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Convergence results for a class of pramarts and superpramarts in Banach spaces

Dedicated to Professor Julian Musielak on the occasion of his 85th birthday

Abstract. Please insert here a short abstract.

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1. Introduction. In this paper we state several convergence results for vector-valued pramarts and convex weakly compact valued pramarts-sub-superpramarts in a separable Banach space. There is an abundant bibliography on this subject, see [2, 9, 13, 15, 16, 18, 23, 24, 26, 27, 28, 29, 32] and the references therein. We provide here a comprehensive presentation of the subject under consideration by using results for real-valued submartingales-pramarts [18, 26], Convex Analysis [11], Young measures [10] and other tools. In particular, this study contains famous a.s. norm convergence results in [15, 16, 18, 23, 26, 28] dealing with vector-valued martingales and vector-valued pramarts in the separable dual of a separable Banach space. Main results are given in Sects 4-5-6-7-8-9 where several convergence results for the aforementioned objects are presented. The paper is organized as follows. In section 2 we summarize some basic properties of the Mosco convergence [25], Hausdorff topology and linear topology [4] in the space of closed convex subsets in a Banach space. In section 3 we state and summarize for reference the multivalued conditional expectation of closed convex valued integrable multifunctions. In section 4 we provide some versions of the biting compactness theorem for both the space L^1_E of Bochner integrable mappings in a separable Banach space E. These results are of importance in the problem of identifying the limits in the pramart convergence problems and further related results. In section 5 we provide the a.s. norm convergence for vector-valued pramarts in separable Banach space via the biting compactness method. Section 6 is devoted to the a.s. convergence with respect to the linear topology for convex weakly compact valued pramarts when the underlying Banach space is separable and its strong dual is separable. Here we also focus on further convergence results related to the structure of the pramarts, namely, the pointwise supremum property and the a.s. pointwise convergence with respect to the Hausdorff excess. In section 7 we discuss the notion of sub-superpramart for convex weakly compact valued mappings and state new convergence results for these objects extending some results in the literature dealing with convex weakly compact valued supermartingales. Section 8 is devoted to a.s. norm convergence of pramarts in the dual of a separable Banach space. In section 9 we state an unusual convergence result for bounded positive submartingales in separable order continuous Banach lattice via a renorming lattice norm [15] and other related tools.Notations and preliminaries Throughout this paper $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(\mathcal{F}_n)_{n\in\mathbf{N}}$ is an increasing sequence of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}$ is the σ-algebra generated by $\cup_{n\in N}\mathcal{F}_n$. E is a separable Banach space and E^* is its topological dual. Let $\overline{B}_E$ (resp. $\overline{B}_{E^*}$)e the closed unit ball of E (resp. E^*) and 2^E the collection of all subsets of E. Let $c(E)$ (resp. $cc(E)$) (resp. $cwk(E)$) (resp. $\mathcal{R}wk(E)$) be the set of nonempty closed (resp. closed convex)resp. convex weak compact) (resp. closed convex *ball-weakly compact*) subsets of E, here a closed convex subset in E is *ball-weakly compact* if its intersection with any closed ball in E is weakly compact. For $A \in cc(E)$, the distance function and the support function associated with A are defined respectively by

$$d(x, A) = \inf\{\|x - y\|\, y \in A\}, (x \in E)$$

$$\delta^*(x^*, A) = \sup\{\langle x^*, y\rangle\ : y \in A\}, (x^* \in E^*).$$

e also define

$$|A| = \sup\{||x||\ : x \in A\}.$$

Let us denote by e_H the Hausdorff excess and d_H the Hausdorff distance defined on $cc(E)$ associated with the topology of the norm in E. Given a sub-σ-algebra $\mathcal{B}$ in Ω, a mapping $X : \Omega \to 2^E$ is $\mathcal{B}$-measurable if for every open set U in E the set

$$X^{-}U := \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$$

is a member of $\mathcal{B}$. A function $f : \Omega \to E$ is a $\mathcal{B}$-measurable selection of X if $f(\omega) \in X(\omega)$ for all $\omega \in \Omega$. A Castaing representation of X is a sequence $(f_n)_{n\in\mathbf{N}}$ of $\mathcal{B}$-measurable selections of X such that

$$X(\omega) = cl\{f_n(\omega), n \in \mathbf{N}\} \quad \forall w \in \Omega$$

where the closure is taken with respect to the topology of associated with the norm in E. It is known that a nonempty closed-valued multifunction $X : \Omega \to c(E)$ is $\mathcal{B}$-measurable iff it admits a Castaing representation. If $\mathcal{B}$ is complete, the $\mathcal{B}$-measurability is equivalent to the measurability in the sense of graph, namely the graph of X is a member of $\mathcal{B} \otimes \mathcal{B}(E)$, here $\mathcal{B}(E)$ denotes the Borel tribe on E. A $cc(E)$-valued $\mathcal{B}$-measurable $X : \Omega \to cc(E)$ is integrable if the set $S^1_X(\mathcal{B})$ of all $\mathcal{B}$-measurable and integrable selections of X is nonempty. We denote by $L^1_E(\mathcal{B})$ the space of E-valued $\mathcal{B}$-measurable and Bochner-integrable functions defined on Ω

and $\mathcal{L}^1_{cwk(E)}(\mathcal{B})$ the space of all $\mathcal{B}$-measurable multifunctions $X : \Omega \to cwk(E)$, such that $|X| \in L^1_{\mathbf{R}}(\mathcal{B})$. A sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is bounded (resp. uniformly integrable) f the sequence $(|X_n|)_{n\in\mathbf{N}}$ is bounded (resp. uniformly integrable) in $L^1_{\mathbf{R}}(\mathcal{F})$. $\cong$ Acc(E)$-valuedequence$(X$_n)_{n\in\mathbf{N}}$osco-convergesMo to $X_\infty \in cc(E)$f

$$X_\infty = s\text{-}li X_n = w\text{-}ls X_n$$

where

$$s\text{-}li\, X_n = \{x \in E : ||x_n - x|| \to 0,\ x_n \in X_n\}$$

and

$$w\text{-}ls\, X_n = \{x \in E : x = w\text{-}\lim_{j\to\infty} x_{n_j},\ x_{n_j} \in X_{n_j}\}$$

and s (resp. w) is the strong (resp. weak) topology in E. If $(X_n)_{n\in\mathbf{N}}$ Mosco-converges to X_∞ in $cc(E)$, we write

$$M\text{-}\lim_{n\to\infty} X_n = X_\infty.$$

A $cc(E)$-valuedequence $(X_n)_{n\in\mathbf{N}}$onvergeso $X_\infty \in cc(E)$ with respect to the linear topology τ_L [4]f

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad \forall x^* \in E^*.$$

$$\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad \forall x \in E.$$

Beer showed that the τ_L-topology is stronger than the Mosco-topology. We refer to [11] for the theory of Measurable Multifunctions and Convex Analysis, and to [18, 26] for basic theory of martingales and adapted sequences.

2. Multivalued conditional expectation theorem. Given a sub-σ-algebra, $\mathcal{B}$ of $\mathcal{F}$ and an integrable $\mathcal{F}$-measurable $cc(E)$-valued multifunction $X : \Omega \rightrightarrows E$: i.e. $d(0, X)$ is integrable, Hiai and Umegaki [22] showed the existence of a $\mathcal{B}$-measurable $cc(E)$-valued integrable mutifunction, denoted by $E^{\mathcal{B}}X$ such that

$$\mathcal{S}^1_{E^{\mathcal{B}}X}(\mathcal{B}) = cl\{E^{\mathcal{B}}f : f \in \mathcal{S}^1_X(\mathcal{F})\}$$

the closure being taken in $L^1_E(\Omega, \mathcal{A}, P)$; $E^{\mathcal{B}}X$ is the multivalued conditional expectation of X relative to $\mathcal{B}$. If $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$, and the strong dual E^*_b is separable, then $E^{\mathcal{B}}X \in \mathcal{L}^1_{cwk(E)}(\mathcal{B})$ with $\mathcal{S}^1_{E^{\mathcal{B}}X}(\mathcal{B}) = \{E^{\mathcal{B}}f : f \in \mathcal{S}^1_X(\mathcal{F})\}$. This result was stated by the first author. A unified approach for general conditional expectation of $cc(E)$-valued integrable multifunctions is given in [31] allowing to recover both the $cc(E)$-valued conditional expectation of $cc(E)$-valued integrable multifunctions in the sense of [22] and the $cwk(E)$-valued conditional expectation of $cwk(E)$-valued integrably bounded multifunctions given in [5]. For more information on multivalued conditional expectation and related subjects we refer to [1, 8, 11, 22, 31]. In the context of this paper we present a specific version of conditional expectation that we summarize below. A fairly general version of conditional expectation for closed convex integrable random sets in the dual of a separable Fréchet space is obtained by Valadier [31, Theorem 3]. Here we need only a special version of this result in the dual space E^*_s, where E^*_s is the vector space E^* endowed with the weak star topology $\sigma(E^*, E)$.

THEOREM 2.1 *Let E be a separable Banach space and let Γ be a closed convex valued integrable random set in E^*_s: i.e. $d_{E^*_b}(0,\Gamma)$ is integrable. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$. Then there exist a closed convex $\mathcal{B}$-measurable mapping Σ in E^*_s such that:*
1) Σ is the smallest closed convex $\mathcal{B}$-measurable mapping Θ such that $\forall u \in S^1_\Gamma$, $E^{\mathcal{B}}u(\omega) \in \Theta(\omega)$ a.s.
2) Σ is the unique closed convex $\mathcal{B}$-measurable mapping Θ such that $\forall v \in L^\infty_E(\mathcal{B})$,

$$\int_\Omega \delta^*(v,\Gamma)dP = \int_\Omega \delta^*(v,\Theta)dP.$$

3) Σ is the unique closed convex $\mathcal{B}$-measurable mapping such that $S^1_\Sigma = cl\, E^{\mathcal{B}}(S^1_\Gamma))$ where cl denotes the closure with respect to $\sigma(L^1_{E^*}(\mathcal{B}), L^\infty_E(\mathcal{B}))$.

Theorem 2.1 allows to treat the conditional expectation of convex weakly compact valued integrably bounded mappings.

PROPOSITION 2.2 *Assume that E^*_b is separable. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and an integrably bounded $\mathcal{F}$-measurable $cwk(E)$-valued multifunction $X : \Omega \Rightarrow E$. Then there is a unique (for the equality a.s.) $\mathcal{B}$-measurable $cwk(E)$-valued multifunction Y satisfying the property*

$$(*) \qquad \forall v \in L^\infty_{E^*}(\mathcal{B}), \int_\Omega \delta^*(v(\omega), Y(\omega))P(d\omega) = \int_\Omega \delta^*(v(\omega), X(\omega))P(d\omega).$$

$E^{\mathcal{B}}X := Y$ is the conditional expectation of X.

PROOF Indeed if $F := E^*_b$ is *separable* and if Γ is a convex weakly compact valued measurable mapping in E with $\Gamma(\omega) \subset \alpha(\omega)\overline{B}_E$ where $\alpha \in L^1_{\mathbf{R}}$, then applying Theorem 2.1 to F^* gives $\Sigma(\omega) = E^{\mathcal{B}}\Gamma(\omega) \subset E^{**}$ with $\Sigma(\omega) \subset E^{\mathcal{B}}\alpha(\omega)\overline{B}_{E^{**}}$ where $\overline{B}_{E^{**}}$ is the closed unit ball in E^{**}. As S^1_Γ is $\sigma(L^1_E, L^\infty_{E^*})$ compact, $S^1_\Sigma = E^{\mathcal{B}}(S^1_\Gamma) \subset L^1_E$. Whence $\Sigma(\omega) \subset E$ a.s. See [31, Remark 4, page 10] for details. ■

3. Biting Compactness Theorem. We state and summarize some biting compactness convergence results in the space $L^1_E(\mathcal{F})$ which will be used in the problem of identification of the limit occurring in the convergence of pramarts in Banach space.

