv} \quad \text{and} \quad t:=v+\sqrt{v^2-uv}$$

in (B) to get

$$H(uv)=G(v)+G(u).$$

Due to the commutativity of the group $(X,+)$ and the resulting symmetry of the roles of $u$ and $v$ in the latter equation we derive its validity for all $u,v\in(0,\infty)$. Setting $\ell:=G-G(1)$ we easily check that the map $\ell$ is logarithmic. It is a straightforward matter to check the validity of the equalities (9). Since the verification of the sufficiency presents no difficulties the proof has been completed.

Remark 1. Equation (B) written in the form

$$H_1(\sqrt{uv}) = G\left(\frac{u+v}{2}\right) + G\left(\frac{2uv}{u+v}\right) \tag{B$_1$}$$

where $H_1 := H \circ (\cdot)^2$ , seems to be of interest since it ties the three basic quasi-arithmetic means: geometric, arithmetic and harmonic ones. Its full pexiderization, that is the equation

$$H_1(\sqrt{uv}) = G\left(\frac{u+v}{2}\right) + K\left(\frac{2uv}{u+v}\right) \tag{B$_2$}$$

may be solved in a similar way as it was done in the case of ($B_1$) which is a version of (B). The general solution of ($B_2$) reads as follows:

$$H_1(u) = 2\ell(u) + c + d, \quad G(u) = \ell(u) + c, \quad K(u) = \ell(u) + d,\ u \in (0, \infty),$$

where $\ell$ stands for an arbitrary logarithmic function and $c, d$ are arbitrary real constants.

Quadratic "polynomialsąre also characterized by means of the following theorem being a variant of M. Kuczma's result from [5, Theorem 1].

Theorem 5. *Let $X$ be a real linear space and let functions $f : \mathbb{R} \longrightarrow X$ and $g : \mathbb{R} \longrightarrow X$ satisfy the equation*

$$\frac{f(y) - f(x)}{y - x} = h(x) + h(y)$$

*for all pairs $(x, y) \in \mathbb{R}^2$ such that $x \neq y$, then there exist points $a, b, c \in X$ such that*

$$f(x) = ax^2 + bx + c, \quad x \in \mathbb{R},$$

*and*

$$h(x) = ax + \frac{1}{2}b, \quad x \in \mathbb{R}.$$

Proof. Fix arbitrarily a real linear functional $\ell$ from the algebraic dual $X'$ of the space $X$ and put $\varphi := \ell \circ f_0$ with $f_0 := f - f(0)$. Let $\psi := \ell \circ h$; then

$$\frac{\varphi(y) - \varphi(x)}{y - x} = \psi(x) + \psi(y)$$

for all pairs $(x, y) \in \mathbb{R}^2$ such that $x \neq y$. By means of Kuczma's characterization theorem from [5], there exist real numbers $a(\ell)$ and $b(\ell)$ such that

$$\varphi(x) = a(\ell)x^2 + b(\ell)x \quad \text{and} \quad \psi(x) = a(\ell)x + \frac{1}{2}b(\ell)$$

for all $x \in \mathbb{R}$. Since $\varphi(1) = a(\ell) + b(\ell)$ and $\psi(0) = \frac{1}{2}b(\ell)$ we infer that

$$\begin{aligned}\ell(f_0(x)) &= (\varphi(1) - 2\psi(0))x^2 + 2\psi(0)x \\ &= (\ell(f_0(1)) - 2\ell(h(0)))\,x^2 + 2\ell(h(0))x = \ell((f_0(1) - 2h(0))x^2 + 2h(0)x)\end{aligned}$$

for every $x \in \mathbb{R}$. Due to the unrestricted choice of the functional $\ell \in X'$ and the fact that $X'$ separates points, we conclude that

$$f_0(x) = ax^2 + bx,\ x \in \mathbb{R}, \quad \text{where} \quad a := f_0(1) - 2h(0) \ \text{ and } \ b := 2h(0).$$

Consequently,

$$f(x) = ax^2 + bx + c,\ x \in \mathbb{R}, \quad \text{where} \quad c := f(0).$$

Moreover, we have also,

$$h(x) + h(0) = \frac{f(x) - c}{x} = \frac{ax^2 + bx}{x} = ax + b \ \text{ for all } \ x \in \mathbb{R} \setminus \{0\},$$

as well as

$$2(b - h(0)) = h(x) + h(-x) = \frac{f(x) - f(-x)}{2x} = b \quad \text{for all} \ x \in \mathbb{R} \setminus \{0\},$$

whence

$$h(0) = \frac{1}{2}b \qquad \text{and} \qquad h(x) = ax + \frac{1}{2}b \ \text{ for all } \ x \in \mathbb{R}.$$

This finishes the proof.

