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## Remark on borderline traces of Besov and Triebel-Lizorkin spaces on noncompact hypersurfaces.

*Dedicated to our teacher Professor Julian Musielak  
on the occasion of his 85 birthday*

**Abstract.** We investigate borderline traces of Besov and Triebel-Lizorkin spaces. The function spaces are defined on noncompact Riemannian manifolds with bounded geometry. We described spaces of traces on noncompact submanifolds that are also of bounded geometry.

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**1. Introduction.** Traces of function spaces of Besov-Triebel-Lizorkin type on hyperplane in  $\mathbb{R}^n$ ,  $n \geq 2$ , has been studied for years. The classical trace problem consists in finding spaces  $X$  and  $Y$ , as subspaces of  $\mathcal{D}'(\mathbb{R}^n)$  and  $\mathcal{D}'(\mathbb{R}^{n-1})$ , respectively, such that a trace operator  $\text{Tr}$  is a continuous, linear surjection

$$\text{Tr} : X \rightarrow Y.$$

In particular it was proved that the trace space for the space  $X = B_{p,q}^s(\mathbb{R}^n)$  is  $Y = B_{p,q}^{s-1/p}(\mathbb{R}^{n-1})$  if  $s > 1/p$ ,  $1 \leq p \leq \infty$ . We refer to Triebel's book [17] for the classical results and historical comments. It was a bit surprising that also in the borderline case  $s = \frac{1}{p}$  one can find the trace space. This case was first investigated by V. I. Burenkov and M. Goldman [2], [6] as well as by M. Frazier and B. Jawerth [3], [4]. It was proved that the corresponding trace space is  $L_p(\mathbb{R}^{n-1})$ .

In 1986 H. Triebel extended the definition of inhomogeneous Besov  $B_{p,q}^s$  and Triebel-Lizorkin  $F_{p,q}^s$  spaces to a noncompact Riemannian manifold  $M^n$  with

bounded geometry, cf. [16] and [18]. Trace problem for the spaces  $B_{p,q}^s(M^n)$  and  $F_{p,q}^s(M^n)$  on noncompact submanifolds  $N^k$  was studied by one of the authors in [14]. However in that paper a strong assumption that  $N^k$  is a totally geodesic submanifold was used. Last time N. Große and C. Schneider proved the trace theorem for the function spaces defined on the manifolds for much wider class of submanifolds, cf. [7]. Their result reads as follows. Let  $N^k$  is  $k$ -dimensional submanifold that satisfies the conditions of Definition 2.3, cf. below. Let  $0 < p, q \leq \infty$  ( $0 < p, q < \infty$  or  $p = q = \infty$  for F-spaces) and

$$s - \frac{n-k}{p} > k \left( \frac{1}{p} - 1 \right)_+ .$$

Then  $\text{Tr}$  is a linear and bounded operator from  $F_{p,q}^s(M^n)$  onto  $B_{p,p}^{s-\frac{n-k}{p}}(N^k)$  and from  $B_{p,q}^s(M^n)$  onto  $B_{p,q}^{s-\frac{n-k}{p}}(N^k)$ . Just as in [14] the borderline case is not considered there. The aim of the paper is to supplement this result by adding the limiting case. Moreover, we want to show that the approach to the trace problem via atomic decomposition, proposed by Frazier and Jawerth in [3], works also in the Riemannian manifold setting.

The main result of the paper asserts that if  $N^k$  is a  $k$ -dimensional submanifold that satisfies the conditions of Definition 2.3 then

$$\text{Tr } B_{p,q}^{\frac{n-k}{p}}(M^n) = L_p(N^k) \quad \text{if } 0 < p < \infty \text{ and } 0 < q < \min\{1, p\}$$

and

$$\text{Tr } F_{p,q}^{\frac{n-k}{p}}(M^n) = L_p(N^k) \quad \text{if } 0 < p \leq 1 \text{ and } 0 < q < \infty .$$

In contrast to the Große and Schneider's proof, that is based on the uniformly localization principle, here we used an atomic decomposition of the function spaces on Riemannian manifolds. The decomposition was constructed in [12].

We assume that the reader is familiar with the basic facts about the Besov and Triebel-Lizorkin spaces on  $\mathbb{R}^n$ . All we need is covered by Triebel's books [17] and [18]. The second book contains also the information about the function spaces on Riemannian manifolds. We also assume some basic knowledge in differential geometry. We refer e.g. to [5] for the needed definitions. One can consult also the books about Sobolev spaces and nonlinear analysis on manifolds [1], [8], where the basic facts of differential geometry needed for analysis are well described.

The paper is organized as follows. In the first section we recall some notation about Riemannian manifolds and submanifolds. In particular for the pair  $(M^n, N^k)$ ,  $N^k$  being the submanifold of  $M^n$ , we introduce the notation of bounded geometry. Here we used the definition of Große and Schneider, cf. [7]. The next section is devoted to the principle of uniform localization. In particular we recall here the definition of the Besov and Triebel-Lizorkin spaces on manifolds with bounded geometry. In Section 4 we recall the atomic decomposition on manifolds and adapt the construction of the atomic decomposition to coverings using Fermi coordinates in a neighbourhood of the submanifolds. These coverings were introduced in [7]. In the last section we formulated and proved the trace theorem.

**2. Manifolds and submanifolds of bounded geometry.** Let  $(M^n, g)$  be a  $n$ -dimensional Riemannian manifold with the Riemannian metric tensor  $g$ . We assume that  $M^n$  is complete and connected. Then the Riemannian distance  $d_M : M^n \times M^n \mapsto [0, \infty)$  is well defined and the exponential map  $\exp_x$  is defined everywhere i. e. for every  $x \in M^n$  and every vector  $v$  belonging to the tangent space  $T_x M^n$ . We recall that the last map is given by

$$\exp_x v = c(x, v, 1), \quad x \in M^n, \quad v \in T_x M^n,$$

where  $t \mapsto c(x, v, t)$  is a geodesic line through  $x$  in direction  $v$ . The exponential map is a diffeomorphism of a ball  $B(0, r) \subset T_x M^n$  onto a geodesic ball  $B_M(x, r)$  in  $M^n$  if the radius  $r$  is sufficiently small. Denoting by  $r_x$  the supremum of all possible radii of such balls we can define an injectivity radius of  $M^n$  as  $r_M = \inf_{x \in M^n} r_x$ . If  $r_M > 0$  then taking  $r \in (0, r_M)$  we see that  $\exp_x : B_x(0, r) \mapsto B_M(x, r)$  is a diffeomorphism for any  $x \in M^n$ .