THEOREM 3.1 *Suppose that E is a separable and $(X_n)_{n\in\mathbf{N}}$ is bounded and $\mathcal{R}wk(E)$-tight sequence $L^1_E(\mathcal{F})$, that is, for every $\varepsilon > 0$, there is a $\mathcal{R}wk(E)$-valued measurable mapping $L_\varepsilon : \Omega \Rightarrow E$ such that*

$$\forall n \in \mathbf{N}, P([\omega \in \Omega : X_n(\omega) \notin L_\varepsilon(\omega)]) \leq \varepsilon.$$

Then there exist an increasing sequence $(A_p)_{p\in\mathbf{N}}$ in $\mathcal{F}$ with $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(Z_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ and $X_\infty \in L^1_E(\mathcal{F})$ such that, for each $p \in \mathbf{N}$, and for each $g \in L^\infty_{E^*}(A_p \cap \mathcal{F})$, the following holds:

$$\lim_{n\to\infty} \int_{A_p} \langle g, Z_n\rangle\, dP = \int_{A_p} \langle g, X_\infty\rangle\, dP.$$

Proof As $(X_n)_{n\in\mathbf{N}}$ is bounded in $L^1_E(\mathcal{F})$, a well-known biting argument cf. [6, 10, 18, 19, 28] provides an increasing sequence $(A_p)_{p\in\mathbf{N}}$ in $\mathcal{F}$ such that $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(Y_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ such that the restrictions $(Y_n|A_p)$ to each A_p is uniformly integrable, so that, by the tightness condition $(Y_n|A_p)$ is relatively weakly compact in $L^1_E(\mathcal{F}\cap A_p)$ (cf. [3], Theorem 1). By successive applications of this weak compactness result, for each $p \in \mathbf{N}$, we provide a sequence $(Y^n_p)_{n\in\mathbf{N}}$ such that $(Y^n_{p+1})_{n\in\mathbf{N}}$ is extracted from $(Y^n_p)_{n\in\mathbf{N}}$ and $W_p \in L^1_E(A_p\cap\mathcal{F})$ and that

$$\lim_{n\to\infty}\int_{A_p}\langle g, Y^n_p\rangle dP = \int_{A_p}\langle g, W_p\rangle dP$$

for each $g \in L^\infty_{E^*}(A_p\cap\mathcal{F})$. Let us set $Z_n = Y^n_n$ for all $n \in \mathbf{N}$. Then, for each $p \in \mathbf{N}$, $(Z_n|A_p)$ converges $\sigma(L^1_E, L^\infty_{E^*})$ to $W_p \in L^1_E(A_p\cap\mathcal{F})$. As $(A_p)_{p\in\mathbf{N}}$ is increasing $W_p = W_{p+1}$ a.s. on A_p. Let us define $X_\infty = W_p$ if $\omega \in A_p$ and $X_\infty = 0$ elsewhere. Clearly X_∞ is $\mathcal{F}$ measurable and belongs to $L^1_E(\mathcal{F})$. Indeed, we have

$$(*)\qquad \int_\Omega |X_\infty| dP = \int_{\cup_p A_p} |X_\infty| dP$$
$$= \sup_{p\in\mathbf{N}}\int_{A_p}|X_\infty|dP = \sup_{p\in\mathbf{N}}\int_{A_p}|W_p|dP. \qquad \blacksquare$$

But, for each $p \in \mathbf{N}$, $(Z_n|A_p)_{i\in\mathbf{N}}$ converges $\sigma(L^1_E, L^\infty_{E^*})$ to W_p, by the weak lower semicontinuity of the functional $f \mapsto \int |f|dP (f \in L^1_E)$ we deduce that

$$(**)\qquad \int_{A_p}|W_p|dP \le \liminf_n \int_{A_p}|Z_n|dP \le \sup_n \int_\Omega ||X_n(\omega)||dP(\omega) < \infty.$$

By $(*)$ and $(**)$ it follows that $\int_\Omega |X_\infty|dP < \infty$. Coming back to Z_n and X_∞ we have

$$\lim_{n\to\infty}\int_{A_p}\langle g, Z_n\rangle\, dP = \int_{A_p}\langle g, W_p\rangle\, dP = \int_{A_p}\langle g, X_\infty\rangle\, dP$$

for each $p \in \mathbf{N}$ and for each $g \in L^\infty_{E^*}(A_p\cap\mathcal{F})$.

There is a useful variant

Theorem 3.2 *Suppose that E^*_b is a separable, E has the Radon Nikodym property, $(X_n)_{n\in\mathbf{N}}$ is a bounded sequence $L^1_E(\mathcal{F})$ such that for each $A \in \mathcal{F}$, the set $\{\int_A X_n dP : n \in \mathbf{N}\}$ is relatively weakly compact in E. Then there exist an increasing sequence $(A_p)_{p\in\mathbf{N}}$ in $\mathcal{F}$ such that $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(Z_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ and $X_\infty \in L^1_E(\mathcal{F})$ such that, for each $p \in \mathbf{N}$ and for each $g \in L^\infty_{E^*}(A_p\cap\mathcal{F})$, the following holds:*

$$\lim_{n\to\infty}\int_{A_p}\langle g, Z_n\rangle\, dP = \int_{A_p}\langle g, X_\infty\rangle\, dP.$$

Proof As $(X_n)_{n\in\mathbf{N}}$ is bounded in $L^1_E(\mathcal{F})$, using a biting argument cf. [6, 10, 18, 19, 28] provides an increasing sequence $(A_p)_{p\in\mathbf{N}}$ in $\mathcal{F}$ such that $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(Y_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ such that the restrictions $(Y_n|A_p)$ to each A_p

is uniformly integrable, so that, by our assumption and the Dunford theorem ([17], Theorem 1, p. 101), $(Y_n|A_p)$ is relatively weakly compact in $L^1_E(\mathcal{F})$. The proof can be finished as that of Theorem 4.2. ∎

The following result is easy, since it can be applied directly in the convergence of convex weakly compact valued pramart, we will provide some details of proof for the sake of completeness. See ([18], Theorem IX.1.1) for a related result dealing with L^1_E-bounded sequences.

THEOREM 3.3 *Assume that E^*_b is separable, E has the Radon-Nikodym property and $(X_n)_{n\in\mathbf{N}}$ is a sequence in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ with the following properties*
(i) $|X_n| \leq g$ for all $n \in \mathbf{N}$ where g is a positive integrable,
(ii) For each $A \in \mathcal{F}$, the set $\{\int_A X_n dP : n \in \mathbf{N}\}$ is relatively weakly compact in E,
(iii) For every $x^* \in E^*$, and for every $A \in \mathcal{F}$, $\lim_{n\to\infty} \delta^*(x^*, \int_A X_n dP)$ exists,[1]
(iv) For every $x^* \in E^*$, $(\delta^*(x^*, X_n))$ converges a.s. to an integrable function φ_{x^*}.
Then there exists $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X)$$

a.s. for all $x^* \in \overline{B}_{E^*}$.

PROOF We may assume that for each $A \in \mathcal{F}$, $\{\int_A X_n dP : n \in \mathbf{N}\}$ is included in a convex weakly compact subset K_A. Set

$$l(A, x^*) := \lim_{n\to\infty} \int_A \delta^*(x^*, X_n) dP = \lim_{n\to\infty} \delta^*(x^*, \int_A X_n dP)$$

for $A \in \mathcal{F}$ and $x^* \in \overline{B}_{E^*}$. Then for any fixed $A \in \mathcal{F}$, $x^* \mapsto l(A, x^*)$ is the support function of a convex weakly compact set $M(A) \subset K_A$. Set

$$\delta^*(x^*, M(A)) = l(A, x^*), x^* \in E^*,\ A \in \mathcal{F}.$$

Then $M : \mathcal{F} \to cwk(E)$ is a convex weakly compact valued multimeasure [14, 20] with bounded variation, that is $A \mapsto \delta^*(x^*, M(A))$ is a real-valued measure and there exists a finite measure ν on $\mathcal{F}$ such that $M(A) \subset \nu(A)\overline{B}_E$ for all $A \in \mathcal{F}$ and that ν is absolutely continuous with respect to P. Indeed we have $|\delta^*(x^*, M(A))| \leq \nu(A) := \int_A g dP$ for all $x^* \in \overline{B}_{E^*}$ and for all $A \in \mathcal{F}$ so that $|M(A)| \leq \nu(A) = \int_A g dP$ for all $A \in \mathcal{F}$. Since E has the RNP, the multimeasure M admits a density $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ [7, 14], that is $M(A) = \int_A X dP$ for $A \in \mathcal{F}$, so that $\delta^*(x^*, M(A)) = \int_A \delta^*(x^*, X) dP$ for all $x^* \in E^*$ and for all $A \in \mathcal{F}$. Whence we have

$$\lim_{n\to\infty} \int_A \delta^*(x^*, X_n) dP = \int_A \delta^*(x^*, X) dP = \int_A \varphi_{x^*} dP$$

for every $A \in \mathcal{F}$ and for every $x^* \in \overline{B}_{E^*}$. Let $D^*_1 = (e^*_m)$ be a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. Then we deduce that $\lim_{n\to\infty} \delta^*(e^*_m, X_n) = \delta^*(e^*_m, X)$ a.s. for all $m \in \mathbf{N}$ so that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X)$$

[1] Actually (i) and (iv) imply (iii).

a.s. for all $x^* \in \overline{B}_{E^*}$. ■

4. Vector-valued pramarts. In this section we present some applications of the biting compactness results to a.s. convergence for vector-valued pramarts in separable Banach space. Let us recall some needed notions in the pramart convergence.

DEFINITION 4.1 A sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $L^1_E(\mathcal{F})$ is an adapted sequence if each X_n is $\mathcal{F}_n$-measurable. An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $L^1_E(\mathcal{F})$ is a pramart if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([\,||E^{\mathcal{F}_\sigma} X_\tau - X_\sigma|| > \varepsilon]) \leq \varepsilon$$

where T denotes the set of bounded stopping times.

We also need the classical notion of subpramarts.

DEFINITION 4.2 An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $L^1_{\mathbf{R}}(\mathcal{F})$ is a subpramart, if for every $\varepsilon > 0$ there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([X_\sigma - E^{\mathcal{F}_\sigma} X_\tau > \varepsilon]) \leq \varepsilon$$

We need the following definition of uniform sequence of subpramarts that is due to Egghe ([18], definition VIII.1.14).

DEFINITION 4.3 Let $(X_n^m, \mathcal{F}_n)$ $(m \in \mathbf{N})$ be a sequence of real-valued subpramarts in $L^1_{\mathbf{R}}(\mathcal{F})$. It is called a uniform sequence of subpramarts if for every $\varepsilon > 0$, there is $\sigma_0 \in T$ such that if $\sigma \in T(\sigma_0)$ and $\tau \in T(\sigma)$, then

$$P([\sup_{m\in\mathbf{N}} (X_\sigma^m - E^{\mathcal{F}_\sigma} X_\tau^m) > \varepsilon]) \leq \varepsilon.$$

THEOREM 4.4 *Assume that E is separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded and $\mathcal{R}wk(E)$-tight pramart in $L^1_E(\mathcal{F})$. Then there is $X_\infty \in L^1_E(\mathcal{F})$ such that*

$$\lim_{n\to\infty} ||E^{\mathcal{F}_n} X_\infty - X_n|| = 0 \quad a.s.$$

so that

$$\lim_{n\to\infty} ||X_n - X_\infty|| = 0 \quad a.s.$$

PROOF the proof is divided in two steps.
Step 1. For any sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$ and for any $X \in L^1_{E^*_s}(\mathcal{F})$, for any $x \in E$, we have

$$||x - E^{\mathcal{B}} X|| \leq E^{\mathcal{B}} ||x - X|| \quad a.s. \tag{5.1.1}$$

Next we deduce that, for every $x \in E$, $(||x - X_n||)_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart. Indeed using (5.1.1) we have

$$||x - X_\sigma|| - E^{\mathcal{F}_\sigma}||x - X_\tau|| \le ||x - X_\sigma|| - ||x - E^{\mathcal{F}_\sigma}X_\tau|| \\ \le ||E^{\mathcal{F}_\sigma}X_\tau - X_\sigma||.$$

Therefore $(|X_n|)_{n\in\mathbf{N}}$ is a L^1-bounded positive subpramart in $L^1_{\mathbf{R}}(\mathcal{F})$. So $(|X_n|)_{n\in\mathbf{N}}$ pointwise converges a.s. to an integrable function by Millet-Sucheston theorem, see ([18], Theorem VIII.1.11). Consequently $(X_n)_{n\in\mathbf{N}}$ is pointwise bounded: $\sup_{n\in\mathbf{N}}|X_n| < \infty$ a.s. As each $(\langle x^*, X_n\rangle)$ $(x^* \in \overline{B}_{E^*})$ is a L^1-bounded pramart in $L^1_{\mathbf{R}}(\mathcal{F})$, it converges a.s. to an integrable function $m_{x^*} \in L^1_{\mathbf{R}}(\mathcal{F})$. By Theorem 4.1, there exist an increasing sequence $(A_p)_{p\in\mathbf{N}}$ in $\mathcal{F}$ such that $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(X'_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ and $X_\infty \in L^1_E(\mathcal{F})$ such that, for each $p \in \mathbf{N}$, for each $A \in A_p \cap \mathcal{F}$ and for each $x^* \in E^*$, the following holds:

$$\lim_{n\to\infty}\int_A \langle x^*, X'_n\rangle = \int_A m_{x^*}dP = \int_A \langle x^*, X_\infty\rangle\, dP.$$

