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On the almost periodicity of a generalized trigonometric polynomial

Dedicated to my teacher Professor Julian Musielak on the occasion of the 85th anniversary of his birthday, in friendship and high esteem

Abstract. This paper presents some remarks on various types of the almost periodicity of a generalized trigonometric polynomial and of the inverse of a generalized trigonometric polynomial of constant sign.

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1. Preliminaries. A non-empty set $E \subset \mathbb{R}$ is called *relatively dense* if there exists a number $l > 0$ such that every open interval in $\mathbb{R}$ of length l contains at least one element of E.

1.1. Given $f, g \in C^{(n)}(\mathbb{R})$, where $n \in \mathbb{N}_0$, define the quantity

$$D^{(n)}(f,g) = \sup_{x\in\mathbb{R}} \left(|f(x) - g(x)| + \sum_{k=1}^{n} |(f-g)^{(k)}(x)| \right).$$

A number $\tau \in \mathbb{R}$ is called a $\big(D^{(n)}, \varepsilon\big)$-*almost period* of a function $f \in C^{(n)}(\mathbb{R})$ whenever $D^{(n)}(f_\tau, f) \leq \varepsilon$, where $\varepsilon > 0$ and $f_\tau(x) \equiv f(x+\tau)$. Let us denote by $E^{(n)}\{\varepsilon; f\}$ the set of all $\big(D^{(n)}, \varepsilon\big)$-almost periods of f. A function $f \in C^{(n)}(\mathbb{R})$ is said to be $C^{(n)}$-*almost periodic* if for an arbitrary $\varepsilon > 0$ the set $E^{(n)}\{\varepsilon; f\}$ is relatively dense [1]. In particular, when $n = 0$ one speaks about an ε-*almost period* of a function $f \in C(\mathbb{R})$ which is *uniformly almost periodic* (B-*almost periodic*) if for all $\varepsilon > 0$ the set $E\{\varepsilon; f\}$ of its ε-almost periods is relatively dense [4], [8], [19]. Let $\widetilde{C^{(n)}}$ denote the space of all $C^{(n)}$-almost periodic functions.

1.2. Given any $f, g \in C^{(\infty)}(\mathbb{R})$ and a sequence $a = (a_k)$ with positive terms, define

$$D_a^{(\infty)}(f,g) = \sup_{x\in\mathbb{R}} \left(|f(x) - g(x)| + \sum_{k=1}^{\infty} a_k \left| (f-g)^{(k)}(x) \right| \right).$$

A number $\tau \in \mathbb{R}$ is called a $\left(D^{(\infty)}, \varepsilon\right)$-*almost period* of $f \in C^{(\infty)}(\mathbb{R})$ if $D_a^{(\infty)}(f_\tau, f) \le \varepsilon$, where $\varepsilon > 0$ and $f_\tau(x) \equiv f(x+\tau)$. Denote by $E_a^{(\infty)}\{\varepsilon; f\}$ the set of all $\left(D_a^{(\infty)}, \varepsilon\right)$-almost periods of f. Let $a = (a_k)$ be a sequence such that $a_k > 0$ and $a_{k+1} \le a_k \le 1$ for all $k \in \mathbb{N}$. Then we say that an $f \in C^{(\infty)}(\mathbb{R})$ is *conditionally locally bounded with respect to* a if for arbitrary closed interval $[c,d]$ the numbers $M_k = M_{k,f}^{[c,d]} = \max\{|f^{(k)}(x)| : x \in [c,d]|\}$ satisfy the condition $\sum_{k=1}^{\infty} a_k M_{k+1} < \infty$. We then shortly write $f \in \left(CBC_{a,loc}^{(\infty)}\right)$. An $f \in \left(CBC_{a,loc}^{(\infty)}\right)$ is called $C_a^{(\infty)}$-*almost periodic* if for each $\varepsilon > 0$ the set $E_a^{(\infty)}\{\varepsilon; f\}$ is relatively dense [2]. The space of all $C_a^{(\infty)}$-almost periodic functions is denoted $\widetilde{C_a^{(\infty)}}$. Every $C_a^{(\infty)}$-almost periodic function is $C^{(n)}$-almost periodic for all $n \in \mathbb{N}_0$, hence B-almost periodic.

