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Between Local Connectedness and Sum Connectedness

Abstract. A new generalization of local connectedness called Z-local connectedness is introduced. Basic properties of Z-locally connected spaces are studied and their place in the hierarchy of variants of local connectedness, which already exist in the literature, is elaborated. The class of Z-locally connected spaces lies strictly between the classes of pseudo locally connected spaces (Commentations Math. 50(2)(2010),183-199) and sum connected spaces ($\equiv$ weakly locally connected spaces) (Math. Nachrichten 82(1978), 121-129; Ann. Acad. Sci. Fenn. AI Math. 3(1977), 185-205) and so contains all quasi locally connected spaces which in their turn contain all almost locally connected spaces introduced by Mancuso (J. Austral. Math. Soc. 31(1981), 421-428). Formulations of product and subspace theorems for Z-locally connected spaces are suggested. Their preservation under mappings and their interplay with mappings are discussed. Change of topology of a Z-locally connected space is considered so that it is simply a locally connected space in the coarser topology. It turns out that the full subcategory of Z-locally connected spaces provides another example of a mono-coreflective subcategory of TOP which properly contains all almost locally connected spaces.

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1. Introduction. The full subcategory of locally connected spaces is a coreflective subcategory of TOP ($\equiv$ the category of topological spaces and continuous maps) [6]. However, the larger category of almost locally connected spaces is not a coreflective subcategory of TOP. The search for the existence of coreflective subcategories of TOP containing all almost locally connected spaces was begun in [20], wherein it is observed that the problem bears an affirmative answer as is inherent in the category of sum connected spaces [11]. Moreover, in [20], the categories of quasi (pseudo) locally connected spaces are introduced and are shown to be coreflective subcategories of TOP containing all almost locally connected spaces. In this paper we introduce the category of 'Z-locally connected spaces' which turns out to be a mono-coreflective subcategory of TOP containing all quasi (pseudo) locally connected spaces and so includes all almost locally connected spaces as well.

Organization of the paper is as follows: Section 2 is devoted to preliminaries and basic definitions. In Section 3 we introduce the notion of a 'Z-locally connected space' and elaborate on its place in the hierarchy of variants of local connectedness that already exist in the literature. Basic properties of Z-locally connected spaces are dealt with in Section 4, while Section 5 is devoted to the interplay between Z-locally connected spaces / sum connected spaces and mappings wherein several preservation results are obtained. Function spaces of Z-locally connected spaces are considered in Section 6 and the conditions for their closedness / compactness in the topology of pointwise convergence are outlined. In Section 7 we consider change of topology of a Z-locally connected space such that it is just a locally connected space in the coarser topology.

Throughout the paper closure of a set A in a space X will be denoted by $\overline{A}$ and the interior of A by A^o . For the definitions of categorical terms used in the paper we refer the reader to Herrlich and Strecker ([7] [8]).

2. Preliminaries, basic definitions. A subset A of a space X is said to be **regular open** if it is the interior of its closure,i.e., $A = \overline{A}^{\circ}$. The complement of a regular open set is referred to as a **regular closed** set. A union of regular open sets is called **δ-open**[38]. The complement of a δ-open set is referred to as a **δ-closed** set. A subset A of a space X is called **regular G_δ-set**[23] if A is an intersection of a sequence of closed sets whose interiors contain A, i.e.,if $A = \bigcap\limits_{n=1}^{\infty} F_n = \bigcap\limits_{n=1}^{\infty} F_n^0$,where each F_n is a closed subset of X (here F_n^0 denotes the interior of F_n). The complement of a regular G_δ-set is called a **regular F_σ-set**. A subset A of a space X is called **z-open (cl-open)** ([12][33]) if for each $x \in A$, there exists a cozero (clopen) set H such that $x \in H \subseteq A$; equivalently A is expressible as a union of cozero (clopen)sets. A point $x \in X$ is called a **θ-adherent point** [38] of $A \subset X$ if every closed neighbourhood of x intersects A. Let $cl_\theta A$ denote the set of all θ-adherent points of A.The set A is called **θ-closed** if $A = cl_\theta A$. The complement of a θ-closed set is referred to as a **θ-open set**.

Definition 2.1 A space X is said to be

(i) **mildly compact** [35][1] if every clopen cover of X has a finite subcover.

(ii) **functionally Hausdorff** if for every pair of distinct points x and y in X, there exists a continuous function $f : X \to [0,1]$ such that f(x)= 0 and f(y)= 1.

[1]Sostak [34] refers mildly compact spaces as clustered spaces.

Definition 2.2 A space X is said to be

(i) **almost locally connected** ([24] [29]) if for each $x \in X$ and each regular open set U containing x there exists a connected open set V containing x such that $V \subset U$.