Now let $(e^*_m)_{m\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_{E^*}$ of E with respect to the weak star topology. By identifying the limits, we have that

$$\lim_{n\to\infty}\langle e^*_m, X_n\rangle = \langle e^*_m, X_\infty\rangle \quad a.s. \quad \forall m \in \mathbf{N}.$$

Step 2. For each $x \in E$ and for each $m \in \mathbf{N}$, we have

$$\langle e^*_m, E^{\mathcal{F}_\sigma}(x - X_\tau)\rangle = E^{\mathcal{F}_\sigma}\langle e^*_m, (x - X_\tau)\rangle$$

so that

$$|\langle e^*_m, E^{\mathcal{F}_\sigma}(x - X_\tau)\rangle| = |E^{\mathcal{F}_\sigma}\langle e^*_m, (x - X_\tau)\rangle| \le E^{\mathcal{F}_\sigma}|\langle e^*_m, (x - X_\tau)\rangle|.$$

We deduce the estimation

$$\text{(5.1.2)} \quad |\langle e^*_m, (x - X_\sigma)\rangle| - E^{\mathcal{F}_\sigma}|\langle e^*_m, (x - X_\tau)\rangle| \\ \le |\langle e^*_m, (x - X_\sigma)\rangle| - |\langle e^*_m, E^{\mathcal{F}_\sigma}(x - X_\tau)\rangle| \\ \le |\langle e^*_m, (x - X_\sigma) - E^{\mathcal{F}_\sigma}(x - X_\tau)\rangle| \\ \le ||E^{\mathcal{F}_\sigma}X_\tau - X_\sigma||.$$

Since this estimation holds for all $m \in \mathbf{N}$ and $(X_n)_{n\in\mathbf{N}}$ is a L^1-bounded pramart, by (5.1.2) we see that the sequence

$$((|\langle e^*_m, x - X_n\rangle|)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a positive L^1-bounded uniform subpramart in the sense of the definition 5.3. Applying Lemma VIII.1.15 in [18] to this sequence yields

$$\lim_{n\to\infty}||x - X_n|| = \lim_{n\to\infty}\sup_{m\in\mathbf{N}}|\langle e^*_m, x - X_n\rangle| = \sup_{m\in\mathbf{N}}\lim_{n\to\infty}\langle e^*_m, x - X_n\rangle| \\ = \sup_{m\in\mathbf{N}}|\langle e^*_m, x - X_\infty\rangle| = ||x - X_\infty|| \quad a.s.$$

which proves (b). We finish the proof as follows. We have

$$||E^{\mathcal{F}_n}X_\infty - X_n|| = \sup_{m\in\mathbf{N}} |\langle e_m^*, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m^*, X_n\rangle|.$$

It is clear that $(\langle e_m^*, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m, X_n\rangle)_{n\in\mathbf{N}}$ are real-valued L^1-bounded pramarts in $L^1_{\mathbf{R}}(\mathcal{F})$ which converges a.s. to 0 when $n\to\infty$. Since $(Y_n = E^{\mathcal{F}_n}X_\infty)_{n\in\mathbf{N}}$ is a martingale: $Y_\sigma = E^{\mathcal{F}_\sigma}Y_\tau, \sigma\le\tau$, by similar computation as in (5.1.2) we see that

$$((|\langle e_m^*, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m^*, X_n\rangle|)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a uniform sequence of positive L^1-bounded subpramarts. Now applying Lemma VIII.1.15 in [18] to this sequence yields

$$\begin{aligned}\lim_{n\to\infty} ||E^{\mathcal{F}_n}X_\infty - X_n|| &= \lim_{n\to\infty}\sup_{m\in\mathbf{N}} |\langle e_m^*, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m^*, X_n\rangle| \\ &= \sup_{m\in\mathbf{N}}\lim_{n\to\infty} |\langle e_m^*, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m^*, X_n\rangle| = 0 \quad a.s.\end{aligned}$$ ■

By Levy's theorem

$$||E^{\mathcal{F}_n}X_\infty - X_\infty|| \to 0 \quad a.s.$$

we deduce that

$$\lim_{n\to\infty} ||X_n - X_\infty|| = 0 \quad a.s.$$

Here is a corollary of Theorem 5.1.

THEOREM 4.5 *Assume that E is separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $L^1_E(\mathcal{F})$ satisfying: there is a closed convex ball-weakly compact valued measurable mapping $K:\Omega\to E$ such that $X_n(\omega)\in K(\omega)$ for all $n\in\mathbf{N}$ and for all $\omega\in\Omega$. Then there is $X_\infty\in L^1_E(\mathcal{F})$ such that*

$$\lim_{n\to\infty} ||E^{\mathcal{F}_n}X_\infty - X_n|| = 0 \quad a.s.$$

so that

$$\lim_{n\to\infty} ||X_n - X_\infty|| = 0 \quad a.s.$$

REMARK. Actually one can replace *closed convex ball-weakly compact* by *convex weakly compact* since by the proof of Theorem 5.1,

$$r(\omega) := \sup_{n\in\mathbf{N}} ||X_n(\omega)|| < \infty \quad a.s.$$

See also Corollary 6.1.

Using Theorem 4.2 and repeating the arguments of the proof of Theorem 5.1 we have a useful variant

THEOREM 4.6 *Suppose that E_b^* is a separable, E has the Radon Nikodym property, (X_n) is a bounded pramart in $L^1_E(\mathcal{F})$ such that: for each $A\in\mathcal{F}$, the set $\{\int_A X_n dP : n\in\mathbf{N}\}$ is relatively weakly compact in E. Then there is $X_\infty\in L^1_E(\mathcal{F})$ such that*

$$\lim_{n\to\infty} ||E^{\mathcal{F}_n}X_\infty - X_n|| = 0 \quad a.s.$$

so that

$$\lim_{n\to\infty} ||X_n - X_\infty|| = 0 \quad a.s.$$

In particular, if E is separable reflexive, Theorem 5.1 and 5.3 are reduced to

THEOREM 4.7 *Assume that E is reflexive separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $L^1_E(\mathcal{F})$. Then there is $X_\infty \in L^1_E(\mathcal{F})$ such that*

$$\lim_{n\to\infty} ||E^{\mathcal{F}_n} X_\infty - X_n|| = 0 \quad a.s.$$

so that

$$\lim_{n\to\infty} ||X_n - X_\infty|| = 0 \quad a.s.$$

5. Multivalued Pramarts. From now we will assume in the majority of this section that the strong dual E^*_b is *separable* in order to ensure the weak compactness of conditional expectation of convex weakly compact valued random sets. This assumption can be removed in some particular cases. Before going further we state and summarize some properties of convex weakly compact valued pramarts. Let us recall and summarize some definitions.

DEFINITION 5.1 A sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is an adapted sequence if each X_n is $\mathcal{F}_n$-measurable. An adapted sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is a pramart if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$\sigma,\, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([d_H(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma) > \varepsilon]) \leq \varepsilon$$

where T denotes the set of bounded stopping times and d_H the Hausdorff distance defined on $cc(E)$.

It is clear that if $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, then, for each $x^* \in \overline{B}_{E^*}$, the adapted sequence $(\delta^*(x^*, X_n), \mathcal{F}_n)_{n\in\mathbf{N}}$ is a real-valued pramart in $L^1_{\mathbf{R}}(\mathcal{F})$ because

$$|\delta^*(x^*, E^{\mathcal{F}_\sigma} X_\tau) - \delta^*(x^*, X_\sigma)| \leq d_H(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma).$$

It is clear that this definition covers the notion of vector-valued pramarts in $L^1_E(\mathcal{F})$.

For the convenience of the reader, we recall and summarize some useful results. See [9, 13].

LEMMA 5.2 *Assume that E^*_b is separable and $D^*_1 := (e^*_m)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Then the following holds:*

$$\begin{aligned}[\delta^*(e^*_m, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)^+] &\leq [\delta^*(e^*_m, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)]^+ \\ &\leq |\delta^*(e^*_m, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e^*_m, X_\tau)| \leq d_H(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) \quad a.s.\end{aligned}$$

for all $m \in \mathbf{N}$ *, for all* σ, $\tau \in T, \tau \geq \sigma$*. Similarly*

$$\begin{aligned}[\delta^*(e_m^*, X_\sigma)^- - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)^-] &\leq [\delta^*(e_m^*, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)]^- \\ &\leq |\delta^*(e_m^*, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)| \leq d_H(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau) \quad a.s.\end{aligned}$$

for all $m \in \mathbf{N}$ *, for all* σ, $\tau \in T, \tau \geq \sigma$*.*

Proof For each $m, n \in \mathbf{N}$, let us set

$$\varphi_{m,n}(\omega) := \delta^*(e_m^*, X_n(\omega)).$$

Let σ, $\tau \in T, \tau \geq \sigma$ and let us set

$$\varphi_{m,\tau}(\omega) := \delta^*(e_m^*, X_\tau(\omega)).$$

$$\varphi_{m,\sigma}(\omega) := \delta^*(e_m^*, X_\sigma(\omega)).$$

From Jensen inequality we have

$$|E^{\mathcal{F}_\sigma}\varphi_{m,\tau}(\omega)| \leq E^{\mathcal{F}_\sigma}|\varphi_{m,\tau}|(\omega) \quad a.s.$$

Then for a.s $\omega \in \Omega$ we have that

$$\begin{aligned}\delta^*(e_m^*, X_\sigma)^+ - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)^+ &= \varphi_{m,\sigma}^+ - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}^+) \\ &= \frac{1}{2}[\varphi_{m,\sigma} + |\varphi_{m,\sigma}| - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) - E^{\mathcal{F}_\sigma}(|\varphi_{m,\tau}|)] \\ &\leq \frac{1}{2}[\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) + |\varphi_{m,\sigma}| - |E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})|] \\ &\leq \frac{1}{2}[\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) + |\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})|] \\ &= [\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})]^+ \\ &= [\delta^*(e_m^*, X_\sigma) - \delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau)]^+ \\ &\leq |\delta^*(e_m^*, X_\sigma) - \delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau)| \leq d_H(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau).\end{aligned}$$

Similarly

$$\begin{aligned}\delta^*(e_m^*, X_\sigma)^- - E^{\mathcal{F}_\sigma}\delta^*(e_m^*, X_\tau)^- &= \varphi_{m,\sigma}^- - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}^-) \\ &= \frac{1}{2}[|\varphi_{m,\sigma}| - \varphi_{m,\sigma} + E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) - E^{\mathcal{F}_\sigma}(|\varphi_{m,\tau}|)] \\ &\leq \frac{1}{2}[|\varphi_{m,\sigma}| - \varphi_{m,\sigma} + E^{\mathcal{F}_\sigma}(\varphi_{m,\tau}) - |E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})|] \\ &\leq \frac{1}{2}[-(\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})) + |\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})|] \\ = [\varphi_{m,\sigma} - E^{\mathcal{F}_\sigma}(\varphi_{m,\tau})]^- &= [\delta^*(e_m^*, X_\sigma) - \delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau)]^- \\ &\leq |\delta^*(e_m^*, X_\sigma) - \delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau)| \leq d_H(X_\sigma, E^{\mathcal{F}_\sigma}X_\tau).\end{aligned}$$

■

By repeating the techniques of Lemma 6.1 we have the following results.

Lemma 5.3 *Assume that E_b^* is separable and $D_1^* := (e_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. Let $x \in E$ and let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Then the following holds:*

$$\begin{aligned}(\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\sigma))^+ - E^{\mathcal{F}_\sigma}(\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\tau))^+ \\ \leq [\delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau) - \delta^*(e_m^*, X_\sigma)]^+ \\ \leq |\delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau) - \delta^*(e_m^*, X_\sigma)| \leq d_H(E^{\mathcal{F}_\sigma}X_\tau, X_\sigma)\end{aligned}$$

a.s. for all $m \in \mathbf{N}$, σ, $\tau \in T, \tau \geq \sigma$. Similarly

$$\begin{aligned}(\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\sigma))^- - E^{\mathcal{F}_\sigma}(\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\tau))^- \\ \leq [\delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau) - \delta^*(e_m^*, X_\sigma)]^- \\ \leq |\delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau) - \delta^*(e_m^*, X_\sigma)| \leq d_H(E^{\mathcal{F}_\sigma}X_\tau, X_\sigma)\end{aligned}$$

a.s. for all $m \in \mathbf{N}$, σ, $\tau \in T, \tau \geq \sigma$.