Using the Riemannian structure we can define a gradient  $\nabla f$  of a sufficiently smooth function  $f$  on  $M^n$  and the divergence  $\operatorname{div} \mathcal{X}$  of the vector field  $\mathcal{X}$  on  $M^n$ . In consequence we can define a Laplace operator by

$$\Delta f = \operatorname{div}(\nabla f).$$

The operator have the following form in the local coordinate system

$$\Delta f = (1/\sqrt{|g|}) \sum_{j,i} \partial_j (\sqrt{|g|} g^{ji} \partial_i f),$$

where  $g^{ji}$  are the components of the inverse of the metric tensor  $g$ . We put also

$$|\nabla^k f|^2 = g^{\alpha_1 \beta_1} \dots g^{\alpha_k \beta_k} \nabla_{\alpha_1} \dots \nabla_{\alpha_k} f \nabla_{\beta_1} \dots \nabla_{\beta_k} \bar{f},$$

(Einstein's summation convention.) So, for any bounded domain  $U \subset M^n$  we can define

$$\|f\|_{C^k(\bar{U})} = \sum_{0 \leq j \leq k} \|\nabla^j f\|_{C(\bar{U})}.$$

**DEFINITION 2.1** The Riemannian manifold  $M^n$  is called a manifold of bounded geometry if the following two conditions are satisfied:

- (a)  $r_M > 0$ ,
- (b)  $|\nabla^k \mathcal{R}_M| \leq C_k$ ,  $k = 0, 1, 2, \dots$ , (i. e. every covariant derivative of the Riemannian curvature tensor is bounded.)

**REMARK 2.2** (i) Note that (a) implies that  $M^n$  is complete.

(ii) The condition (b) is equivalent to any one stated below:

- (b') for every  $0 < r < r_M$  and every multi-index  $\alpha$  there exist positive constants  $C_\alpha$  and  $C$  such that the components of the metric tensor  $g_{j,i}$  and the

components of its inverse  $g^{j,i}$  satisfy in every normal coordinate system on  $B_M(x, r)$  the inequalities

$$|\nabla^\alpha g_{j,i}| \leq C_\alpha \quad |g^{j,i}| \leq C.$$

(b'') Let us fix  $0 < r < r_M$  and let  $B_M(x, r)$  and  $B_M(x', r)$  be two geodesic balls with the normal coordinates  $y : B_M(x, r) \mapsto \mathbb{R}^n$ ,  $y' : B_M(x', r) \mapsto \mathbb{R}^n$  such that  $B_M(x, r) \cap B_M(x', r) \neq \emptyset$ . Consider the mapping  $y' \circ y^{-1} : y(B_M(x, r) \cap B_M(x', r)) \mapsto \mathbb{R}^n$ . Then for every multi-index  $\alpha$  there is a positive constant  $C_\alpha$  independent of  $x, x'$  such that the inequalities

$$|\partial_y^\alpha y' \circ y^{-1}| \leq C_\alpha$$

holds.

The condition (b'') is often more useful in applications than condition (b).

- (iii) Examples of manifolds of bounded geometry include all compact manifolds, all homogeneous spaces i. e. manifolds with a transitive group of isometries (symmetric spaces, Lie groups with left (right) Riemannian structure), as well as leaves of foliations of compact manifolds.

Let  $N^k \subset M^n$  be an embedded submanifold, meaning, there is a  $k$ -dimensional manifold  $N'$  and an injective immersion  $f : N' \rightarrow M^n$  with  $f(N') = N^k$ ,  $1 \leq k < n$ . Let  $x \in N^k$ . A normal ball  $B^\perp(x, r)$  centered at  $x$  with radius  $r > 0$  is defined in the following way

$$B^\perp(x, r) := \{z \in M^n : d_M(x, z) \leq r, \exists \varepsilon_o \forall \varepsilon < \varepsilon_o \ d_M(x, z) = d_M(B_N(x, \varepsilon), z)\}$$

with

$$B_N(x, \varepsilon) = \{u \in N^k : d_N(u, x) \leq \varepsilon\},$$

and  $d_M$  and  $d_N$  denote the distance functions in  $M^n$  and  $N^k$ , respectively.

The following definition of submanifolds of bounded geometry where introduced in [7].

**DEFINITION 2.3** Let  $(M^n, g)$  be a Riemannian manifold with a  $k$ -dimensional embedded submanifold  $(N^k, g|_{N^k})$ . We say that  $(M^n, N^k)$  is of bounded geometry if the following is fulfilled

- (i)  $(M^n, g)$  is of bounded geometry.
- (ii) The injectivity radius  $r_N$  of  $(N^k, g|_{N^k})$  is positive.
- (iii) There is a collar around  $N^k$  (a tubular neighbourhood of fixed radius), i.e., there is  $r_c > 0$  such that for all  $x, y \in N^k$  with  $x \neq y$  the normal balls  $B^\perp(x, r_c)$  and  $B^\perp(y, r_c)$  are disjoint.
- (iv) The mean curvature  $\mathcal{L}$  of  $N^k$  given by

$$\mathcal{L}(\mathcal{X}, \mathcal{Y}) := \nabla_{\mathcal{X}}^M \mathcal{Y} - \nabla_{\mathcal{X}}^N \mathcal{Y}, \quad \mathcal{X}, \mathcal{Y} \in TN^k,$$

and all its covariant derivatives are bounded. Here,  $\nabla^M$  is the Levi-Civita connection of  $(M^n, g)$  and  $\nabla^N$  the one of  $(N^k, g|_{N^k})$ .

REMARK 2.4 (i) If the normal bundle of  $N^k$  in  $M^n$  is trivial then condition (iii) in Definition 2.3 simply means that  $\{z \in M^n : \text{dist}_M(z, N^k) \leq r_c\}$  is isomorphic to  $B^{n-k}(0, r_c) \times N^k$ . Then

$$(1) \quad F : B^{n-k}(0, r_c) \times N^k \ni (t, z) \mapsto \exp_z^M(t^i \nu_i) \in M^n$$

is a diffeomorphism onto its image, where  $(t^1, \dots, t^{n-k})$  are the coordinates for  $t$  with respect to a standard orthonormal basis on  $\mathbb{R}^{n-k}$  and  $(\nu_1, \dots, \nu_{n-k})$  is an orthonormal frame for the normal bundle of  $N^k$  in  $M^n$ . If the normal bundle is not trivial  $F$  still exists locally, which means that for all  $x \in N^k$  and  $\varepsilon$  smaller than the injectivity radius of  $N^k$ , the map

$$F : B^{n-k}(0, r_c) \times B_N(x, \varepsilon) \ni (t, z) \mapsto \exp_z^M(t^i \nu_i) \in M^n$$

is a diffeomorphism onto its image. All included quantities are as in the case of a trivial vector bundle, but  $\nu_i$  is now just a local orthonormal frame of the normal bundle. By abuse of notation, we suppress here and in the following the dependence of  $F$  on  $\varepsilon$  and  $x$ .