Keeping in mind that we wish to characterize quadratic "polynomials", by means of Theorem 5 above, we may force $f$ to satisfy the functional equation

$$\frac{f(y) - f(x)}{y - x} = h(x) + h(y)$$

with some new unknown function $h$ which jointly with (2) leads to yet another Pexider type functional equation

$$(\mathrm{E}_{3,4}) \qquad h(x) + h(y) = g(m_3(x, y)) + g(m_4(x, y)).$$

Before stating a suitable result we have to prove the following

Lemma. *Let $\varphi : (0, \infty) \longrightarrow \mathbb{R}$ be either monotonic or continuous solution to the functional equation*

$$(\mathrm{B}') \qquad \varphi(s) + \varphi(t) = \varphi\left(\frac{s + t}{2}\right) + \varphi\left(\frac{2st}{s + t}\right) \quad \textit{for all} \quad s, t \in (0, \infty).$$

*Then there exist real constants $\alpha$ and $\beta$ such that*

$$\varphi(t) = \alpha \log t + \beta \quad \textit{for all} \quad t \in (0, \infty).$$

Proof. Observe that each increasing (resp. decreasing) solution of equation (B′) is concave (resp. convex) and hence automatically continuous because of the openness of the interval $(0,\infty)$. Indeed, assuming, without loss of generality, that a solution $\varphi : (0,\infty) \longrightarrow \mathbb{R}$ is increasing and taking into account that the aritmetic and harmonic means

$$A(s,t) := \frac{s+t}{2}, \qquad H(s,t) := \frac{2st}{s+t}, \qquad s,t \in (0,\infty),$$

are comparable in the sense that $H \le A$, we infer that

$$\varphi(s)+\varphi(t) = \varphi(A(s,t)) + \varphi(H(s,t)) \le 2\varphi(A(s,t)) = 2\varphi\left(\frac{s+t}{2}\right), \quad s,t \in (0,\infty).$$

This proves that $\varphi$ is Jensen-concave and being Lebesgue measurable (because of the monotonicity) enjoys the continuity property. Therefore, in what follows we may restrict ourselves to the continuous solutions of equation (B′). A natural iterating procedure applied for this equation leads to

$$\begin{aligned}\varphi(s)+\varphi(t) &= \varphi(A(s,t)) + \varphi(H(s,t))\\ &= \varphi(A(A(s,t)),H(s,t))) + \varphi(H(A(s,t),H(s,t))) = ...\end{aligned}$$

Since both means in question are strict, their Gausian-compositions $A \otimes H = H \otimes A$, i.e. the limits of the Gauss-iterations do exist and both are equal to the geometric mean $(0,\infty)^2 \ni (s,t) \longmapsto \sqrt{st}$ (see e.g. Z. Dar/oczy-Zs. Páles dissertation [2, Theorem 1.5]). Consequently, due to the continuity of $\varphi$ we conclude that

$$\varphi(s)+\varphi(t) = 2\varphi\left(\sqrt{st}\right) \quad \text{for all} \quad s,t, \in (0,\infty).$$

Setting here $\beta := \varphi(1)$ we obtain easily that $\varphi(t) = 2\varphi\left(\sqrt{t}\right) - \beta$ whence $\varphi(t^2) = 2\varphi(t) - \beta$ for all $t \in (0,\infty)$. This forces $\varphi$ to satisfy the equalities

$$2\varphi(s) + 2\varphi(t) = \varphi(s^2) + \beta + \varphi(t^2) + \beta = 2\varphi(st) + 2\beta \quad \text{for all} \quad s,t \in (0,\infty)\,,$$

stating that the map $\ell := \varphi - \beta$ is logarithmic. It is well known that each continuous real logarithmic function on positive half-line has to have the form

$$\ell(t) = \alpha \log t, \quad t \in (0,\infty).$$

Finally,

$$\varphi(t) = \alpha \log t + \beta \quad \text{for all} \quad t \in (0,\infty),$$

as claimed.

Observe that for any real constants $\alpha, \beta$ and each logarithmic function $\ell : (0,\infty) \longrightarrow \mathbb{R}$ the map $\alpha\ell + \beta$ yields a solution to (B′). Unfortunately, at present we are unable to prove that there are no other solutions unless some regularity conditions (e.g. monotonicity or continuity) are imposed upon the unknown function.

Remark 2. Without any regularity conditions, equation (B′) admits no nonlogarithmic (modulo compositions with affine functions) solutions $\varphi : (0,\infty) \longrightarrow \mathbb{R}$ provided that:

a) $\varphi\left(\frac{1}{t}\right) = -\varphi(t)$, $x \in (0,\infty)$ (oddness with respect to the multiplicative structure of the domain),

b) the map $(0,\infty) \ni t \longmapsto \varphi(2t) - \varphi(t)$ is constant (a kind of 2-homogeneity of order 0).