Lemma 5.4 *Assume that E_b^* is separable and $D_1^* := (e_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a martingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Then the following holds:*

$$\begin{aligned}[\delta^*(e_m^*, Y_\sigma) - \delta^*(e_m^* X_\sigma)]^+ - E^{\mathcal{F}_\sigma}[\delta^*(e_m^*, Y_\tau) - \delta^*(e_m^*, X_\tau)]^+ \\ \leq [\delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau) - \delta^*(e_m^*, X_\sigma)]^+ \\ \leq |\delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau) - \delta^*(e_m^*, X_\sigma)| \leq d_H(E^{\mathcal{F}_\sigma}X_\tau, X_\sigma)\end{aligned}$$

a.s. for all $m \in \mathbf{N}$, σ, $\tau \in T, \tau \geq \sigma$. Similarly

$$\begin{aligned}[\delta^*(e_m^*, Y_\sigma) - \delta^*(e_m^* X_\sigma)]^- - E^{\mathcal{F}_\sigma}[\delta^*(e_m^*, Y_\tau) - \delta^*(e_m^*, X_\tau)]^- \\ \leq [\delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau) - \delta^*(e_m^*, X_\sigma)]^- \\ \leq |\delta^*(e_m^*, E^{\mathcal{F}_\sigma}X_\tau) - \delta^*(e_m^*, X_\sigma)| \leq d_H(E^{\mathcal{F}_\sigma}X_\tau, X_\sigma)\end{aligned}$$

a.s. for all $m \in \mathbf{N}$, σ, $\tau \in T, \tau \geq \sigma$.

Proof Since $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a martingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ we have

$$Y_\sigma = E^{\mathcal{F}_\sigma}Y_\tau \quad \forall\, \sigma, \tau \in T, \quad \tau \geq \sigma.$$

Hence the result follows by applying again the techniques of Lemma 6.1. ■

Remarks. According to this definition, Lemma 6.1-6.2-6.3 show that the sequence

$$((\delta^*(e_m^*, X_n)^+_{n\in\mathbf{N}})_{m\in\mathbf{N}}, \quad ((\delta^*(e_m^*, X_n)^-_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

$$(([\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}, \ (([\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n)]^-)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

$$(([\delta^*(e_m^*, Y_n) - \delta^*(e_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}, \ (([\delta^*(e_m^*, Y_n) - \delta^*(e_m^*, X_n)]^-)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

are uniform sequence of positive subpramarts.

Now we proceed to convergence results for $cwk(E)$-valued pramarts.

LEMMA 5.5 *Assume that E is separable. Let $x \in E$ and $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that $\sup_{n\in\mathbf{N}} \int_\Omega d(x, X_n)dP < \infty$. Then $(d(x, X_n))_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart converging a.s. to an integrable function in $L^1_{\mathbf{R}}(\mathcal{F})$.*

PROOF By ([21], Lemma 4.3) for any sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$ and for any $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ we have

$$(6.4.1) \qquad d(x, E^{\mathcal{B}}X) \le E^{\mathcal{B}} d(x, X) \quad a.s.$$

Since $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a pramart, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$(6.4.2) \qquad \sigma, \tau \in T, \quad \tau \ge \sigma \ge \sigma_\varepsilon \Rightarrow P([d_H(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) > \varepsilon]) \le \varepsilon.$$

By (6.4.1) for $\sigma, \tau \in T, \quad \tau \ge \sigma \ge \sigma_\varepsilon$, we have

$$(5.4.3) \qquad d(x, X_\sigma) - E^{\mathcal{F}_\sigma} d(x, X_\tau) \le d(x, X_\sigma) - d(x, E^{\mathcal{F}_\sigma} X_\tau) \quad a.s.$$

By (6.4.3) and ([4], Lemma 1.5.1, p. 29) we deduce

$$(6.4.4) \quad d(x, X_\sigma) - E^{\mathcal{F}_\sigma} d(x, X_\tau) \le \sup_{x\in E} |d(x, X_\sigma) - d(x, E^{\mathcal{F}_\sigma} X_\tau)| = d_H(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) \quad a.s.$$

Using (6.4.4) and (6.4.2) and the definition 5.2 (of subpramart) it is easy to conclude that $(d(x, X_n))_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart which converges a.s. to an integrable function by virtue of Millet-Sucheston theorem, see ([18], Theorem VIII.1.11). ■

REMARKS. In particular, Lemma 6.1 and Lemma 6.4 show that if (X_n) is an L^1-bounded real-valued pramart, then (X_n^+), (X_n^-) and $(|X_n|)$ are L^1-bounded positive subpramarts.

Here is a useful corollary.

COROLLARY 5.6 *Assume that E is separable, $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a pramart in $L^1_E(\mathcal{F})$ such that $\sup_{n\in\mathbf{N}} \int_\Omega |X_n| dP < \infty$. Then $(|X_n|)_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart converging a.s. to an integrable function in $L^1_{\mathbf{R}}(\mathcal{F})$, so that $\sup_{n\in\mathbf{N}} |X_n| < \infty$ a.s.*

The following result is new, it generalizes a partial result obtained in ([9], Theorem 4.1).

THEOREM 5.7 *Assume that E^*_b is separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that*

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}.$$

Then the following hold:
(a) $\lim_{n\to\infty} d_H(E^{\mathcal{F}_n} X_\infty, X_n) = 0 \quad a.s.$
(b) $M\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$
(c) $\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad a.s. \quad \forall x \in E.$
(d) $\sup_{n\in N} |X_n| < \infty$ *a.s.*

Proof (a) $\lim_{n\to\infty} d_H(E^{\mathcal{F}_n}X_\infty, X_n) = 0$ a.s.
Let $D_1^* = (e_m^*)_{m\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_{E^*}$ with respect to the Mackey topology $\tau(E^*, E)$. We have

$$d_H(E^{\mathcal{F}_n}X_\infty, X_n) = \sup_{m\in\mathbf{N}} |\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)|.$$

It is clear that $((\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n))_{n\in\mathbf{N}}$ constitutes a real-valued L^1-bounded pramart which converges a.s. to 0. Since $(E^{\mathcal{F}_n}X_\infty)$ is a regular martingale, Lemma 6.3 shows that

$$(([\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a uniform sequence of positive L^1-bounded subpramarts and so is the sequence

$$(([\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)]^-)_{n\in\mathbf{N}})_{m\in\mathbf{N}}.$$

Note that

$$\begin{aligned} &|\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)| = \\ &[\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)]^+ \\ &+[\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)]^-. \end{aligned}$$

Now applying Lemma VIII.1.15 in [18] to both these positive and negative part of subpramarts yields

$$\begin{aligned} &\sup_{m\in\mathbf{N}} [\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)]^+ \to \\ \sup_{m\in\mathbf{N}} \lim_{n\to\infty} &[\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)]^+ = 0 \quad a.s. \end{aligned}$$

and

$$\begin{aligned} &\sup_{m\in\mathbf{N}} [\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)]^- \to \\ \sup_{m\in\mathbf{N}} \lim_{n\to\infty} &[\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)]^- = 0 \quad a.s. \end{aligned}$$

so that

$$\sup_{m\in\mathbf{N}} |\delta^*(e_m^*, E^{\mathcal{F}_n}X_\infty) - \delta^*(e_m^*, X_n)| \to 0 \quad a.s.$$

that is

$$d_H(E^{\mathcal{F}_n}X_\infty, X_n) \to 0 \quad a.s.$$

(b) M-$\lim_{n\to\infty} E^{\mathcal{F}_n}X_\infty = X_\infty$ a.s. See Theorem 3.1 in [1].
(c) By (a), (b) and Proposition 3.1 in [1], we conclude that

$$M\text{-}\lim_{n\to\infty} X_n = X_\infty \quad a.s.$$

Further (a), (b) and Proposition 3.2 in [1] imply (c)

$$\lim_{n\to\infty} d(x, X_n(\omega)) = d(x, X_\infty(\omega)) \quad a.s. \quad \forall x \in E.$$

Property (d) follows by using the same lines as those of (a). Lemma 5.1 shows that both the positive part $(\delta^*(e_m^*, X_n)^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ and the negative part $(\delta^*(e_m^*, X_n)^-_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ are positive uniform subpramarts so that by applying Lemma VIII.1.15 in [18] to these sequences yields

$$\begin{aligned}\lim_{n\to\infty} \sup_{x^*\in\overline{B}_{E^*}} \delta^*(x^*, X_n)^+ &= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} \delta^*(e_m^*, X_n)^+ \\ &= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} \delta^*(e_m^*, X_n)^+ = \sup_{m\in\mathbf{N}} \delta^*(e_m^*, X_\infty)^+ \\ &= \sup_{x^*\in\overline{B}_{E^*}} \delta^*(x^*, X_\infty)^+ \quad a.s.\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \sup_{x^*\in\overline{B}_{E^*}} \delta^*(x^*, X_n)^- &= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} \delta^*(e_m^*, X_n)^- \\ &= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} \delta^*(e_m^*, X_n)^- = \sup_{m\in\mathbf{N}} \delta^*(e_m^*, X_\infty)^- \\ &= \sup_{x^*\in\overline{B}_{E^*}} \delta^*(x^*, X_\infty)^- \quad a.s.\end{aligned}$$ ■

There is a variant of the Wijsmann convergence result in Theorem 6.1(c).

THEOREM 5.8 *Assume that E_b^* is separable. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the following condition:*
(i) For each $x \in E$, $\sup_{n\in\mathbf{N}} \int_\Omega d(x, X_n)dP < \infty$
(ii) $X_\infty \subset s\text{-}liX_n$
(iii) $\lim_{n\to\infty} \delta^*(x^*, X_n) = \delta^*(x^*, X_\infty) \quad a.s. \quad \forall x^* \in \overline{B}_{E^*}$.
Then the following holds

$$\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad a.s. \quad \forall x \in E$$

PROOF From (i) and Lemma 6.4, for each $x \in E$, $(d(x, X_n))_{n\in\mathbf{N}}$ is a L^1-bounded subpramart which converges a.s. to an integrable function f_x. Let $D_1^* = \{e_j^* : j \in \mathbf{N}\}$ be a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. Then we have

$$d(x, X_n) = \sup_{j\in\mathbf{N}} [\langle e_j^*, x\rangle - \delta^*(e_j^*, X_n)]$$

and

$$d(x, X_\infty) = \sup_{j\in\mathbf{N}} [\langle e_j^*, x\rangle - \delta^*(e_j^*, X_\infty)].$$

By (iii) we see that, for each $j \in \mathbf{N}$, the sequence $(\langle e_j^*, x\rangle - \delta^*(e_j^*, X_n))_{n\in\mathbf{N}}$ converges a.s. to $\langle e_j^*, x\rangle - \delta^*(e_j^*, X_\infty)$. Passing to the limit when n goes to ∞ in the equality

$$\langle e_j^*, x\rangle - \delta^*(e_j^*, X_n) \le d(x, X_n)$$

yields

$$d(x, X_\infty) \le f_x \quad a.s.$$

From Tsukada ([30], Theorem 2.2) and (ii) we deduce that

$$\limsup_{n\to\infty} d(x, X_n) \le d(x, s\text{-}liX_n) \le d(x, X_\infty).$$

Whence $f_x \le d(x, X_\infty)$ a.s. for each $x \in E$, so that

$$\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad a.s.$$

for each $x \in E$. By equicontinuity of the distance function and the sepa rability of E we conclude that

$$\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty) \quad a.s.$$

The aforementioned results show that $cwk(E)$-valued pramarts enjoy good convergence properties. We finish this section with an application illustrating this fact.