(ii) If  $(M^n, N^k)$  is of bounded geometry then  $N^k$  is a manifold with bounded geometry, cf. [7].

**3. Coverings, trivializations and function spaces.** Following H. Triebel [16] we define the Triebel-Lizorkin spaces  $F_{p,q}^s(M^n)$  via uniform localization principle and then the Besov spaces  $B_{p,q}^s(M^n)$  by interpolation. To simplify the notation we assume that  $M^n$  is noncompact. The changes for compact manifold are obvious.

Let  $\{U_j\}_j$  be a covering of  $M^n$  by open bounded sets. The maximal number of the sets with non-empty intersection in the covering is called a *multiplicity of the covering*. A covering with finite multiplicity is called *uniformly locally finite*.

Let  $\mathcal{A} = \{(U_j, \kappa_j)\}_j$  be an atlas of the manifold  $M^n$  and  $\{\varphi_j\}_j \subset C_0^\infty(M^n)$  a resolution of unity subordinated to  $\mathcal{A}$ . Following [7] we will call  $\mathcal{T} = \{(U_j, \kappa_j, \varphi_j)\}_j$  a *trivialization of the manifold  $M^n$*  if the covering  $\{U_j\}_j$  is uniformly locally finite.

Two atlases  $\mathcal{A} = \{(U_j, \kappa_j)\}_j$  and  $\tilde{\mathcal{A}} = \{(V_k, \eta_k)\}_k$  of the manifold  $M^n$  are called *compatible* if for any  $\ell \in \mathbb{N}_0$  there is a constant  $C_\ell > 0$  such that for all  $j$  and  $k$  with  $U_j \cap V_k \neq \emptyset$  and all multi-index  $\alpha$ ,  $|\alpha| \leq \ell$  we have

$$|D^\alpha(\eta_k^{-1} \circ \kappa_j)| \leq C_\ell \quad \text{and} \quad |D^\alpha(\kappa_j^{-1} \circ \eta_k)| \leq C_\ell$$

Two trivializations  $\mathcal{T} = \{(U_j, \kappa_j, \varphi_j)\}_j$  and  $\tilde{\mathcal{T}} = \{(V_k, \eta_k, \psi_k)\}_k$  of the manifold  $M^n$  are called compatible if the corresponding atlases  $\mathcal{A} = \{(U_j, \kappa_j)\}_j$  and  $\tilde{\mathcal{A}} = \{(V_k, \eta_k)\}_k$  are compatible.

REMARK 3.1 For the manifold  $M^n$  of bounded geometry there exist a number  $0 < r_0 < r_M$  such that if  $r \in (0, r_0)$  then there exists a countable uniformly locally finite covering  $\{B_M(x_j, r)\}_j$  of  $M^n$  by balls of radius  $r$ , cf. [9, Lemma 1.2]. Let  $\mathcal{A}^{geo} = \left\{ (B_M(x_j, r), \exp_{x_j}) \right\}_j$  denotes the atlas such that the corresponding

covering by geodesic balls is uniformly locally finite. Any two such atlases are compatible.

It can be easily proved that, for the atlas  $\mathcal{A}^{geo}$  there exists a corresponding resolution of unity  $\{\varphi_j\} \subset C_0^\infty(M^n)$  with the following properties:

$$0 \leq \varphi_j \leq 1, \quad \text{supp } \varphi_j \subset B_M(x_j, r), \quad j = 1, 2, \dots, \quad \sum_j \varphi_j = 1 \quad \text{on } M^n;$$

for any multi-index  $\alpha$  there exists a positive number  $b_\alpha$  with

$$\left| D^\alpha(\varphi_j \circ \exp_{x_j}) \right| \leq b_\alpha, \quad j = 1, 2, \dots$$

We will call the trivialization  $\mathcal{T}^{geo} = \left\{ (B_M(x_j, r), \exp_{x_j}, \varphi_j) \right\}_j$  a *geodesic trivialization*.

One can easily show that if  $0 < r < r_M/3$  then any two geodesic trivialization are compatible.

We are ready to define the function spaces via uniform localization principle.

**DEFINITION 3.2** Let  $\mathcal{T}^{geo} = \left\{ (B_M(x_j, r), \exp_{x_j}, \varphi_j) \right\}_j$  be a geodesic trivialization of connected manifold  $M^n$  with bounded geometry,  $0 < r < r_M/3$ .

1. Let either  $0 < p < \infty$ ,  $0 < q \leq \infty$ , or  $p = q = \infty$ . Let  $-\infty < s < \infty$ . Then

$$F_{p,q}^s(M^n) = \left\{ f \in \mathcal{D}'(M^n) : \|f\|_{F_{p,q}^s(M^n)} = \left( \sum_j \|\varphi_j f \circ \exp_{x_j}\|_{F_{p,q}^s(\mathbb{R}^n)}^p \right)^{1/p} < \infty \right\}$$

(with the usual modification if  $p = \infty$ ).

2. Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ . Let  $-\infty < s_0 < s < s_1 < \infty$ . Then

$$B_{p,q}^s(M^n) = (F_{p,p}^{s_0}(M^n), F_{p,p}^{s_1}(M^n))_{\theta,q}$$

with  $s = (1-\theta)s_0 + \theta s_1$ . Here  $(\cdot, \cdot)_{\theta,q}$  denotes the real method of interpolation.

The properties of Triebel-Lizorkin function spaces justify the following definition, cf. [7].

**DEFINITION 3.3** Let  $(M^n, g)$  be a Riemannian manifold of bounded geometry. Moreover, let a trivialization  $\mathcal{T} = \{(U_j, \kappa_j, \varphi_j)\}_j$  be given. We say that  $\mathcal{T}$  is admissible if the following conditions are fulfilled:

(B1)  $\mathcal{T}$  is compatible with a geodesic trivialization  $\mathcal{T}^{geo}$  defined in Remark 3.1.

(B2) For all  $\ell \in \mathbb{N}$  there exists  $C_\ell > 0$  such that for all  $j$  and all multi-index  $\alpha$  with  $|\alpha| \leq \ell$

$$|D^\alpha(\varphi_j \circ \kappa_j)| \leq C_\ell.$$

PROPOSITION 3.4 *Let  $-\infty < s < \infty$  and either  $0 < p < \infty$ ,  $0 < q \leq \infty$ , or  $p = q = \infty$ . Let  $\mathcal{T} = \{(U_j, \kappa_j, \varphi_j)\}_j$  be an admissible trivialization of the Riemannian manifolds  $M^n$  with bounded geometry then*

$$F_{p,q}^s(M^n) = \left\{ f \in \mathcal{D}'(M^n) : \|f\|_{F_{p,q}^s(M^n)}^{\mathcal{T}} = \left( \sum_j \|\varphi_j f \circ \kappa_j\|_{F_{p,q}^s(\mathbb{R}^n)}^p \right)^{1/p} < \infty \right\}$$

and  $\|f\|_{F_{p,q}^s(M^n)}^{\mathcal{T}}$  is an equivalent (quasi)-norm in  $F_{p,q}^s(M^n)$ .