1.3. Denote by R_0 the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Given $f \in R_0, x \in \mathbb{R}$, we define the variation of f on $[x-1, x+1]$ by

$$V(f;x) = \sup_{\Pi} \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|,$$

where $\Pi = \{x - 1 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = x+1\}$. For $f, g \in R_0$ we put

$$V(f,g) = \sup_{x\in\mathbb{R}} \left(|f(x) - g(x)| + V(f-g;x) \right).$$

Denote by BV_{loc} the set of all $f \in R_0$ of locally finite variation, meaning $V(f;x) < \infty$ for all $x \in \mathbb{R}$. Let $f \in BV_{loc}$. If $V(f_\tau, f) \le \varepsilon$, where $\varepsilon > 0$ and $f_\tau(x) \equiv f(x+\tau)$, then $\tau \in \mathbb{R}$ is called a (V,ε)-*almost period* of f. The set of all (V,ε)-almost periods of f is denoted $E_V\{\varepsilon; f\}$. A continuous function $f \in BV_{loc}$ is said to be *almost periodic in variation* or V-*almost periodic* if $E_V\{\varepsilon; f\}$ is relatively dense for all $\varepsilon > 0$ [14],[19]. The space of all V-almost periodic functions is denoted $\widetilde{V}$.

1.4. Let F_Δ be the class of subsets of the plane Oxy whose projections onto the x-axis coincide with the closed interval (bounded or not) and such that the intersection of each line $x = x_0$, where $x_0 \in \Delta$, with every $F \in F_\Delta$ is a bounded closed interval or unbounded interval: $(-\infty, a], [a, +\infty), (-\infty, +\infty)$. Let $A, B \in F_\Delta$. The *Hausdorff distance* [7] between A and B is the quantity

$$r_\Delta(A,B) = \max\left(\sup_{X\in A} \inf_{Y\in B} ||X-Y||_0, \sup_{X\in B} \inf_{Y\in A} ||X-Y||_0 \right),$$

where

$$||X-Y||_0 = ||X(x_1,y_1) - Y(x_2,y_2)||_0 = \max\left(|x_1-x_2|, |y_1-y_2|\right).$$

If $\Delta = \mathbb{R}$, we write $r_\Delta = r$. It is easily checked that for all $A, B, C \in F_\Delta$ the following hold:

$$r_\Delta(A,B) = 0 \Leftrightarrow A = B,$$
$$r_\Delta(A,B) = r_\Delta(B,A),$$
$$r_\Delta(A,B) \leq r_\Delta(A,C) + r_\Delta(C,B).$$

The following lemmas are useful in estimating Hausdorff distance:

LEMMA 1.1 *Let $A, B \in F_\Delta$ and $\delta > 0$. Then*

$$r_\Delta(A,B) \leq \delta$$

if and only if the following condtitions hold:

(a) for every $X \in A$ there is a $Y \in B$ such that $||X - Y||_0 \leq \delta$,

(b) for every $X \in B$ there is a $Y \in A$ such that $||X - Y||_0 \leq \delta$.

LEMMA 1.2 *Let $A, B \in F_\Delta$. If there is an $X_0 \in A$ such that $||X_0 - Y||_0 > \delta$ for all $Y \in B$, then*

$$r_\Delta(A,B) > \delta.$$

Proofs can be found in [9] and [10].