THEOREM 5.9 *Assume that E^*_b is separable, E has the Radon-Nikodym property and $(X_n)_{n\in\mathbf{N}}$ is pramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ with the following properties:*
(i) $|X_n| \le g$ for all $n \in N$ where g is a positive integrable,
(ii) For each $A \in \mathcal{F}$, the set $\{\int_A X_n dP : n \in \mathbf{N}\}$ is relatively weakly compact in E.
Then there exists $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that
(a) $\lim_{n\to\infty} d_H(E^{\mathcal{F}_n}X, X_n) = 0 \quad a.s.$
(b) $M\text{-}\lim_{n\to\infty} X_n = X \quad a.s.$
(c) $\lim_{n\to\infty} d(x, X_n) = d(x, X) \quad a.s. \quad \forall x \in E.$

PROOF Let $D^*_1 = (e^*_m)_{m\in\mathbf{N}}$ be a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. By virtue of Theorem 6.1 we only need to prove that there exists $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(e^*_m, X_n) = \delta^*(e^*_m, X) \quad a.s.$$

for all $m \in \mathbf{N}$. Since $(X_n)_{n\in\mathbf{N}}$ is bounded in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$

$$\sup_{n\in\mathbf{N}} \int_\Omega |X_n| dP = \int_\Omega g|dP < \infty$$

for each $x^* \in \overline{B}_{E^*}$, the L^1-bounded pramart $(\delta^*(x^*, X_n))_{n\in\mathbf{N}}$ converges a.s. to an integrable function $m_{x^*} \in L^1_{\mathbf{R}}(\mathcal{F})$. Taking account of this fact and the condition (i), we may apply Theorem 4.3 to the bounded sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ which gives $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = m_{x^*} = \delta^*(x^*, X) \quad a.s.$$

for each $x^* \in \overline{B}_{E^*}$. Hence

$$\lim_{n\to\infty} \delta^*(e^*_m, X_n) = \delta^*(e^*_m, X) \quad a.s.$$

for all $m \in \mathbf{N}$. Now (a)-(b)-(c) follows as in the proof of Theorem 6.1. ∎

Remarks. We guess that Theorem 6.3 is true for uniformly integrable $cwk(E)$-valued pramarts.

6. A new class of superpramarts in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. In this section we introduce a new class of sub-superpramarts including the classical sub-supermartingales, see [8, 21] and the references therein. Let E be a separable Banach with separable dual space E^*_b. Let us denote by $e_H(A,B)$ the Hausdorff excess associated with the norm of E between two convex weakly compact subsets A, B of E. By ([11], Theorem II-18) we have that

$$e_H(A,B) = \sup_{x^* \in \overline{B}_{E^*}} [\delta^*(x^*, A) - \delta^*(x^*, B)]$$

Definition 6.1 An adapted sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is a $cwk(E)$-valued subpramart if for every $\varepsilon > 0$ there is $\sigma_\varepsilon \in T$ such that

$$\sigma,\, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([e_H(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau) > \varepsilon]) \leq \varepsilon.$$

(a) If $(X_n)_{n\in\mathbf{N}}$ is a $cwk(E)$-valued subpramart, then, for every $x^* \in \overline{B}_{E^*}$, $(\delta^*(x^*, X_n))_{n\in\mathbf{N}}$ is a subpramart, since

$$\begin{aligned}\delta^*(x^*, X_\sigma) - E^{\mathcal{F}_\sigma}\delta^*(x^*, X_\tau) &= \delta^*(x^*, X_\sigma) - \delta^*(x^*, E^{\mathcal{F}_\sigma} X_\tau)\\ &\leq e_H(X_\sigma, E^{\mathcal{F}_\sigma} X_\tau).\end{aligned}$$

(b) it is clear that a $cwk(E)$-valued submartingale

$$X_\sigma \subset E^{\mathcal{F}_\sigma} X_\tau, \forall \tau \geq \sigma$$

is a $cwk(E)$-valued subpramart, further, according to Definition 5.2 a real-valued submartingale: $X_\sigma \leq E^{\mathcal{F}_\sigma} X_\tau, \forall \tau \geq \sigma$ is a real-valued subpramart.

Definition 6.2 An adapted sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ is a $cwk(E)$-valued superpramart if for every $\varepsilon > 0$ there is $\sigma_\varepsilon \in T$ such that

$$\sigma,\, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([e_H(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma) > \varepsilon]) \leq \varepsilon.$$

(a) If $(X_n)_{n\in\mathbf{N}}$ is a $cwk(E)$-valued superpramart, then, for every $x^* \in \overline{B}_{E^*}$, $(\delta^*(x^*, X_n))_{n\in\mathbf{N}}$ is a superpramart, since

$$\begin{aligned}E^{\mathcal{F}_\sigma}\delta^*(x^*, X_\tau) - \delta^*(x^*, X_\sigma) &= \delta^*(x^*, E^{\mathcal{F}_\sigma} X_\tau) - \delta^*(x^*, X_\sigma)\\ &\leq e_H(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma).\end{aligned}$$

(b) it is clear that a $cwk(E)$-valued supermartingale

$$E^{\mathcal{F}_\sigma} X_\tau \subset X_\sigma, \quad \forall \sigma \leq \tau$$

is a $cwk(E)$-valued superpramart. A real-valued supermartingale

$$E^{\mathcal{F}_\sigma} X_\tau \leq X_\sigma, \quad \forall \sigma \leq \tau$$

is a real-valued superpramart.

The following results follow from the definition 7.2 and the techniques developed in Lemma 6.1–6.3.

Lemma 6.3 *Assume that E_b^* is separable and $D_1^* := (e_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. Let $x \in E$ and let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded superpramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Then the following holds:*

$$\begin{aligned}(\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\sigma))^+ - E^{\mathcal{F}_\sigma}(\langle e_m^*, x\rangle - \delta^*(e_m^*, X_\tau))^+ \\ \leq [\delta^*(e_m, E^{\mathcal{F}_\sigma} X_\tau) - \delta^*(e_m, X_\sigma)]^+ \\ \leq e_H(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma)\end{aligned}$$

a.s. for all $m \in \mathbf{N}$, $\sigma, \tau \in T, \tau \geq \sigma$. Consequently, the sequence $((([\langle e_m^*, x\rangle - \delta^*(e_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ is uniform sequence of L^1-bounded subpramarts.

Lemma 6.4 *Assume that E_b^* is separable and $D_1^* := (e_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology. Let $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ be a bounded superpramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and $(Y_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ is a bounded martingale in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$. Then the following holds:*

$$\begin{aligned}(\delta^*(e_m^*, Y_\sigma) - \delta^*(e_m^*, X_\sigma))^+ - E^{\mathcal{F}_\sigma}(\delta^*(e_m^*, Y_\tau) - \delta^*(e_m^*, X_\tau))^+ \\ \leq [\delta^*(e_m^*, E^{\mathcal{F}_\sigma} X_\tau) - \delta^*(e_m^*, X_\sigma)]^+ \\ \leq e_H(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma)\end{aligned}$$

a.s. for all $m \in \mathbf{N}$, $\sigma, \tau \in T, \tau \geq \sigma$. Consequently, the sequence

$$(([\delta^*(e_m^*, Y_n) - \delta^*(e_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is uniform sequence of L^1-bounded subpramarts.
Similarly, let $A \in cwk(E)$, then the following holds:

$$\begin{aligned}(\delta^*(e_m^*, A) - \delta^*(e_m^*, X_\sigma))^+ - E^{\mathcal{F}_\sigma}(\delta^*(e_m^*, A) - \delta^*(e_m^*, X_\tau))^+ \\ \leq [\delta^*(e_m^*, E^{\mathcal{F}_\sigma} X_\tau) - \delta^*(e_m^*, X_\sigma)]^+ \\ \leq e_H(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma)\end{aligned}$$

a.s. for all $m \in \mathbf{N}$, $\sigma, \tau \in T, \tau \geq \sigma$. Consequently, the sequence

$$(([\delta^*(e_m^*, A) - \delta^*(e_m^*, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is uniform sequence of L^1-bounded subpramarts.

Theorem 6.5 *Suppose that E_b^* is separable, $D_1^* = (e_m^*)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology and (X_n) is a bounded superpramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying:*
$(^*)$ *There is a mapping $K \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that $X_n(\omega) \subset K(\omega)$ $\forall n \in \mathbf{N}, \forall\omega \in \Omega$.*
Then there exist $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that
(a) $\lim_{n\to\infty} \delta^*(e_m^*, X_n) = \delta^*(e_m^*, X_\infty)$ *a.s.* $\forall m \in \mathbf{N}$
(b) $\lim_{n\to\infty} d(x, X_n) = d(x, X_\infty)$ *a.s.* $\forall x \in E$
(c) $\lim_{n\to\infty} e_H(E^{\mathcal{F}_n} X_\infty, X_n) = 0$ *a.s.*
(d) $\lim_{n\to\infty} e_H(A, X_n) = e(A, X_\infty)$ *a.s. for each $A \in cwk(E)$.*

Proof Since $(X_n)_{n\in\mathbf{N}}$ is bounded in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, for each $x^* \in \overline{B}_{E^*}$, the L^1-bounded superpramart $(\delta^*(x^*, X_n))_{n\in\mathbf{N}}$ converges a.s. to an integrable function $\varphi_{x^*} \in L^1_{\mathbf{R}}(\mathcal{F})$. Thanks to (*) and the boundedness assumption: $\sup_n \int_\Omega |X_n| dP < \infty$, arguing as in Proposition 5.1 in [21] yields $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(e^*_m, X_n) = \varphi_{e^*_m} = \delta^*(e^*_m, X_\infty) \quad a.s. \quad \forall m \in \mathbf{N}.$$

(b) By Lemma 7.1, $(([\langle e^*_m, x\rangle - \delta^*(e^*_m, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$ is uniform sequence of L^1-bounded subpramarts. Applying Lemma VIII.1.15 in [18] to this sequence and using (a) yields

$$\begin{aligned}
\lim_{n\to\infty} d(x, X_n) &= \lim_{n\to\infty} \sup_{x^*\in\overline{B}_{E^*}} [\langle x^*, x\rangle - \delta^*(x^*, X_n)]^+ \\
&= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} [\langle e^*_m, x\rangle - \delta^*(e^*_m, X_n)]^+ \\
&= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} [\langle e^*_m, x\rangle - \delta^*(e^*_m, X_n)]^+ \\
&= \sup_{m\in\mathbf{N}} [\langle e^*_m, x\rangle - \delta^*(e^*_m, X_\infty)]^+ \\
&= \sup_{x^*\in\overline{B}_{E^*}} [\langle x^*, x\rangle - \delta^*(x^*, X_\infty)]^+ = d(x, X_\infty)
\end{aligned}$$

(c) By Lemma 7.2 the sequence

$$(([\delta^*(e^*_m, E^{\mathcal{F}_n} X_\infty) - \delta^*(e^*_m, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is uniform sequence of L^1-bounded subpramarts. Applying Lemma VIII.1.15 in [18] to this sequence and using (a) yields

$$\begin{aligned}
\lim_{n\to\infty} e(E^{\mathcal{F}_n} X_\infty, X_n) &= \lim_{n\to\infty} \sup_{x^*\in\overline{B}_{E^*}} [\delta^*(x^*, E^{\mathcal{F}_n} X_\infty) - \delta^*(x^*, X_n)]^+ \\
&= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} [\delta^*(e^*_m, E^{\mathcal{F}_n} X_\infty) - \delta^*(e^*_m, X_n)]^+ \\
&= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} [\delta^*(e^*_m, E^{\mathcal{F}_n} X_\infty) - \delta^*(e^*_m, X_n)]^+ = 0 \quad a.s.
\end{aligned}$$

(d) Similarly, let $A \in cwk(E)$, by Lemma 7.3

$$(([\delta^*(e^*_m, A) - \delta^*(e^*_m, X_n)]^+)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is uniform sequence of L^1-bounded subpramarts. By Lemma VIII.1.15 in [18], we have

$$\begin{aligned}\lim_{n\to\infty} e_H(A, X_n) &= \lim_{n\to\infty} \sup_{x^*\in\overline{B}_{E^*}} [\delta^*(x^*, A) - \delta^*(x^*, X_n)]^+ \\ &= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} [\delta^*(e^*_m, A) - \delta^*(e^*_m, X_n)]^+ \\ &= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} [\delta^*(e^*_m, A) - \delta^*(e^*_m, X_n)]^+ \\ &= \sup_{m\in\mathbf{N}} [\delta^*(e^*_m, A) - \delta^*(e^*_m, X_\infty)]^+ \\ &= \sup_{x^*\in\overline{B}_{E^*}} [\delta^*(x^*, A) - \delta^*(x^*, X_\infty)]^+ \\ &= e(A, X_\infty) \quad a.s.\end{aligned}$$

■

Here is a variant of Theorem 7.1.