PROOF The proposition follows immediately from the Definition 3.3, diffeomorphism properties and pointwise multipliers for Triebel-Lizorkin spaces, cf. [15]. ■

**4. Atomic decompositions on manifolds.** We start with the recalling of the definition of atoms. We follows the ideas from [12] and [11]. For an open set  $Q \subset M^n$  and  $r \in \mathbb{R}_+$  we put  $rQ = \{x \in M^n : \text{dist}_M(x, Q) < r\}$ .

DEFINITION 4.1 (CF.[12]) Let  $s \in \mathbb{R}$  and  $0 < p \leq \infty$ . Let  $L$  and  $K$  be integers such that  $L \geq 0$  and  $K \geq -1$ . Let  $r > 0$  and  $C \geq 1$  be positive constants. Let  $Q \subset M^n$  be an open connected set with  $\text{diam } Q = r$ .

(a) A smooth function  $a(x)$  is called an  $1_L$ -atom centered in  $Q$  if

$$(2) \quad \text{supp } a \subset \frac{r}{2}Q,$$

$$(3) \quad \sup_{y \in M^n} |\nabla^k a(y)| \leq C \quad \text{for any } k \leq L.$$

(b) A smooth function  $a(x)$  is called an  $(s, p)_{L,K}$ -atom centered in  $Q$  if

$$(4) \quad \text{supp } a \subset \frac{r}{2}Q,$$

$$(5) \quad \sup_{y \in M^n} |\nabla^k a(y)| \leq C r^{s-k-\frac{n}{p}}, \quad \text{for any } k \leq L,$$

$$(6) \quad \left| \int_{M^n} a(y) \psi(y) dy \right| \leq C r^{s+K+1+n/p'} \left\| \psi|_{C^{K+1}(\overline{rQ})} \right\|$$

holds for any  $\psi \in C_0^\infty(M^n)$ .

If  $K = -1$  then (6) means that no moment conditions are required.

To work with an atomic decomposition for  $p > 1$  we should control location of atoms. This can be done by the sequence of regular atlases.

DEFINITION 4.2 Let  $(M^n, g)$  be a Riemannian manifold of bounded geometry. Let  $0 < \lambda < 1$  and  $R > 0$ .

(i) An atlas  $\mathcal{A} = \{(U_i, \kappa_i)\}, i \in I$  of  $M^n$  is called  $(\lambda, R)$ -regular if it is compatible with  $\mathcal{A}^{geo}$  and if for any  $i \in I$  there exist a point  $x_i \in M^n$  such that

$$B_M(x_i, \lambda R) \subset U_i \subset B_M(x_i, R)$$

(ii) Let a sequence  $\mathcal{A}_j = \{(U_{j,i}, \kappa_{j,i})\}, i \in I_j, j = 0, 1, 2, \dots$  of atlases of  $M^n$  be given. We say that the sequence  $\mathcal{A}_j, j = 0, 1, 2, \dots$ , is  $(\lambda, R)$ -regular if the atlas  $\mathcal{A}_j$  is  $(\lambda, 2^{-j}R)$ -regular for any  $j = 0, 1, \dots$

Now we define the families of atom we need for the decompositions of Besov and Triebel-Lizorkin spaces.

DEFINITION 4.3 Let  $\mathcal{A}_j = \{(U_{j,i}, \kappa_{j,i})\}, i \in I_j, j = 0, 1, \dots$ , be  $(\lambda, R)$ -regular sequence of atlases of the manifold  $M^n$ .

Let  $s \in \mathbb{R}$  and  $0 < p \leq \infty$ . Let  $L$  and  $K$  be integers satisfying the assumption of Definition 4.1. A family  $\mathcal{A}_{s,p}^{L,K}$  of  $1_L$ -atoms and  $(s, p)_{L,K}$ -atoms is called a building family of atoms corresponding to the sequence  $\{\mathcal{A}_j\}$  if:

- (a) all atoms belonging to the family are centered at the sets of the atlases  $\mathcal{A}_j$
- (b) all atoms belonging to the family satisfy the conditions (2) - (6) with the same positive constant  $C$ ,
- (c) the family contains all atoms satisfying (a)-(b).

One can construct the  $(\lambda, R)$ -regular sequence of atlases using the notations of separations and discretizations of manifolds, cf. [13].

DEFINITION 4.4 (CF. [13]) Let  $V$  be a nonempty subset of a connected Riemannian manifold  $M^n$  of dimension  $n$ . Let  $R > 0$  be a positive number,  $\beta = 1, 2, \dots$  be a positive integer.

A subset  $\mathcal{H}$  of  $V$  is said to be  $R$ -separation of  $M^n$ , if the distance between any two distinct points of  $\mathcal{H}$  is greater than or equal to  $R$ .

A subset  $\mathcal{H}$  of  $V$  is called an  $(R, \beta)$ -discretization of  $M^n$  if it is an  $R$ -separation of  $V$  and

$$V \subset \bigcup_{x \in \mathcal{H}} B_M(x, \beta R).$$

The following lemma can be easily proved by the Zorn Lemma and the volume argument, cf.[13].

LEMMA 4.5 Let  $M^n$  be a connected Riemannian manifold with bounded geometry. Let  $0 < R < r_M/3$  and  $\beta \in \mathbb{N}$ .

(a) For any connected nonempty open subset  $V$  of the Riemannian manifold  $M^n$  there is an  $(R, 1)$ -discretization of  $V$ .

(b) Let  $\ell$  be a positive integer. If  $\mathcal{H}$  is an  $(R, \beta)$ -discretization of  $M^n$  and  $\ell \geq \beta$  then the family  $\{B_M(x, \ell R)\}_{x \in \mathcal{H}}$  is a uniformly locally finite covering of  $V$  with multiplicity that can be estimated from above by constant depending on  $\dim M^n$  and  $\ell$ , but independent of  $R$ .

REMARK 4.6 Let  $M^n$  be a manifold with bounded geometry and  $0 < R < r_M/3$ . Let  $\{x_{j,i}\}_{i \in I_j}$  be a  $(2^{-j}R, 1)$ -discretization of the manifold  $M^n$ ,  $j = 0, 1, 2, 3, \dots$ . Then the sequence  $\mathcal{A}_j = \{(B_M(x_{j,i}, 2^{-j}R), \exp_{x_{j,i}})\}$ ,  $i \in I_j$ ,  $j = 0, 1, \dots$  is a  $(\frac{1}{2}, R)$ -regular sequence of trivializations cf. [13].