Let $f : \Delta \to \mathbb{R}$. The *lower* and *upper Baire functions* of f are defined by

$$I_f(x) = \lim_{\delta \to 0} \inf_{|x-x'| \leq \delta} f(x') \quad \text{and} \quad S_f = \lim_{\delta \to 0} \sup_{|x-x'| \leq \delta} f(x'),$$

resp. A *complete graph* of f is the set

$$\widetilde{f} = \{(x,y) : x \in \Delta \quad \text{and} \quad I_f(x) \leqq y \leqq S_f(x)\}.$$

REMARK 1.3

$$a \leqq b \Leftrightarrow (a \leq b \quad \text{if} \quad a, b \in \mathbb{R} \quad \text{or} \quad a < b \quad \text{if} \quad a = -\infty \quad \text{or} \quad b = +\infty).$$

Then $\widetilde{f} \in F_\Delta$. The *Hausdorff distance between functions* $f, g : \Delta \to \mathbb{R}$ is defined to be the Hausdorff distance their complete graphs, i.e. $r_\Delta(f,g) = r_\Delta\left(\widetilde{f}, \widetilde{g}\right)$.

Let $f : \mathbb{R} \to \mathbb{R}$. If for $\varepsilon > 0$ the Hausdorff distance between f and f_τ, where $f_\tau(x) \equiv f(x + \tau)$, satisfies $r(f_\tau, f) \leq \varepsilon$, then τ is called an (H, ε)-*almost period* of f. The set of all (H, ε)-almost periods of f is denoted $E_H\{\varepsilon; f\}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called H-*almost periodic* if $E_H\{\varepsilon; f\}$ is relatively dense for all $\varepsilon > 0$ [5], [11]-[13], [17]-[20]. Denote by $\widetilde{H}$ the space of all H-almost periodic functions. In the following, in case when the function f is discontinuous at the point x_0, we denote with $f(x_0)$ one of the members from interval $[I_f(x_0), S_f(x_0)]$ (or: $(-\infty, S_f(x_0)], [I_f(x_0), +\infty), (-\infty, +\infty)$).

1.5. Given $f, g \in C^{(n)}(\mathbb{R}), n \in \mathbb{N}_0$ and $N > 0$, we define the quantity

$$\left(ND^{(n)}\right)(f,g) = \max_{-N\leq x\leq N}\left(|f(x)-g(x)| + \sum_{k=1}^{n}\left|(f-g)^{(k)}(x)\right|\right).$$

A number $\tau \in \mathbb{R}$ is called an $\big((ND^{(n)}), \varepsilon\big)$-*almost period* of $f \in C^{(n)}(\mathbb{R})$, where $\varepsilon > 0$ and $N > 0$, if $(ND^{(n)})(f_\tau, f) \leq \varepsilon$, where $f_\tau \equiv f(x+\tau)$. Denote by $NE^{(n)}\{\varepsilon; f\}$ the set of all $\big((ND^{(n)}), \varepsilon\big)$-almost periods of f. An $f \in C^{(n)}(\mathbb{R})$ is called $(NC^{(n)})$-*almost periodic* if there exists a $C^{(n)}$-almost periodic function φ, so-called *majorant* of f, such that for each $\varepsilon > 0$ and $N > 0$ there is a $\delta > 0$ such that each $(D^{(n)}, \delta)$-almost period of φ is an $((ND^{(n)}), \varepsilon)$-almost period of f [3]. The space of all $(NC^{(n)})$-almost periodic functions is denoted $(\widetilde{NC^{(n)}})$. In particular, for $n = 0$ we obtain the N-almost periodic functions [8].

2. Main results. We shall consider the generalized trigonometric polynomial

$$T(x) = \sum_{k=1}^{m}\left(\alpha_k\cos(\nu_k x) + \beta_k\sin(\nu_k x)\right), \quad x \in \mathbb{R}, \tag{1}$$

where α_k, β_k and ν_k are given real numbers.

Of course, T is B-almost periodic. It is known [19] that f is $C^{(n)}$-almost periodic if and only if $f, f', \ldots, f^{(n)}$ are B-almost periodic. Hence T is $C^{(n)}$-almost periodic for all $n \in \mathbb{N}$.