(ii) **sum connected**[11] if each $x \in X$ has a connected neighbourhood, or equivalently each component of X is open.

Definition 2.3 ([20]) A space X is said to be **quasi (pseudo) locally connected** at $x \in X$ if for each θ-open set (regular F_σ-set) U containing x there is an open connected set V such that $x \in V \subset U$. The space X is said to be quasi (pseudo) locally connected if it is quasi(pseudo) locally connected at each $x \in X$.

The notions of almost local connectedness as well as quasi (pseudo) local connectedness represent generalizations of local connectedness while the notion of sum connectedness represents a simultaneous generalization of connectedness as well as local connectedness. The category of sum connected spaces is precisely the epi-coreflective hull of the category of connected spaces and contains all connected as well as all locally connected spaces (see [11]). The disjoint topological sum of two copies of topologist's sine curve [36] is an example of a sum connected space which is neither connected nor locally connected. Sum connected spaces have also been referred to as weakly locally connected spaces by some authors (see [25] [27]).

Definition 2.4 A function $f : X \to Y$ from a topological space X into a topological space Y is said to be

(i) **strongly continuous** [21] if $f(\overline{A}) \subset f(A)$ for each subset A of X.

(ii) **perfectly continuous** [28] if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.

(iii) **cl-supercontinuous** [33] ($\equiv$ **clopen continuous**[31]) if for each $x \in X$ and each open set V containing $f(x)$, there is a clopen set U containing x such that $f(U) \subset V$.

(iv) **z-supercontinuous** [12] if for each $x \in X$ and each open set V containing $f(x)$, there exists a cozero set U containing x such that $f(U) \subset V$.

(v) **almost z-supercontinuous** (**almost D_δ-supercontinuous**) [17] if for each x $\in$ X and each regular open set V containing $f(x)$, there exists a cozero set (regular F_σ- set) U containing x such that f(U)$\subset$V.

(vi) **z-continuous** (**cl-continuous**[2]) ([32] [14] [19]) if for each x $\in$ X and each cozero (clopen) set V containing $f(x)$ there is an open set U containing x such that f(U)$\subset$V.

[2] cl-continuous functions are called slightly continuous functions in [9].

(vii) **connected** (**Darboux**) [30] if for each connected set $C \subset X$, $f(C)$ is connected.

DEFINITION 2.5 Let $p : X \to Y$ be a function from a topological space X onto a set Y. The collection of all subsets A⊂Y such that $p^{-1}(A)$ is a

(i) z-open set in X is a topology on Y and is called the **z-quotient topology** [12] and the map p is called **z-quotient map**;

(ii) cl-open set in X is a topology on Y and is called the **cl-quotient topology** [33] and the map p is called **cl-quotient map**.

For the detailed study of interrelations and interplay of z-quotient topology and cl-quotient topology with other variants of quotient topology we refer the interested reader to ([16] [18]).

3. Z-locally connected spaces. A space X is said to be **Z-locally connected** at $x \in X$ if for each cozero set U containing x there is an open connected set V such that $x \in V \subset U$. The space X is said to be Z-locally connected if it is Z-locally connected at each $x \in X$.

The following diagram extends the diagram in [20] and well illustrates the interrelations that exist among Z-local connectedness and other variants of local connectedness that already exist in the literature (see Definitions 2.2 and 2.3).

locally connected → almost locally connected → quasi locally connected

↓

pseudo locally connected

↓

Z-locally connected

↓

sum connected
(≡weakly locally connected)

However, none of the above implications is reversible as is well illustrated by the examples in ([11] [20] [24]) and the following examples/observations.

Examples /Observations

PROPOSITION 3.1 *Every completely regular, Z-locally connected space is locally connected.*

PROOF In a completely regular space cozero sets constitute a base for the topology and so the result is immediate in view of definition. ■

PROPOSITION 3.2 *Every Z-locally connected space is sum connected.*

PROOF This is immediate since every space X is a cozero subset of itself and so each $x \in X$ has a connected open neighbourhood containing x. ■

However, the converse of Proposition 3.2 is false. For example topologist's sine curve [36] is a connected space and hence sum connected but not Z-locally connected.

COROLLARY 3.3 ([20, PROPOSITION 3.6]) *Every quasi (pseudo) locally connected space is sum connected.*

PROPOSITION 3.4 *In a Z-locally connected space, components and quasicomponents coincide in every cozero subset.*

PROOF Let X be a Z-locally connected space and let U be a cozero set in X. Then any component of U is open in X. Thus every quasicomponent of U, being union of components is open. Result follows since every open quasicomponent is a component. ■

PROPOSITION 3.5 *For a space X the following statements are equivalent.*

(a) X is sum connected.