For  $c \in \mathbb{R}$  let  $[c]$  stand for the largest integer less than or equal to  $c$  and  $c_+ = \max(c, 0)$ . Moreover, for the characteristic function  $\chi_{j,i}$  of the set  $U_{j,i}$  we put  $\chi_{j,i}^{(p)} = 2^{jn/p} \chi_{j,i}$ . Moreover we put

$$\sigma_p = n \left( \frac{1}{p} - 1 \right)_+ \quad \text{and} \quad \sigma_{p,q} = n \left( \max \left( \frac{1}{p} - 1; \frac{1}{q} - 1 \right) \right)_+$$

THEOREM 4.7 ([12]) Let  $s \in \mathbb{R}$ ,  $0 < q \leq \infty$ , let  $0 < p < \infty$  or  $p = q = \infty$  in the case of the  $F_{p,q}^s$ -scale and  $0 < p \leq \infty$  in the case of  $B_{p,q}^s$ -scale. Let  $L$  and  $K$  be fixed integers satisfying the following conditions  $L \geq ([s] + 1)_+$  and

$$K \geq \max([\sigma_p - s], -1) \text{ (in } B\text{-case)} \quad \text{or} \quad K \geq \max([\sigma_{p,q} - s], -1) \text{ (in } F\text{-case)}.$$

Let  $\mathcal{A}_j$ ,  $j = 0, 1, \dots$ , be  $(\lambda, R)$ -regular sequence of atlases of  $M^n$ ,  $0 < R < r_M$ . There exists a building family of atoms  $\mathcal{A}_{s,p}^{L,K}$  corresponding to the sequence  $\mathcal{A}_j$  with the following properties:

(a) each  $f \in F_{p,q}^s(M^n)$  ( $f \in B_{p,q}^s(M^n)$ ) can be decomposed as follows

$$(7) \quad f = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} s_{j,i} a_{j,i} \quad (\text{convergent in } \mathcal{D}'(M^n)),$$

$$(8) \quad \left\| \left( \sum_{j,i=0}^{\infty} (|s_{j,i}| \chi_{j,i}^{(p)}(\cdot))^q \right)^{1/q} \right\|_p < \infty, \quad \left( \left( \sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} |s_{j,i}|^p \right)^{q/p} \right)^{1/q} < \infty \right)$$

with atoms belonging to  $\mathcal{A}_{s,p}^{L,K}$ .

(b) Conversely, suppose that  $f \in \mathcal{D}'(M^n)$  can be represented as in (7) and (8) with atoms belonging to  $\mathcal{A}_{s,p}^{L,K}$ . Then  $f \in F_{p,q}^s(M^n)$  ( $f \in B_{p,q}^s(M^n)$ ).

Furthermore, the infimum of (8) with respect to all admissible representations (for fixed sequence of coverings and fixed integers  $L, K$ ) is an equivalent norm in  $F_{p,q}^s(M^n)$  ( $B_{p,q}^s(M^n)$ ).

PROOF The above theorem was proved for sequences of atlases defined in Remark 4.6, but the proof can be repeated literally for  $(\lambda, R)$ -regular sequences of atlases. ■

Now we describe the sequence of regular atlases suitable for investigation the traces of function spaces on submanifolds via atomic decomposition. We adopt the approach via Fermi coordinates system due to N. Große and C. Schneider, cf. [7]. We recall that the maps  $F$  are defined in (1).

DEFINITION 4.8 Let  $(M^n, N^k)$  be of bounded geometry. Let moreover  $0 < R \leq \min\{\frac{1}{2}r_N, \frac{1}{4}r_M, \frac{1}{2}r_c\}$ , where  $r_N$  is the injectivity radius of  $N^k$  and  $r_M$  the one of  $M^n$  and  $r_c$  is as in Definition 2.3. For any  $j = 0, 1, 2, \dots$  we put

$$\begin{aligned} U_j(N^k) &= \bigcup_{x \in N^k} B^\perp(x, 2^{-j}R), \\ \{x_{j,i}\}_{i \in I} &\text{ be an } (2^{-j}R, 1)\text{-discretization of the manifold } N^k \\ &\quad \text{(with the induced metric } g|_N), \\ \{x_{j,i}\}_{i \in \tilde{I}} &\text{ be an } (2^{-j}R, 1)\text{-discretization of the subset } M^n \setminus U_j(N^k) \\ &\quad \text{of the manifold } M^n. \end{aligned}$$

We consider the covering  $\{U_{j,i}\}$  of the manifold  $M^n$  with

$$\begin{aligned} U_{j,i} &= B_M(x_{j,i}, R) \quad \text{for some } i \in \tilde{I}, \\ U_{j,i} &= F(B^{n-k}(0, 2R) \times B_N(x_{j,i}, R)) \quad \text{for some } i \in I. \end{aligned}$$

Coordinates  $\kappa_{j,i}$  on  $U_{j,i}$  are chosen to be geodesic coordinates  $\exp_{x_{j,i}}^M$  if  $U_{j,i} = B_M(x_{j,i}, 2^{-j}R)$  with  $i \in \tilde{I}$  and to be Fermi coordinates

$$(9) \quad \kappa_{j,\ell} : B^{n-k}(0, 2R) \times B_N(x_{j,i}, 2^{-j}R) \ni (t, x) \mapsto \exp_{\exp_{x_{j,i}}^N(\lambda_{j,i}(x))}^M(t^m \nu_m)$$

if  $i \in I$ . Here  $(t_1, \dots, t_{n-k})$  are the coordinates for  $t$  with respect to a standard orthonormal basis on  $\mathbb{R}^{n-k}$ ,  $(\nu_1, \dots, \nu_{n-k})$  is an orthonormal frame for the normal bundle of  $B_N(x_{j,i}, R)$  in  $M^n$ ,  $\exp^N$  is the exponential map on  $N^k$  with respect to the induced metric  $g|_N$ , and  $\lambda_{j,i} : \mathbb{R}^k \rightarrow T_{x_{j,i}}N^k$  is the choice of an orthonormal frame on  $T_{x_{j,i}}N^k$ .

PROPOSITION 4.9 Let  $(M^n, N^k)$  be of bounded geometry.

- (i) For any  $j = 0, 1, 2, \dots$  there is a partition of unity subordinated to the atlas  $\mathcal{A}_j^{FC} = \{(U_{j,i}, \kappa_{j,i})\}$  introduced in Definition 4.8 fulfilling condition (B2) of Definition 3.3.
- (ii) Let  $\mathcal{T}_j^{FC}$  be a trivialization  $M^n$  given by the atlas  $\mathcal{A}_j^{FC}$  together with the subordinated partition of unity. Then,  $\mathcal{T}_j^{FC}$  is an admissible trivialization.
- (iii) The sequence  $\mathcal{A}_j^{FC}$ ,  $j = 0, 1, 2, \dots$  is  $(\lambda, R)$ -regular for some  $0 < \lambda < 1$ .