THEOREM 2.1 *If for the generalized trigonometric polynomial T of the form (1) the sequence $a = (a_n)$, where $0 < a_{n+1} \leq a_n \leq 1$ for $n \in \mathbb{N}$, satisfies the following condition*

$$\sum_{n=1}^{\infty}\left(a_n\sum_{k=1}^{m}\max(|\nu_k|^n, |\nu_n|^{n+1})(|\alpha_k| + |\beta_k|)\right) < \infty,$$

then T is $C_a^{(\infty)}$-almost periodic.

PROOF Since for $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have

$$T^{(n)}(x) = \sum_{k=1}^{m}\nu_k^n\left(\alpha_k\cos\left(\nu_k x + n\frac{\pi}{2}\right) + \beta_k\sin\left(\nu_k x + n\frac{\pi}{2}\right)\right),$$

we seen that for every closed interval $[c, d]$

$$\sum_{n=1}^{\infty}a_n M_{n+1} = \sum_{n=1}^{\infty}a_n\max\left\{|T^{(n+1)}(x)| : x \in [c,d]\right\} \leq$$

$$\sum_{n=1}^{\infty}a_n\left(\sum_{k=1}^{m}|\nu_k|^{n+1}(|\alpha_k| + |\beta_k|)\right) < \infty,$$

and so $T \in \left(CBC^{(\infty)}_{a,loc}\right)$.

Of course, the trigonometric polynomial T is $C^{(p)}$-almost periodic for all $p \in \mathbb{N}$. For an arbitrary $\varepsilon > 0$ and for each $\tau \in E^{(p)}\left\{\frac{\varepsilon}{2}, T\right\}$ we obtain

$$\begin{aligned}(2)\qquad &\sup_{x\in\mathbb{R}}\left(|T_\tau(x)-T(x)|+\sum_{n=1}^{p}a_n|(T_\tau-T)^{(n)}(x)|\right)\le\\ &\sup_{x\in\mathbb{R}}\left(|T_\tau(x)-T(x)|+\sum_{n=1}^{p}|(T_\tau-T)^{(n)}(x)|\right)\le\frac{\varepsilon}{2}\end{aligned}$$

and

$$\sup_{x\in\mathbb{R}}\sum_{n=p+1}^{\infty}a_n|(T_\tau-T)^{(n)}(x)|\le 2\sum_{n=p+1}^{\infty}\left(a_n\sum_{k=1}^{m}|\nu_k|^n(|\alpha_k|+|\beta_k|)\right)$$

By the assumption it follows that there is a natural p for which

$$(3)\qquad \sum_{n=p+1}^{\infty}\left(a_n\sum_{k=1}^{m}|r_k|^n(|\alpha_k|+|\beta_k|)\right)<\frac{\varepsilon}{4}.$$

Thus, by (2) and (3) we have $E^{(p)}\left\{\frac{\varepsilon}{2};T\right\}\subset E^{(\infty)}_a\{\varepsilon;T\}$. Hence T is $C^{(\infty)}_a$-almost periodic. ■

THEOREM 2.2 *If $f \in \widetilde{C^{(2)}}, |f(x)| > 0$ for $x \in \mathbb{R}$ and $\inf\{f(x)sgnf(x) : x \in \mathbb{R}\} = 0$, then $g = \frac{1}{f}$ is $(NC^{(2)})$-almost periodic.*

PROOF Let for example $f(x) > 0$ for $x \in \mathbb{R}$. Since g is unbounded, hence g is not $C^{(2)}$-almost periodic. Let $\varepsilon > 0$ and $N > 0$. Denote

$$M=\max\left(\sup_{x\in\mathbb{R}}|f^{(i)}(x)|, i=0,1,2\right),\quad m_N=\min_{-N\le x\le N}f(x)$$

and let

$$\delta\in\left(0,\min\left(\frac{m_N}{2},\varepsilon\left(\frac{2}{(m_N)^2}+\frac{12M^2}{(m_N)^3}+\frac{336M^6}{(m_N)^8}\right)^{-1}\right)\right).$$