In fact, b) with the corresponding constant function $c$ implies that $\varphi(\frac{1}{2}t) = \varphi(t) - c$, $t \in (0,\infty)$, which allows to transform equation (B′) to the following form:

$$\varphi(s)+\varphi(t) = \varphi(s+t)-c+\varphi\left(\frac{st}{s+t}\right)+c = \varphi(s+t)+\varphi\left(\frac{st}{s+t}\right) \text{ for all } s,t \in (0,\infty).$$

Therefore, by means of a), one has

$$\varphi(s)+\varphi(t)-\varphi(s+t) = \varphi\left(\frac{1}{\frac{1}{x}+\frac{1}{y}}\right) = -\varphi\left(\frac{1}{x}+\frac{1}{y}\right),$$

i.e.

$$\varphi(s+t)-\varphi(s)+\varphi(t) = \varphi\left(\frac{1}{x}+\frac{1}{y}\right) \text{ for all } x,y \in (0,\infty).$$

It remains to apply a nice result of K. Heuvers [4] to conclude that $\varphi$ has to be a logarithmic function.

It seems us that the assumptions a) and b) might be derived from equation (B′) but at present we are unable to settle it.

Theorem 6. *Let $X$ stand for a real locally convex linear topological space and let $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ and $\eta : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and strictly monotonic functions generating the quasi-arithmetic and not arithmetic means*

$$m_3(x,y) := \xi^{-1}\left(\frac{\xi(x)+\xi(y)}{2}\right) \quad \text{and} \quad m_4(x,y) := \eta^{-1}\left(\frac{\eta(x)+\eta(y)}{2}\right), \; x,y \in \mathbb{R}.$$

*If*

$$m_3(x,y) + m_4(x,y) = x + y \quad \text{for all} \quad x,y \in \mathbb{R},$$

*then each pair $(g,h)$ of continuous functions from $\mathbb{R}$ into $X$ satisfying equation $(E_{3,4})$ has to be of the form*

$$g(x) = h(x) = ax + b \quad \text{for all} \quad x \in \mathbb{R}, \tag{10}$$

*where $a$ and $b$ are constant vectors from $X$.*

*Conversely, for arbitrarily fixed vectors $a$ and $b$ from $X$ the functions $g$ and $h$ given by* (10) *satisfy equation* ($\mathrm{E}_{3,4}$) *with the means $m_3, m_4$ spoken of.*

Proof. Applying Z. Daróczy's and Zs. Páles' fundamental result from [2, Theorem 4.12 ] again, and proceeding along the same lines as in the proof of Theorem 4, we arrive at

$$h(x) + h(y) = g\left(\frac{1}{p}\log\left(\frac{e^{px}+e^{py}}{2}\right)\right) + g\left(\frac{1}{p}\log\left(\frac{e^{-px}+e^{-py}}{2}\right)^{-1}\right)$$

with some nonzero real constant $p$.

Setting here $H := h\circ\frac{1}{p}\log$, $G := g\circ\frac{1}{p}\log$ and $s := e^{px}$, $t := e^{py}$ we obtain the equation

$$H(s) + H(t) = G\left(\frac{s+t}{2}\right) + G\left(\frac{2st}{s+t}\right)$$

valid for all $s, t \in (0,\infty)$. On putting here $s = t$ we infer that $H = G$, i.e. we are faced to the equation

$$(\mathrm{B}') \qquad G(s) + G(t) = G\left(\frac{s+t}{2}\right) + G\left(\frac{2st}{s+t}\right),$$

satisfied for all $s, t \in (0,\infty)$.

Put $G_0 := G - G(1)$ and fix arbitrarily a member $x^*$ of the space $X^*$ dual to $X$. Then the real map $\varphi := x^*\circ G_0$ yields a continuous solution to the equation (B$'$). On account of the Lemma, there exist real constants $\alpha(x^*)$ and $\beta(x^*)$ such that

$$\varphi(t) = \alpha(x^*)\log t + \beta(x^*)$$

for all $t \in (0,\infty)$. Since $G_0$ vanishes at 1, so does the functional $\varphi$ which forces $\beta(x^*)$ to vanish as well. Consequently,

$$\varphi(t) = \alpha(x^*)\log t \quad \text{for all} \quad t \in (0,\infty).$$

Obviously, $\alpha(x^*) = \varphi(e)$ which implies that

$$x^*(G_0(t)) = \varphi(t) = \varphi(e)\log t = x^*(G_0(e))\log t = x^*(G_0(e)\log t) \quad \text{for all} \quad t \in (0,\infty).$$

Due to the local convexity of $X$ the family $X^*$ separates points and the unrestricted choice of the functional $x^*$ jointly with the latter equality imply that for every $t \in (0,\infty)$ one has

$$G(t) - G(1) = G_0(t) = G_0(e)\log t, \quad \text{i.e.} \quad G(t) = a_0\log t + b,$$

where we have put $a_0 := G_0(e)$ and $b := G(1)$. Thus $a_0\log t + b = G(t) = g\left(\frac{1}{p}\log t\right)$ for all $t \in (0,\infty)$ and, finally, we get $g(x) = ax + b$, $x \in \mathbb{R}$, with $a := p\,a_0$.

Since the sufficiency is trivial, the proof has been completed.

Remark 3. Like in J. Ger's paper [3], in a similar context several of the results discussed may be proved with the aid of her methods for functions defined on a proper subinterval of $\mathbb{R}$ or $(0, \infty)$.

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