PROOF The points (i) and (ii) were proved in [7] so it remains to prove the point (iii).

The couple  $(M^n, N^k)$  is of bounded geometry therefore there exists a constant  $C > 0$  such that  $|\mathcal{R}_M| \leq C$  and  $|\mathcal{L}| < C$ . Let

$$(10) \quad U = F(B^{n-k}(0, 2R) \times B_N(x, 2R)), \quad x \in N^k,$$

and  $R$  satisfies the condition of Definition 4.8. Let  $\kappa$  be a chart for  $U$  defined as in (9). Then there is a constant  $C' > 0$  depending on  $n, k$  and  $C$  but independent of

$x$  such that the metric tensor  $g$  with respect to the maps  $\kappa$  defined by (9) can be estimated uniformly i.e.

$$|g_{j,i}| \leq C' \quad \text{and} \quad |g^{j,i}| \leq C',$$

with constant  $C'$  cf. [7]. But this implies that there are constants  $c'', C''' > 0$  such that

$$(11) \quad c'' d_e(0, y) \leq d_M(x, \kappa(y)) \leq C''' d_e(0, y), \quad y \in U.$$

Moreover the constants are independent of  $U$ . But any set  $U_{j,i}$ ,  $i \in I$  is subset of some set  $U$  and  $\kappa_{j,i}$  is the restriction of  $\kappa$  to the set  $U_{j,i}$ , therefore the last estimates implies that the sequence  $\mathcal{A}_j^{FC}$ ,  $j = 0, 1, 2, \dots$  is  $(\lambda, R)$ -regular for some  $0 < \lambda < 1$ . ■

**5. Traces.** Let  $M^n$  be a connected Riemannian manifold with bounded geometry and  $N^k$  its connected submanifold such that the pair  $(M^n, N^k)$  is of bounded geometry. If  $f$  is a continuous function on  $M^n$  we can define its trace  $\text{Tr } f$  by restriction

$$\text{Tr } f = f|_N.$$

If  $f \in C(M^n)$  then of course  $\text{Tr } f \in C(N^k)$ . Since the spaces of test functions  $C_0^\infty(M^n)$  is dense in  $B_{p,q}^s(M^n)$  and  $F_{p,q}^s(M^n)$  if  $p < \infty$  and  $q < \infty$  we can extend the trace operator to the whole spaces if there is a constant  $C > 0$  such that

$$\|\text{Tr } f|X(N^k)\| \leq C \|f|A_{p,q}^s(M^n)\|, \quad \text{for all } f \in C_0^\infty(M^n)$$

where  $X(N^k)$  is a function space on  $N^k$  and  $A_{p,q}^s(M^n)$  stands for  $B_{p,q}^s(M^n)$  or  $F_{p,q}^s(M^n)$ .

Now we prove the main result of the paper.

**THEOREM 5.1** *Let  $M^n$  be an  $n$ -dimensional connected Riemannian manifold and let  $N^k$  be a connected  $k$ -dimensional submanifold,  $1 \leq k \leq n - 1$ . Let  $(M^n, N^k)$  be of bounded geometry.*

(i) *If  $0 < p < \infty$ ,  $0 < q \leq \min(1, p)$ , and  $s = \frac{n-k}{p}$  then there exists a linear and bounded trace operator  $\text{Tr}$  from  $B_{p,q}^s(M^n)$  onto  $L_p(N^k)$ , and*

$$\text{Tr}(B_{p,q}^s(M^n)) = L_p(N^k).$$

(ii) *If  $0 < p \leq 1$ ,  $0 < q < \infty$  and  $s = \frac{n-k}{p}$  then there exists a linear and bounded trace operator  $\text{Tr}$  from  $F_{p,q}^s(M^n)$  onto  $L_p(N^k)$ ,*

$$\text{Tr}(F_{p,q}^s(M^n)) = L_p(N^k).$$

**PROOF Step 1** We prove the theorem for Besov spaces. The proof is based on the atomic decomposition. This approach was used by Frazier and Jawerth in [3] and [4] in the case of borderline traces on hyperplains in  $\mathbb{R}^n$ , cf. also [10]. For Triebel-Lizorkin spaces the theorem can be proved analogously. However in that

case one can use also the uniform localization principle as it was done in [7] for nonlimiting traces. Please note that in contrast to the case  $s > \frac{n-k}{p}$  now the result for Besov spaces does not follow by interpolation from the statement for Triebel-Lizorkin spaces.

*Step 2* In this step we prove that the trace operator can be correctly defined and that  $\text{Tr } f \in L_p(N^k)$  if  $f \in B_{p,q}^s(M^n)$ ,  $s = \frac{n-k}{p}$ .

*Substep 2.1* First we consider an atomic decomposition of an element  $f \in B_{p,q}^s(M^n)$ ,  $s = \frac{n-k}{p}$  and show that the decomposition leads to some trace on  $N^k$ . Let

$$(12) \quad f = \sum_j \sum_i s_{j,i} a_{j,i} \quad \text{convergence in } \mathcal{D}'(M^n),$$

$$\text{with } \left( \sum_j \left( \sum_i |s_{j,i}|^p \right)^{q/p} \right)^{1/q} < \infty$$

be an atomic decomposition of  $f$  subordinated to the sequence of covering  $\mathcal{A}_j^{FC}$  described in Proposition 4.9. The functions

$$f_\ell = \sum_{j=0}^{\ell} \sum_i s_{j,i} a_{j,i}$$

are continuous so the trace  $\text{Tr } f_\ell$  is well defined by restriction  $\text{Tr } f_\ell = f|_N$ . We show that the sequence  $\text{Tr } f_\ell$  converges in  $L_p(N^k)$ .