THEOREM 6.6 *Assume that E^*_b is separable, E has the Radon-Nikodym property, $D^*_1 = (e^*_m)_{m\in\mathbf{N}}$ is a dense sequence in $\overline{B}_{E^*}$ with respect to the Mackey topology and (X_n) is a bounded superpramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ with the following properties:*
(i) $|X_n| \leq g$ for all $n \in N$ where g is a positive integrable,
(ii) For each $A \in \mathcal{F}$, the set $\{\int_A X_n dP : n \in \mathbf{N}\}$ is relatively weakly compact in E.
Then there exists $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that
(a) $\lim_{n\to\infty} \delta^*(e^*_m, X_n) = \delta^*(e^*_m, X)$ a.s. $\forall m \in \mathbf{N}$
(b) $\lim_{n\to\infty} d(x, X_n) = d(x, X)$ a.s. $\forall x \in E$
(c) $\lim_{n\to\infty} e_H(E^{\mathcal{F}_n} X, X_n) = 0$ a.s.
(d) $\lim_{n\to\infty} e_H(A, X_n) = e(A, X)$ a.s. for each $A \in cwk(E)$.

PROOF By virtue of Theorem 7.1 we only need to prove that there exists $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(e^*_m, X_n) = \delta^*(e^*_m, X) \quad a.s.$$

for all $m \in \mathbf{N}$. Since $(X_n)_{n\in\mathbf{N}}$ is bounded in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$

$$\sup_{n\in\mathbf{N}} \int_\Omega |X_n| dP = \int_\Omega g| dP < \infty$$

for each $x^* \in \overline{B}_{E^*}$, the L^1-bounded superpramart $(\delta^*(x^*, X_n))_{n\in\mathbf{N}}$ converges a.s. to an integrable function $m_{x^*} \in L^1_{\mathbf{R}}(\mathcal{F})$. Taking account of this fact and the condition (i), we may apply Theorem 4.3 to the bounded sequence $(X_n, \mathcal{F}_n)_{n\in\mathbf{N}}$ which gives $X \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ such that

$$\lim_{n\to\infty} \delta^*(x^*, X_n) = m_{x^*} = \delta^*(x^*, X) \quad a.s.$$

for each $x^* \in \overline{B}_{E^*}$. Hence

$$\lim_{n\to\infty} \delta^*(e^*_m, X_n) = \delta^*(e^*_m, X) \quad a.s.$$

for all $m \in \mathbf{N}$. Now (a)-(b)-(c)-(d) follows as in the proof of Theorem 7.1. ■

It is possible to introduce a general class of closed convex integrable superpramarts taking account into the existence of conditional expectation of $cc(E)$-valued integrable mappings, cf. Section 3.

Definition 6.7 Let E be a separable Banach spaces. An adapted sequence $(X_n)_{n\in\mathbf{N}}$ of $cc(E)$-valued integrable mappings is a $cc(E)$-valued superpramart if for every $\varepsilon > 0$ there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \geq \sigma \geq \sigma_\varepsilon \Rightarrow P([e_H(E^{\mathcal{F}_\sigma} X_\tau, X_\sigma) > \varepsilon]) \leq \varepsilon$$

here e_H denotes the Hausdorff excess on closed convex subsets. Actually only few partial results of convergence on these classes of superpramarts are available.

Applications. Assume that E is separable and reflexive.

1) Let $Y \in \mathcal{L}^1_{cwk(E)}(\mathcal{F})$ and let (Z_n) is a bounded superpramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$, then $(X_n = E^{\mathcal{F}_n} Y + Z_n)$ is a $cwk(E)$-valued superpramart.

(2) Let (z_n) be an L^1-bounded pramart in $L^1_E(\mathcal{F})$, and let $h \in S^1_Y$ where S^1_Y is the set of integrable selections of Y, then $(f_n = E^{\mathcal{F}_n} h + z_n)$ is a pramart selection of $(E^{\mathcal{F}_n} Y + z_n)$.

(3) Let X be a closed convex integrable $(S^1_X \neq \emptyset)$ multifunction, then closed convex valued martingale $(X_n = E^{\mathcal{F}_n} X)$ is an unbounded superpramart in the sense of the definition 7.3 because

$$E^{\mathcal{F}_\sigma} X_\tau = X_\sigma,\ \tau, \sigma \in T, \sigma \leq \tau.$$

Now, let $f \in S^1_X$ and $f_n = E^{\mathcal{F}_n} f$, then, for each $k \in \mathbf{N}$, the truncated multifunction

$$X^k_n := X_n \cap [f_n + E^{\mathcal{F}_n}(|f| + k)\overline{B}_E]$$

constitutes a bounded $cwk(E)$-valued superpramart in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ satisfying the τ_L-convergence in given Theorem 7.1. This allows prove that (X_n) Mosco converges to a closed convex integrable multifunction X_∞. See [8, 21] for details. A closed convex integrable supermartingale

$$(X_n, \mathcal{F}_n)) : E^{\mathcal{F}_n} X_{n+1} \subset X_n, \forall n \in \mathbf{N}$$

is a closed convex integrable superpramart:

$$E^{\mathcal{F}_\sigma} X_\tau \subset X_\sigma;\ \tau, \sigma \in T, \sigma \leq \tau.$$

The Mosco convergence for unbounded closed convex integrable superpramart is an open problem.

(4) If (X_n) is L^1-bounded pramart in $L^1_E(\mathcal{F})$, then $X_n = Y_n + Z_n$ where Y_n is a L^1-bounded regular martingale in $L^1_E(\mathcal{F})$ and Z_n is a L^1-bounded pramart in $L^1_E(\mathcal{F})$ and where $(|Z_n|)$ is a subpramart converging a.s. to 0, see Theorem 5.4. Taking account of the above decomposition, it is convenient to introduce the following definition. An L^1-bounded adapted sequence in $L^1_E(\mathcal{F})$ is a *pseudo-pramart* in $L^1_E(\mathcal{F})$, if $X_n = Y_n + Z_n$ where (Y_n) is an L^1-bounded regular martingale in $L^1_E(\mathcal{F})$ and where (Z_n) is a L^1-bounded in $L^1_E(\mathcal{F})$ such that $|Z_n| \leq T_n$ where T_n is a positive subpramart converging a.s. to 0.

(5) In the context of Theorem 7.2 (with E_b^* separable and E having the RNP), we show that the convergence

$$\lim_{n\to\infty} e_H(E^{\mathcal{F}_n}X, X_n) = 0$$

in Theorem 7.2(c) gives the existence of pseudo-pramart selections for the bounded $cwk(E)$-valued superpramart (X_n) in $\mathcal{L}^1_{cwk(E)}(\mathcal{F})$ therein. Let $f \in S^1_X$. Then $(E^{\mathcal{F}_n}f)$ is a regular martingale selection of the $cwk(E)$-valued martingale $E^{\mathcal{F}_n}X$, further by the definition of the excess e_H, we have $d(E^{\mathcal{F}_n}f, X_n) \leq e_H(E^{\mathcal{F}_n}X, X_n)$. By Lemma 7.2 and the Remark VIII.1.16 1 in [18], it is ready shown that the sequence $(e_H(E^{\mathcal{F}_n}X, X_n))$ is a subpramart. Take a $\mathcal{F}_n$-measurable selection g_n of X_n such that

$$||E^{\mathcal{F}_n}f - g_n|| = d(E^{\mathcal{F}_n}f, X_n) \leq e(E^{\mathcal{F}_n}X, X_n).$$

Let us set $h_n = g_n - E^{\mathcal{F}_n}f$. Then $|h_n| \leq e_H(E^{\mathcal{F}_n}X, X_n) \to 0$ a.s. Consequently, the above construction shows that $(g_n = E^{\mathcal{F}_n}f + h_n)$ is a pseudo-pramart selection of X_n. The above considerations hold with Theorem 7.1.

(6) Existence of pramart selection for closed convex integrable superpramart in an open problem. The above result is a pramart variant of the martingale selection theorem for closed convex integrable supermartingale. See e.g. [12, 21, 22] and the references therein.

7. Pramarts in the dual space. We present a new pramart convergence problem in a dual space of a separable Banach space. The following lemma is of importance in Gelfand integration and will be used in the pramart convergence in this context. We denote by $L^1_{E_s^*}(\mathcal{F})$ the space of all scalarly integrable (Gelfand-integrable) mappings $X : \Omega \to E_s^*$ such that $||X||_{E_b^*} \in L^1_{\mathbf{R}}(\mathcal{F})$, here $||.||_{E_b^*}$ (or $||.||$ for simplicity) denotes the norm of E_b^*. If E_b^* is separable, we denote by $L^1_{E_b^*}(\mathcal{F})$ the space of all Lebesgue-Bochner-integrable mappings $X : \Omega \to E_b^*$.

LEMMA 7.1 *Assume that E is separable Banach space and $(X_n)_{n\in\mathbf{N}}$ is a sequence in $L^1_{E_s^*}(\mathcal{F})$ satisfying*

(i) $X_n(\omega) \in r(\omega)\overline{B}_{E^*}$ for all $n \in \mathbf{N}$ and for all $\omega \in \Omega$ where r is a positive measurable function,

(ii) $\sup_n \int_\Omega ||X_n(\omega)||dP(\omega) < \infty$.

Then there exist an increasing sequence of measurable sets $(A_p)_{p\in\mathbf{N}}$ with $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(Z_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ and a scalarly integrable mapping X_∞ such that, for each $p \in \mathbf{N}$, for each $g \in L^\infty_E(A_p \cap \mathcal{F})$

$$\lim_{n\to\infty} \int_{A_p} \langle g, Z_n\rangle dP = \int_{A_p} \langle g, X_\infty\rangle dP.$$

PROOF By (i) there is an increasing sequence of measurable sets $(A_p)_{p\in\mathbf{N}}$ with $\lim_{p\to\infty} P(A_p) = 1$ such that the restriction of X_n to each A_p is bounded: $X_n|A_p \subset r_p\overline{B}_{E^*}$ for $n \in \mathbf{N}$ and for all $\omega \in A_p$ where r_p is a positive constant. By ([10], Corollary 6.5.10), for each $p \in \mathbf{N}$, the sequence $(X_n|A_p)_{n\in\mathbf{N}}$ is relatively sequentially $\sigma(L^1_{E_s^*}, L^\infty_E)$ compact. By successive applications of this weak compactness result, for each $p \in \mathbf{N}$, we provide a sequence $(Y_p^n)_{n\in\mathbf{N}}$ such that $(Y_{p+1}^n)_{n\in\mathbf{N}}$ is extracted from $(Y_p^n)_{n\in\mathbf{N}}$ and $W_p \in L^1_{E_s^*}(A_p \cap \mathcal{F})$ and that

$$\lim_{n\to\infty}\int_{A_p}\langle g, Y_p^n\rangle dP = \int_{A_p}\langle g, W_p\rangle dP$$

Let us set $Z_n = Y_n^n$ for all $n \in \mathbf{N}$. Then, for each $p \in \mathbf{N}$, $(Z_n|A_p)$ converges $\sigma(L^1_{E_s^*}, L^\infty_E)$ to $W_p \in L^1_{E_s^*}(A_p \cap \mathcal{F})$. As $(A_p)_{p\in\mathbf{N}}$ is increasing $W_p = W_{p+1}$ a.s. on A_p. Let us $X_\infty = W_p$ if $\omega \in A_p$ and $X_\infty = 0$ elsewhere. Clearly X_∞ is $\mathcal{F}$ measurable and scalarly integrable. Indeed, for each $e \in E$, we have

$$(^*) \int_\Omega |\langle e, X_\infty\rangle| dP = \int_{\cup_p A_p} |\langle e, X_\infty\rangle| dP$$
$$= \sup_{p\in\mathbf{N}}\int_{A_p} |\langle e, X_\infty\rangle| dP = \sup_{p\in\mathbf{N}}\int_{A_p} |\langle e, W_p\rangle| dP$$

But, for each $p \in \mathbf{N}$, W_p is the $\sigma(L^1_{E_s^*}, L^\infty_E)$-limit ot the sequence $(Z_n|A_p)_{n\in\mathbf{N}}$, in particular, $(\langle e, Z_n|A_p\rangle)_{n\in\mathbf{N}}$ converges $\sigma(L^1, L^\infty)$ to $\langle e, W_p\rangle$, it follows that

$$(^{**}) \int_{A_p} |\langle e, W_p\rangle| dP \le \liminf_n \int_{A_p} |\langle e, Z_n\rangle| dP$$
$$\le \sup_{n\in\mathbf{N}}\int_\Omega ||e||_E ||X_n(\omega)|| dP(\omega) < \infty$$