For $\tau \in E^{(2)}\{\delta; f\}$ we have

$$\begin{aligned}
&\left(ND^{(2)}\right)(g_\tau, g)\\
&\le \max_{-N\le x\le N} |g_\tau(x)-g(x)| + \max_{-N\le x\le N} |g'_\tau(x)-g'(x)| + \max_{-N\le x\le N} |g''_\tau(x)-g''(x)|\\
&= \max_{-N\le x\le N} \frac{|f_\tau(x)-f(x)|}{f_\tau(x)f(x)} + \max_{-N\le x\le N} \frac{|f'_\tau(x)f^2(x)-f'(x)f^2_\tau(x)|}{f^2_\tau(x)f^2(x)}\\
&\quad + \max_{-N\le x\le N} \frac{1}{f^4_\tau(x)f^4(x)}\big(|f''_\tau(x)f^2_\tau(x)f^4(x) - f''(x)f^4_\tau(x)f^2(x)\\
&\quad + 2(f^{'2}(x)f^4_\tau(x)f(x) - f^{'2}_\tau(x)f_\tau(x)f^4(x))|\big)\\
&\le \delta\left(\frac{1}{(m_N-\delta)m_N} + \frac{3M^2}{(m_N-\delta)^2 m_N} + \frac{21M^6}{(m_N-\delta)^4 m_N^4}\right) < \varepsilon.
\end{aligned}$$

Thus, g is $(NC^{(2)})$-almost periodic, because it has a $C^{(2)}$-almost periodic majorant f whose each $(D^{(2)},\delta)$-almost period is an $\big((ND^{(2)}),\varepsilon\big)$-almost period of g. ■

Assume that the polynomial T of the form (1) satisfies the following condition

$$(4)\qquad |T(x)|>0 \quad \text{for} \quad x\in\mathbb{R} \quad \text{and} \quad \inf\{T(x)\operatorname{sgn}T(x) : x\in\mathbb{R}\}=0.$$

Then, from the above theorem it follows that $\frac{1}{T}$ is $(NC^{(2)})$-almost periodic. Moreover, $\frac{1}{T}$ is N-almost periodic [8], [16] and μ-almost periodic [15], [20].

S. Hartman expressed supposition [6] that the inverse of the generalized trigonometric polynomial of constant sign that fulfills the condition (4) is an unbounded function H-almost periodic.

Theorem 2.3 *If T is the generalized trigonometric polynomial that fulfills the condition (4), then $f=\frac{1}{T}$ is not H-almost periodic.*

Proof Let for example $T(x)>0$ for $x\in\mathbb{R}$. Of course, the function f is unbounded. Now, we shall show that f fails to be H-almost periodic. This means that for some $\varepsilon_0>0$ there exists a sequence (I_n) of open intervals of length, resp., $d_1,d_2,\dots$ and with $d_n\to\infty$, none of which contains an (H,ε_0)-almost period of f. Let us denote $\varepsilon_0=1$. For $a_0>2$ let us put

$$\max\{f(x) : x\in[-a_0,a_0]\} = f(x_0) > 0$$

and

$$d_n = a_n - a_{n-1} > 0, \quad n=1,2,\dots$$

We assume that $\lim_{n\to\infty} d_n = \infty$. Let

$$\max\{f(x) : x\in[-a_n,-a_{n-1}]\cup[a_{n-1},a_n]\} = f(x_n) > f(x_{n-1})+1$$

for $n=1,2,\dots$ In the following way we construct the sequence (I_n). Let $n=2,3,\dots$ If $x_n\in[-a_n,-a_{n-1}]$, then

$$I_n = (-a_{n-1}-x_n+1, -a_{n-2}-x_n-1).$$

However, if $x_n \in [a_{n-1}, a_n]$, then

$$I_n = (a_{n-2} - x_n + 1, a_{n-1} - x_n - 1).$$

Then, by Lemma 1.2, for $\tau \in I_n$ the following estimation holds

$$r(f_\tau, f) > 1.$$

Hence f is not H-almost periodic. ∎

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