Let  $\widehat{I}_j = I_j \cup \{i \in \widetilde{I}_j \text{ such that } 2^{-j}U_{j,i} \cap N^k \neq \emptyset\}$  cf. Definition 4.8. Moreover if  $i \in \widehat{I}_j \setminus I_j$  then  $2^{-j}U_{j,i} \cap N^k \subset U$  for some  $U$  defined by (10). In consequence (11) implies  $\text{diam}_N(2^{-j}U_{j,i} \cap N^k) \leq C2^{-j}$ , for some constant  $C$  independent of  $j$  and  $i$ . The sequence of covering is uniformly locally finite therefore

$$\text{Tr} \left( \sum_{j=0}^{\ell} \sum_i s_{j,i} a_{j,i} \right) = \sum_{j=0}^{\ell} \sum_{i \in \widehat{I}_j} s_{j,i} \text{Tr } a_{j,i}$$

and

$$(13) \quad \left( \int_{N^k} \left| \sum_{j=m}^{\ell} \sum_i s_{j,i} \text{Tr } a_{j,i}(x) \right|^p dx \right)^{1/p}$$

$$\leq C \left( \int_{N^k} \left( \sum_{j=m}^{\ell} \left( \sum_i |s_{j,i}|^p 2^{jk} \widetilde{\chi}_{j,i}(x) \right)^{1/p} \right)^p dx \right)^{1/p}$$

Where  $\widetilde{\chi}_{j,i}(x)$  is a characteristic function of the ball in  $N$  that contains the set  $2^{-j}U_{j,i} \cap N^k$ ,  $i \in \widehat{I}_j$ . The radius of the ball is uniformly bounded by  $C2^{-j}$ . But

$q \leq \min(1, p)$  therefore by monotonicity of  $\ell_q$  spaces and triangle inequality we get

$$\begin{aligned}
 & \left( \int_{N^k} \left( \sum_{j=m}^{\ell} \left( \sum_i |s_{j,i}|^p 2^{jk} \widetilde{\chi}_{j,i}(x) \right)^{\frac{1}{p}} \right)^p dx \right)^{\frac{1}{p}} \\
 & \leq \left( \int_{N^k} \left( \sum_{j=m}^{\ell} \left( \sum_i |s_{j,i}|^p 2^{jk} \widetilde{\chi}_{j,i}(x) \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\
 & \leq C \left( \sum_{j=m}^{\ell} \left( \int_{N^k} \sum_i |s_{j,i}|^p 2^{jk} \widetilde{\chi}_{j,i}(x) dx \right)^{q/p} \right)^{1/q} \\
 (14) \quad & \leq C \left( \sum_{j=m}^{\ell} \left( \sum_i |s_{j,i}|^p \right)^{q/p} \right)^{1/q}.
 \end{aligned}$$

Now it follows from (12) - (14) that the sequence  $f_\ell|_N$  is a Cauchy sequence in  $L_p(N^k)$  so it is convergent. Moreover

$$(15) \quad \left\| \lim_{\ell \rightarrow \infty} \text{Tr } f_\ell|_{L_p(N^k)} \right\| \leq C \left( \sum_j \left( \sum_i |s_{j,i}|^p \right)^{q/p} \right)^{1/q}.$$

*Substep 2.2* The atomic decomposition of the  $f \in B_{p,q}^s(M^n)$ ,  $s = \frac{n-k}{p}$  is not unique, therefore we should show that  $\lim_{\ell \rightarrow \infty} \text{Tr } f_\ell$  is independent of the given decomposition, at least for continuous function  $f$ . The rest will follow by density argument. Let  $f \in B_{p,q}^s(M^n)$ ,  $s = \frac{n-k}{p}$ , be a continuous function. Let us take the atomic decomposition of  $f$  given by (12). Arguing as in (13) and (14) we can show that

$$\left( \int_{M^n} \left| \sum_{j=m}^{\ell} \sum_i s_{j,i} a_{j,i}(x) \right|^p dx \right)^{1/p} \leq C \left( \sum_{j=m}^{\ell} \left( \sum_i |s_{j,i}|^p \right)^{q/p} \right)^{1/q}.$$

Thus  $f_\ell$  is a Cauchy sequence in  $L_p(M^n)$ . But it converges in  $\mathcal{D}'(M^n)$  to  $f$  so it converges to  $f$  also in  $L_p(M^n)$ . In consequence it contains a subsequence converging to  $f$  a.e. But all functions  $f_\ell$  and  $f$  are continuous therefore the subsequence converges to  $f$  everywhere. This means that the sequence  $\text{Tr } f_\ell$  contains the subsequence converging to  $f|_N$  pointwise. So  $\text{Tr } f_\ell$  converges to  $f|_N$  in  $L_p(N^k)$ . Now (15) implies

$$\left\| \text{Tr } f|_{L_p(N^k)} \right\| \leq C \|f|_{B_{p,q}^s(M^n)}\|,$$

if  $f \in B_{p,q}^s(M^n)$  is a continuous function and  $s = \frac{n-k}{p}$ . *Step 3* To prove that the operator  $\text{Tr}$  is onto we construct an extension operator. We follow the ideas of [3].

Let  $h \in L_p(N^k)$ . We may assume that  $h$  is a nonnegative function since any complex-valued function is a sum of two real-valued functions and any real-valued function is a sum of two nonnegative functions. Once more we use the Fermi coordinates associated to the covering  $\mathcal{A}_j^{FC}$ ,  $j = 0, 1, \dots$ , constructed in Proposition 4.9, cf. also Definition 4.8. Let  $\{B_N(x_{j,i}, 2^{-j}R)\}_{i \in I}$  be a covering of  $N^k$  related to  $\mathcal{A}_j^{FC}$ . The coverings are uniformly locally finite with the uniformly bounded

multiplicity constants therefore there exist corresponding resolutions of unity  $\{\varphi_{j,i}\}_i$  with uniformly bounded derivatives. We may assume that the functions  $\varphi_{j,i}$  are non-negative.

Now one can find for sufficiently large  $j_1 \in \mathbb{N}$  a sequence  $\lambda_{j_1,i}$  of positive numbers such that the function

$$(16) \quad e_1(x) = \sum_{i \in I} \lambda_{j_1,i} \varphi_{j_1,i}(x), \quad x \in N^k,$$

satisfies the following inequality

$$(17) \quad \|h - e_1|_{L_p(N^k)}\| \leq 2^{-\max(1,1/p)} \|h|_{L_p(\mathbb{R}^n)}\|.$$

Moreover (16) and (17) and the properties of the covering imply

$$(18) \quad \begin{aligned} \left( \sum_{i \in I} 2^{-j_1 k} |\lambda_{j_1,i}|^p \right)^{1/p} &\leq C \|e_1|_{L_p(N^k)}\| \\ &\leq C \left( \|h|_{L_p(N^k)}\| + \|h - e_1|_{L_p(N^k)}\| \right) \\ &\leq C 2^{-\max(1,1/p)} (1 + 2^{\max(1,1/p)}) \|h|_{L_p(N^k)}\|, \end{aligned}$$

where the constant  $C$  depends on the multiplicity of the covering and the constant from (quasi)-triangle inequality.