By $(^*) - (^{**})$ and (ii), we see that X_∞ is scalarly integrable with the required property

$$\int_{A_p}\langle g, X_\infty\rangle dP = \int_{A_p}\langle g, W_p\rangle dP = \lim_{n\to\infty}\int_{A_p}\langle g, Z_n\rangle dP$$

for all $g \in L^\infty_E(A_p \cap \mathcal{F})$. ■

Thanks to the existence and the properties of conditional expectation for closed convex Gelfand-integrable mappings (see Theorem 3.1) and the specific properties of the space $L^1_{E_s^*}(\mathcal{F})$, it is possible to introduce the notions of martingales and pramarts in this space. See [8, 9] for more details. An adapted sequence $(X_n)_{n\in\mathbf{N}}$ in $L^1_{E_s^*}(\mathcal{F})$ is

- a bounded martingale if $\sup_{n\in\mathbf{N}}\int_\Omega ||X_n(\omega)||_{E_b^*} dP(\omega) < \infty$ and $X_n = E^{\mathcal{F}_n} X_{n+1}$, $\forall n \in \mathbf{N}$,
- a bounded pramart if $\sup_n \int_\Omega ||X_n(\omega)||_{E_b^*} dP(\omega) < \infty$ and if, for every $\varepsilon > 0$, there is $\sigma_\varepsilon \in T$ such that

$$\sigma, \tau \in T, \quad \tau \ge \sigma > \sigma_\varepsilon \Rightarrow P([\,||E^{\mathcal{F}_\sigma} X_\tau - X_\sigma||_{E_b^*} > \varepsilon]) \le \varepsilon.$$

THEOREM 7.2 *Assume that E is separable Banach space and $(X_n)_{n\in\mathbf{N}}$ is a bounded pramart in $L^1_{E_s^*}(\mathcal{F})$, then there exists $X_\infty \in L^1_{E_s^*}(\mathcal{F})$ such that*
(a) $\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle$ *a.s. for all* $x \in \overline{B}_E$,
(b) $\lim_{n\to\infty} ||x^* - X_n|| = ||x^* - X_\infty||$ *a.s. for each* $x^* \in E^*$,
(c) $\lim_{n\to\infty} ||E^{\mathcal{F}_n} X_\infty - X_n|| = 0$ *a.s.*

Proof The proof is divided in two steps.
Step 1. Arguing as in Lemma 5.1 in ([8] it is not difficult to prove that for any sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$ and for any w^*-closed convex $\mathcal{F}$-measurable and integrable multifunction $\Gamma : \Omega \Rightarrow E^*$ and for any $x^* \in E^*$, we have

$$d_{E_b^*}(x^*, E^{\mathcal{B}}\Gamma) \leq E^{\mathcal{B}} d_{E_b^*}(x^*, \Gamma) \quad a.s.$$

In particular, if $X \in L^1_{E_s^*}(\mathcal{F})$, for any $x^* \in E^*$, we have

$$||x^* - E^{\mathcal{B}}X|| \leq E^{\mathcal{B}}||x^* - X|| \quad a.s. \tag{8.1.1}$$

here $||.|| := ||.||_{E_b^*}$ denotes the dual norm of E_b^*. Next we deduce that, for every $x^* \in E^*$, $(||x^* - X_n||)_{n\in\mathbf{N}}$ is a positive L^1-bounded subpramart. Indeed using (8.1.1) we have

$$\begin{aligned} ||x^* - X_\sigma|| - E^{\mathcal{F}_\sigma}||x^* - X_\tau|| &\leq ||x^* - X_\sigma|| - ||x^* - E^{\mathcal{F}_\sigma}X_\tau|| \\ &\leq ||E^{\mathcal{F}_\sigma}X_\tau - X_\sigma||. \end{aligned}$$

Therefore $(|X_n|)_{n\in\mathbf{N}}$ is a L^1-bounded positive subpramart in $L^1_{\mathbf{R}}(\mathcal{F})$. So $(|X_n|)_{n\in\mathbf{N}}$ pointwise converges a.s. to an integrable function by Millet-Sucheston theorem, see ([18], Theorem VIII.1.11). Consequently $(X_n)_{n\in\mathbf{N}}$ satisfies the pointwise supremum property: $\sup_{n\in\mathbf{N}}|X_n| < \infty$ a.s. For simplicity we may assume that $X_n(\omega) \subset r(\omega)\overline{B}_{E^*}$ for all $n \in \mathbf{N}$ and for all $\omega \in \Omega$ where r is a positive measurable mapping. By Lemma 8.1 there exist an increasing sequence of measurable set $(A_p)_{p\in\mathbf{N}}$ with $\lim_{p\to\infty} P(A_p) = 1$, a subsequence $(Z_n)_{n\in\mathbf{N}}$ of $(X_n)_{n\in\mathbf{N}}$ and a scalarly integrable (alias Gelfand-integrable) mapping X_∞ such that, for each $p \in \mathbf{N}$,

$$\lim_{n\to\infty}\int_{A_p}\langle g, Z_n\rangle dP = \int_{A_p}\langle g, X_\infty\rangle dP$$

for all $g \in L^\infty_E(A_p \cap \mathcal{F})$.
Step 2. For each $x \in E$, $(\langle x, X_n\rangle)_{n\in\mathbf{N}}$ is a real-valued L^1-bounded pramart, so it converges a.s. to an integrable function φ_x

$$\lim_{n\to\infty}\langle x, X_n\rangle = \varphi_x \quad a.s.$$

so that using the compactness result given in Step 1 and identifying the limits we get

$$\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle \quad a.s.$$

for each $x \in \overline{B}_E$. Now let $(e_m)_{m\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_E$ of E. Then we have

$$\lim_{n\to\infty}\langle e_m, X_n\rangle = \langle e_m, X_\infty\rangle \quad a.s. \quad \forall m \in \mathbf{N}.$$

Since $\sup_{n\in\mathbf{N}}|X_n| < \infty$ a.s, by density we get (a)

$$\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle \quad a.s. \quad \forall x \in \overline{B}_E.$$

For each $x^* \in E^*$ and for each $m \in \mathbf{N}$, we have

$$\langle e_m, E^{\mathcal{F}_\sigma}(x^* - X_\tau)\rangle = E^{\mathcal{F}_\sigma}\langle e_m, (x^* - X_\tau)\rangle$$

so that

$$|\langle e_m, E^{\mathcal{F}_\sigma}(x^* - X_\tau)\rangle| = |E^{\mathcal{F}_\sigma}\langle e_m, (x^* - X_\tau)\rangle| \le E^{\mathcal{F}_\sigma}|\langle e_m, (x^* - X_\tau)\rangle|.$$

We deduce the estimation

$$\begin{aligned}(8.1.2)\quad & |\langle e_m, (x^* - X_\sigma)\rangle| - E^{\mathcal{F}_\sigma}|\langle e_m, (x^* - X_\tau)\rangle| \\ &\le |\langle e_m, (x^* - X_\sigma)\rangle| - |\langle e_m, E^{\mathcal{F}_\sigma}(x^* - X_\tau)\rangle| \\ &\le |\langle e_m, (x^* - X_\sigma) - E^{\mathcal{F}_\sigma}(x^* - X_\tau)\rangle| \\ &\le ||E^{\mathcal{F}_\sigma}X_\tau - X_\sigma||.\end{aligned}$$

Since this estimation holds for all $m \in \mathbf{N}$ and $(X_n)_{n\in\mathbf{N}}$ is a bounded pramart, by (8.1.2) we see that the sequence

$$((|\langle e_m, x^* - X_n\rangle|)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a positive L^1-bounded uniform subpramart in the sense of the definition 5.3. Applying Lemma VIII.1.15 in [18] to this sequence yields

$$\begin{aligned}\lim_{n\to\infty} ||x^* - X_n|| &= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} |\langle e_m, x^* - X_n\rangle| \\ &= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} \langle e_m, x^* - X_n\rangle| \\ &= \sup_{m\in\mathbf{N}} |\langle e_m, x^* - X_\infty\rangle| = ||x^* - X_\infty|| \quad a.s.\end{aligned}$$

which proves (b). In particular, we have that $||X_n|| \to ||X_\infty||$ a.s. so that $X_\infty \in L^1_{E^*_s}(\mathcal{F})$ by Fatou Lemma

$$\int_\Omega |X_\infty| dP \le \liminf_n \int_\Omega |X_n| dP \le \sup_{n\in\mathbf{N}} {}_\Omega |X_n| dP < \infty.$$

We finish the proof as in that of Theorem 5.1. We have

$$||E^{\mathcal{F}_n}X_\infty - X_n|| = \sup_{m\in\mathbf{N}} |\langle e_m, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m, X_n\rangle|.$$

It is clear that $(\langle e_m, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m, X_n\rangle)_{n\in\mathbf{N}}$ are real-valued L^1-bounded pramarts in $L^1_{\mathbf{R}}(\mathcal{F})$ which converges a.s. to 0 when $n \to \infty$. Since $(Y_n = E^{\mathcal{F}_n}X_\infty)_{n\in\mathbf{N}}$ is a martingale: $Y_\sigma = E^{\mathcal{F}_\sigma}Y_\tau, \sigma \le \tau$, by similar computation as in (8.1.2) we see that

$$((|\langle e_m, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m, X_n\rangle|)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a uniform sequence of positive L^1-bounded subpramarts. Now applying Lemma VIII.1.15 in [18] to this sequence yields

$$\begin{aligned}\lim_{n\to\infty} ||E^{\mathcal{F}_n}X_\infty - X_n|| &= \lim_{n\to\infty} \sup_{m\in\mathbf{N}} |\langle e_m, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m, X_n\rangle| \\ &= \sup_{m\in\mathbf{N}} \lim_{n\to\infty} |\langle e_m, E^{\mathcal{F}_n}X_\infty\rangle - \langle e_m, X_n\rangle| = 0 \quad a.s.\end{aligned}$$

∎

When E^*_b is separable, Theorem 8.1 yields the a.s. norm convergence for bounded pramarts in $L^1_{E^*_b}(\mathcal{F})$

THEOREM 7.3 *Assume that E^*_b is separable and $(X_n)_{n\in\mathbf{N}}$ is a bounded pramart in $L^1_{E^*_b}(\mathcal{F})$, then there exists $X_\infty \in L^1_{E^*_b}(\mathcal{F})$ such that*
(a) $\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle$ *a.s. for all* $x \in \overline{B}_E$,
(b) $\lim_{n\to\infty} ||x^* - X_n|| = ||x^* - X_\infty||$ *a.s. for all* $x^* \in E^*$,
(c) $\lim_{n\to\infty} ||E^{\mathcal{F}_n} X_\infty - X_n|| = 0$ *a.s.*

PROOF As $(X_n)_{n\in\mathbf{N}}$ is a bounded pramart in $L^1_{E^*_b}(\mathcal{F}) \subset L^1_{E^*_s}(\mathcal{F})$, by Theorem 8.1 there exists $X_\infty \in L^1_{E^*_s}(\mathcal{F})$ such that
(i) $\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle$ *a.s.* for all $x \in \overline{B}_E$,
(ii) $\lim_{n\to\infty} ||x^* - X_n|| = ||x^* - X_\infty||$ *a.s.* for each $x^* \in E^*$,
(iii) $\lim_{n\to\infty} ||E^{\mathcal{F}_n} X_\infty - X_n|| = 0$ *a.s.*
Since E^*_b is separable Banach space, by equicontinuity and density and (ii) we get

$$\lim_{n\to\infty} ||x^* - X_n|| = ||x^* - X_\infty|| \quad a.s. \quad \forall x^* \in E^*_b.$$

It follows that

$$\lim_{n\to\infty} ||X_\infty - X_n|| = 0 \quad a.s.$$

Whence we deduce that $X_\infty \in L^1_{E^*_b}(\mathcal{F})$. ■

Here is a martingale variant of Theorem 8.1.