Next we can find  $j_2 > j_1$  such that the function  $e_2$  defined in the way similar to  $e_1$  satisfies

$$\|h - e_1 - e_2|_{L_p(N^k)}\| \leq 2^{-2 \max(1,1/p)} \|h|_{L_p(\mathbb{R}^n)}\|.$$

and

$$\left( \sum_{i \in I} 2^{-j_2 k} |\lambda_{j_2,i}|^p \right)^{1/p} \leq C \|e_2|_{L_p(N^k)}\| \leq C 2^{-2 \max(1,1/p)} \|h|_{L_p(N^k)}\|.$$

By induction we find the sequence of positive integers  $j_1 < j_2 < \dots < j_\ell < \dots$ , and the sequence of positive numbers  $\lambda_{j_\ell}$  such that the functions

$$e_\ell(x) = \sum_{i \in I} \lambda_{j_\ell,i} \varphi_{j_\ell,i}(x), \quad x \in N^k,$$

satisfy

$$(19) \quad \left\| h - \sum_{i=1}^{\ell} e_i|_{L_p(N^k)} \right\| \leq 2^{-\ell \max(1,1/p)} \|h|_{L_p(\mathbb{R}^n)}\|$$

and

$$(20) \quad \left( \sum_{i \in I} 2^{-j_\ell k} |\lambda_{j_\ell,i}|^p \right)^{1/p} \leq C \|e_\ell|_{L_p(N^k)}\| \leq C 2^{-\ell \max(1,1/p)} \|h|_{L_p(N^k)}\|.$$

Let us fix  $\eta \in C_0^\infty(\mathbb{R}^{n-k})$ ,  $\text{supp } \eta \subset B^{n-k}(0, R)$ ,  $\eta(0) = 1$  such that

$$(21) \quad \int_{\mathbb{R}^{n-k}} \eta(y)y^\alpha dy = 0 \quad \text{for any multi-index } |\alpha| \leq K.$$

We put

$$(22) \quad a_{j_\ell, i}(x) = 2^{j_\ell \frac{k}{p}} \eta(2^{j_\ell} \pi_1 \circ \kappa_{j_\ell}^{-1}(x)) \varphi_{j_\ell, i}(\pi_2 \circ \kappa_{j_\ell}^{-1}(x)), \quad i \in I,$$

where  $\pi_1$  is the orthogonal projection onto  $\mathbb{R}^{n-k}$  in  $\mathbb{R}^n$  and  $\pi_2$  is the orthogonal projection onto  $\mathbb{R}^k$  in  $\mathbb{R}^n$ . It is not hard to prove that the functions  $a_{j_\ell, i}$  belongs to a family  $\mathcal{A}_{s,p}^{L,K}$  for  $s = \frac{n-k}{p}$  and suitable  $L$  and  $K$ , cf. Definition 4.3. The conditions (2) - (5) are clear, so it remains to prove the moment condition (6). We have

$$(23) \quad \left| \int_{M^n} a_{j_\ell, i}(x) \psi(x) dx \right| = C \left| \int_{\mathbb{R}^n} a_{j_\ell, i}(\kappa_{j_\ell, i}(y)) \psi(\kappa_{j_\ell, i}(y)) \sqrt{|g|}(y) dy \right| \leq 2^{j_\ell \frac{k}{p}} \int_{\text{supp } \varphi_{j_\ell, i}} \varphi_{j_\ell, i}(y'') \left( \int_{B^{n-k}(0, 2^{-j_\ell} R)} \eta(2^{j_\ell} y') \psi(\kappa_{j_\ell, i}(y', y'')) \sqrt{|g|}(y', y'') dy' \right) dy''$$

for any  $\psi \in C^{K+1}(M^n)$ . Using the Taylor expansion of the function  $\beta(y', y'') = \psi(\kappa_{j_\ell, i}(y', y'')) \sqrt{|g|}(y', y'')$  with respect to the origin and (21) we can easily prove that

$$(24) \quad \left| \int_{B^{n-k}(0, 2^{j_\ell} R)} \beta(y', y'') dy' \right| \leq \frac{1}{(K+1)!} \int_{B^{n-k}(0, 2^{-j_\ell} R)} |((y', y'') \cdot \nabla \beta)^{K+1}(\theta(y', y''))| dy' \leq C 2^{-j_\ell(K+1+\frac{n-k}{p})} \left\| \beta \right\|_{C^{K+1}(\overline{B^{n-k}(0, 2^{-j_\ell+1} R)})}.$$

Now (23) - (24) and the boundedness of the geometry of the manifolds imply

$$\left| \int_{M^n} a_{j_\ell, i}(x) \psi(x) dx \right| \leq C 2^{-j_\ell(n-\frac{k}{p}+K+1)} \left\| \beta \right\|_{C^{K+1}(\overline{2^{-j_\ell} U_{j_\ell, i}})}.$$

Thus  $a_{j_\ell, i}$  is the  $(\frac{n-k}{p}, p)$ -atom on  $M^n$  satisfying  $K$  moment conditions.

We defined

$$(25) \quad f_\ell(x) = \sum_{i \in I} 2^{-j_\ell \frac{k}{p}} \lambda_{j_\ell, i} a_{j_\ell, i}(x), \quad x \in N^k.$$

Then  $f_\ell$  is a continuous function and  $f_\ell|_N = e_\ell$ .

It follows from (20) that

$$(26) \quad \left( \sum_\ell \left( \sum_{i \in I} 2^{-j_\ell k} |\lambda_{j_\ell, i}|^p \right)^{q/p} \right)^{1/q} \leq C \|h\|_{L_p(N^k)},$$

for any positive  $q$ . Now it follows from Theorem 4.7 that the series

$$f = \sum_{\ell} f_{\ell}$$

converge in  $\mathcal{D}'(M^n)$  and  $f \in B_{p,q}^{\frac{n-k}{p}}(M^n)$ , cf. (22)-(26). Moreover,

$$h = \lim_{\ell \rightarrow \infty} \sum_{i=1}^{\ell} e_i = \lim_{\ell \rightarrow \infty} \sum_{i=1}^{\ell} f_i|_N = \text{Tr } f \quad (\text{convergence in } L_p(N^k)),$$

cf. (19). This finishes the proof.  $\blacksquare$

REMARK 5.2 1. We proved a bit more that stated in the theorem. Namely it follows from the Step 3 of the proof that there exists a bounded extension operator

$$\text{Ext} : L_p(N^k) \rightarrow B_{p,q}^{\frac{n-k}{p}}(M^n),$$

such that  $\text{Tr} \circ \text{Ext} = \text{id}$ .

2. The theorem seems to be new also for noncompact  $k$ -dimensional hypersurfaces in  $\mathbb{R}^n$ , that satisfies the conditions of Definition 2.3, this means the submanifolds of bounded geometry and with bounded means curvature and its derivatives.

3. The method used in the proof works also for the traces of Besov and Triebel-Lizorkin spaces with smoothness  $s > \frac{n-k}{p}$  described in [7].

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