THEOREM 7.4 *Assume that E is separable Banach space and $(X_n)_{n\in\mathbf{N}}$ is a bounded martingale in $L^1_{E^*_s}(\mathcal{F})$, then there exists $X_\infty \in L^1_{E^*_s}(\mathcal{F})$ such that*
(a) $\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle$ *a.s. for all* $x \in \overline{B}_E$,
(b) $\lim_{n\to\infty} ||x^* - X_n|| = ||x^* - X_\infty||$ *a.s. for each* $x^* \in E^*$,
(c) $\lim_{n\to\infty} ||E^{\mathcal{F}_n} X_\infty - X_n|| = 0$ *a.s.*

PROOF Follows the same lines of the proof of Theorem 8.1 using convergence of submartingales. Note first that $(|X_n|)_{n\in\mathbf{N}}$ is a L^1-bounded positive submartingale in $L^1_{\mathbf{R}}(\mathcal{F})$: $||X_n|| = ||E^{\mathcal{F}_n} X_{n+1}|| \leq E^{\mathcal{F}_n}||X_{n+1}||, \forall n \in \mathbf{N}$. As $(X_n)_{n\in\mathbf{N}}$ is a L^1-bounded martingale, for each $x \in E$, $(\langle x, X_n\rangle)_{n\in\mathbf{N}}$ is a scalar L^1-bounded martingale. By Doob's theorem, $(\langle x, X_n\rangle)_{n\in\mathbf{N}}$ converges a.s to an integrable function φ_x. Also the L^1-bounded positive submartingale $(|X_n|, \mathcal{F}_n)_{n\in\mathbf{N}}$ converges a.s. to an integrable function so that $r(\omega) := \sup_{n\in\mathbf{N}} ||X_n(\omega)|| < \infty$ a.s. Hence $X_n(\omega) \subset r(\omega)\overline{B}_{E^*}$ a.s. By applying Lemma 8.1, and by identifying the limits, we provide a Gelfand-integrable X_∞ satisfying $\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle = \varphi_x$ a.s. Now let $(e_m)_{m\in\mathbf{N}}$ be a dense sequence in the closed unit ball $\overline{B}_E$ of E. Then we have

$$\lim_{n\to\infty}\langle e_m, X_n\rangle = \langle e_m, X_\infty\rangle \quad a.s. \quad \forall m \in \mathbf{N}.$$

As $\sup_{n\in\mathbf{N}} |X_n| < \infty$ a.s, by density we get (a)

$$\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle \quad a.s. \quad \forall x \in \overline{B}_E.$$

Since $(X_n)_{n\in\mathbf{N}}$ is a bounded martingale, the sequence

$$((|\langle e_m, x^* - X_n\rangle|)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a positive L^1-bounded submartingale. Applying Lemma V.2.9 in [26] to this sequence yields

$$\begin{aligned}\lim_{n\to\infty} \|x^* - X_n\| &= \lim_{n\to\infty}\sup_{m\in\mathbf{N}} |\langle e_m, x^* - X_n\rangle| \\ &= \sup_{m\in\mathbf{N}}\lim_{n\to\infty} \langle e_m, x^* - X_n\rangle| \\ &= \sup_{m\in\mathbf{N}} |\langle e_m, x^* - X_\infty\rangle| = \|x^* - X_\infty\| \quad a.s.\end{aligned}$$

which proves (b). In particular, we have that $\|X_n\| \to \|X_\infty\|$ a.s. so that $X_\infty \in L^1_{E^*_s}(\mathcal{F})$ by Fatou Lemma

$$\int_\Omega |X_\infty| dP \le \liminf_n \int_\Omega |X_n| dP \le \sup_{n\in\mathbf{N}} \int_\Omega |X_n| dP < \infty.$$

It is clear that $(\langle e_m, E^{\mathcal{F}_n} X_\infty\rangle - \langle e_m, X_n\rangle)_{n\in\mathbf{N}}$ are real-valued L^1-bounded martingale in $L^1_{\mathbf{R}}(\mathcal{F})$ which converges a.s. to 0 when $n\to\infty$. Since $(Y_n = E^{\mathcal{F}_n} X_\infty)_{n\in\mathbf{N}}$ is a L^1-bounded martingale in $L^1_{E^*_s}(\mathcal{F})$

$$((|\langle e_m, E^{\mathcal{F}_n} X_\infty\rangle - \langle e_m, X_n\rangle|)_{n\in\mathbf{N}})_{m\in\mathbf{N}}$$

is a sequence of positive L^1-bounded submartingales. Now applying Lemma V.2.9 in [26] to this sequence yields

$$\begin{aligned}\lim_{n\to\infty} \|E^{\mathcal{F}_n} X_\infty - X_n\| &= \lim_{n\to\infty}\sup_{m\in\mathbf{N}} |\langle e_m, E^{\mathcal{F}_n} X_\infty\rangle - \langle e_m, X_n\rangle| \\ &= \sup_{m\in\mathbf{N}}\lim_{n\to\infty} |\langle e_m, E^{\mathcal{F}_n} X_\infty\rangle - \langle e_m, X_n\rangle| = 0 \quad a.s.\end{aligned}$$ ■

In this vein we obtain also a martingale variant in a separable dual that is the martingale convergence result in ([26], Proposition V.2.8).

Theorem 7.5 *Assume that E^*_b is separable and $(X_n)_{n\in\mathbf{N}}$ is a bounded martingale in $L^1_{E^*_b}(\mathcal{F})$, then there exists $X_\infty \in L^1_{E^*_b}(\mathcal{F})$ such that*
(a) $\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle$ a.s. for all $x \in \overline{B}_E$,
(b) $\lim_{n\to\infty} \|x^* - X_n\| = \|x^* - X_\infty\|$ a.s. for all $x^* \in E^*$,
(c) $\lim_{n\to\infty} \|E^{\mathcal{F}_n} X_\infty - X_n\| = 0$ a.s.

Proof As $(X_n)_{n\in\mathbf{N}}$ is a bounded martingale in $L^1_{E^*_b}(\mathcal{F}) \subset L^1_{E^*_s}(\mathcal{F})$, by Theorem 8.2 there exists $X_\infty \in L^1_{E^*_s}(\mathcal{F})$ such that
(i) $\lim_{n\to\infty}\langle x, X_n\rangle = \langle x, X_\infty\rangle$ *a.s.* for all $x \in \overline{B}_E$,
(ii) $\lim_{n\to\infty} \|x^* - X_n\| = \|x^* - X_\infty\|$ *a.s.* for each $x^* \in E^*$,
(iii) $\lim_{n\to\infty} \|E^{\mathcal{F}_n} X_\infty - X_n\| = 0$ *a.s.*
Since E^*_b is separable Banach space, by equicontinuity and density and (ii) we get

$$\lim_{n\to\infty} \|x^* - X_n\| = \|x^* - X_\infty\| \quad a.s. \quad \forall x^* \in E^*_b.$$

It follows that

$$\lim_{n\to\infty} ||X_\infty - X_n|| = 0 \quad a.s.$$

Whence we deduce that $X_\infty \in L^1_{E_b^*}(\mathcal{F})$. ■

8. Submartingale in Banach lattice. We finish the paper with an application to almost sure convergence of lattice-valued submartingales and subpramarts. In the sequel E is a separable order continuous Banach lattice. By Davis-Ghoussoub-Lindenstrauss renorming theorem [15], there exist an equivalent lattice norm $|||.|||$ and a countable norming subset $D := (f_m^*)_{m\in\mathbf{N}}$ of the the positive cone E_+^* in the dual E^* of E, i.e. $|||x||| = \sup_{m\in\mathbf{N}}\langle f_m^*, |x|\rangle$, $\forall x \in E$ such that $x_n \to x$ in E whenever $\langle x^*, x_n\rangle \to \langle x^*, x\rangle$ for every $x^* \in D$ and $|||x_n||| \to |||x|||$.

THEOREM 8.1 *Let E be a separable order continuous Banach lattice. Let $(X_n)_{n\in\mathbf{N}}$ be a bounded positive $\mathcal{R}wk(E)$-tight submartingale:*

$$X_n \le E^{\mathcal{F}_n} X_{n+1}, \forall n \in \mathbf{N}$$

in $L^1_E(\mathcal{F})$. Then there is $X_\infty \in L^1_E(\mathcal{F})$ such that $X_n \to X_\infty$ strongly a.s.

PROOF As $(X_n)_{n\in\mathbf{N}}$ is a L^1-bounded submartingale, for each $f^* \in E_+^*$, $(\langle f^*, X_n\rangle)_{n\in\mathbf{N}}$ is a scalar L^1-bounded submartingale. By Doob's theorem, $(\langle f^*, X_n\rangle)_{n\in\mathbf{N}}$ converges a.s to an integrable function φ_{f^*}. By our assumption, $(X_n)_{n\in\mathbf{N}}$ is a bounded and $\mathcal{R}wk(E)$-tight sequence in $L^1_E(\mathcal{F})$. Applying the biting compactness Theorem 4.1 to $(X_n)_{n\in\mathbf{N}}$ provides $X_\infty \in L^1_E(\mathcal{F})$ such that

$$(^*) \qquad \lim_{n\to\infty} \langle f^*, X_n\rangle = \langle f^*, X_\infty\rangle = \varphi_{f^*} \quad a.s.$$

Note that $(||X_n||, \mathcal{F}_n)_{n\in\mathbf{N}}$ is L^1-bounded positive submartingale. Applying Lemma V.2.9 in [26] to the L^1-bounded submartingales $(\langle f_m^*, X_n\rangle)_{n\in\mathbf{N}, m\in\mathbf{N}}$ and using the equivalent norm $|||.|||$, we get

$$\lim_{n\to\infty} |||X_n||| = \lim_{n\to\infty} \sup_{m\in\mathbf{N}} \langle f_m^*, X_n\rangle$$

$$(^{**}) \qquad = \sup_{m\in\mathbf{N}} \lim_{n\to\infty} \langle f_m^*, X_n\rangle = \sup_{m\in\mathbf{N}} \langle f_m^*, X_\infty\rangle = |||X_\infty||| \quad a.s.$$

By $(^*)$ and $(^{**})$ and by the properties of the norm $|||.|||$ we conclude that $X_n \to X_\infty$ strongly a.s. ■

Taking account of the above consideration, we mention the following variant

THEOREM 8.2 *Let E be a separable order continuous Banach lattice. Let $(X_n)_{n\in\mathbf{N}}$ be a bounded positive $\mathcal{R}wk(E)$-tight subpramart in $L^1_E(\mathcal{F})$. Then there is $X_\infty \in L^1_E(\mathcal{F})$ such that $X_n \to X_\infty$ strongly a.s.*

Proof The proof follows the same line as that of Theorem 9.1. By Egghe lemma ([18], Lemma VIII.1.12), for each $f^* \in E^*_+$,

$$(\langle f^*, X_n\rangle)_{n\in\mathbf{N}}$$

is a L^1- bounded subpramart in $L^1_{\mathbf{R}}(\mathcal{F})$ and so is $(||X_n||, \mathcal{F}_n)_{n\in\mathbf{N}}$. By virtue of Millet-Sucheston subpramart convergence theorem ([18], Theorem VIII.1.11), $(\langle f^*, X_n\rangle)_{n\in\mathbf{N}}$ converges a.s to an integrable function φ_{f^*}. Applying the biting compactness Theorem 4.1 provides $X_\infty \in L^1_E(\mathcal{F})$ such that

$$(^*) \lim_{n\to\infty} \langle f^*, X_n\rangle = \langle f^*, X_\infty\rangle = \varphi_{f^*}\ a.s.$$

As $(||X_n||, \mathcal{F}_n)$ is a L^1-bounded subpramart, using Lemma VIII. 1.15 in [18] and the equivalent norm $|||.|||$ we get

$$\lim_{n\to\infty} |||X_n||| = \lim_{n\to\infty} \sup_{m\in\mathbf{N}} \langle f^*_m, X_n\rangle$$

$$(^{**}) \qquad = \sup_{m\in\mathbf{N}} \lim_{n\to\infty} \langle f^*_m, X_n\rangle = \sup_{m\in\mathbf{N}} \langle f^*_m, X_\infty\rangle = |||X_\infty||| \quad a.s.$$

By $(^*)$ and $(^{**})$ and by the properties of the norm $|||.|||$ we conclude that $X_n \to X_\infty$ strongly a.s. ∎

We refer to [15, 18, 23, 28] where several results on a.s. convergence for submartingales and subpramarts in Banach lattice can be found. In particular, if E is a separable Banach lattice with the (RNP) and $(X_n)_{n\in\mathbf{N}}$ is a L^1_E-bounded positive submartingale, then $(X_n)_{n\in\mathbf{N}}$ strongly converges a.s. to an $Y_\infty \in L^1_E(\mathcal{F})$ (Heinich theorem, [18], Theorem III.2.2). An alternative proof can be given using the biting compactness theorem 4.2 and again the above Kadec property of the norm.

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