(b) For each $x \in X$ and each clopen set U containing x there exists an open connected set V such that $x \in V \subset U$.

(c) Components of clopen sets in X are open in X.

PROOF $(a) \Rightarrow (b)$. Let U be a clopen subset of X and let $x \in U$. Let C be the component of X containing x. Since X is sum connected,$C \cap U$ is a clopen subset of X containing x and C is a connected open set such that $x \in C \subset U$.
$(b) \Rightarrow (c)$. Obvious.
$(c) \Rightarrow (a)$. Since X is a clopen subset of itself, its components are open and so X is sum connected. ■

PROPOSITION 3.6 *Every mildly compact sum connected space has at most finitely many components.*

COROLLARY 3.7 ([11, PROPOSITION 2.16]) *Every pseudocompact sum connected space has at most finitely many components.*

COROLLARY 3.8 *Every mildly compact Z-locally connected space has at most finitely many components.*

COROLLARY 3.9 ([20, PROPOSITION 3.5]) *Every mildly compact quasi (pseudo) locally connected space has at most finitely many components.*

PROPOSITION 3.10 *A T_o zero dimensional, sum connected space X is discrete.*

PROOF Let X be a T_o zero dimensional, sum connected space. We claim that every component of X is a singleton. For if A is subset of X having more than one point. Let $x, y \in A$, $x \neq y$. By T_o-property there exists an open set U containing one of the points x and y but not both. To be precise, assume that $x \in U$. In view of zero dimensionality of X there is a clopen set V such that $x \in V \subset U$. Then $(A \cap V) \cup ((X \setminus V) \cap A)$ is a partition of A exhibiting that A is disconnected. Again, since components are open in X, every singleton is open and so X is discrete. ■

4. Basic properties of Z-locally connected spaces.

THEOREM 4.1 *For a zero dimensional space X the following statements are equivalent.*

(a) X is locally connected.

(b) X is almost locally connected

(c) X is quasi locally connected.

(d) X is Z-locally connected.

(e) X is sum connected.

PROOF The equivalence of (a), (b) and (c) is discussed in [20,Theorem 4.1]. The equivalence of (a) and (d) follows from Proposition 3.1.
$(d) \Rightarrow (e)$. Since every zero dimensional space is completely regular and since every clopen set is a cozero set, in view of Proposition 3.5, X is sum connected.
$(e) \Rightarrow (d)$. Suppose X is sum connected and let U be a cozero set in X. Let $x \in U$. By zero dimensionality of X, there exists a clopen set V such that $x \in V \subset U$. So in view of Proposition 3.5 there exists an open connected set W such that $x \in W \subset V \subset U$ and hence X is Z-locally connected. ■

THEOREM 4.2 *A space X is Z-locally connected if and only if for each component of every cozero subset of X is open in X.*

Proof Suppose that X is Z-locally connected. Let U be a cozero set in X and let C be a component of U. Let $x \in C$. In view of Z-local connectedness of X there exists an open connected set V such that $x \in V \subset U$. Since V is connected and in view of maximality of C, $V \subset C$. Thus C is open being a neighbourhood of each of its points. Conversely, suppose that components of cozero sets in X are open. Let U be a cozero set in X containing x and let C be the component of x in U. By hypothesis C is an open connected set and so X is Z-locally connected at x. ∎

Definition 4.3 A subset S of a space X is said to be **z-embedded** in X [2] if every cozero set in S is the intersection of a cozero set in X with S.

Theorem 4.4 *Every z-embedded cozero subspace of a Z-locally connected space X is Z-locally connected.*

Proof Let A be a z-embedded cozero subspace of a Z-locally connected space X and let B be a cozero subset of A. Then B is a cozero set in X. Let $b \in B$. Now, since X is Z-locally connected, there exists a connected open set C containing b in X such that $C \subset B$. Clearly, C is a connected open subset of A and so A is Z-locally connected. ∎

Theorem 4.5 *Every cozero cover of Z-locally connected space X has a refinement consisting of open connected sets.*

Proof Let X be a Z-locally connected space and let $\vartheta = \{V_\alpha : \alpha \in \Lambda\}$ be a cozero cover of X. Since components of cozero sets in a Z-locally connected space are open, components of members of ϑ constitute a refinement of ϑ consisting of open connected sets. ∎

Theorem 4.6 *Disjoint topological sum of any family of Z-locally connected spaces is Z-locally connected.*

Proof Let $\{X_\alpha : \alpha \in \Lambda\}$ be any family of Z-locally connected spaces and let $X = \oplus_{\alpha\in\Lambda} X_\alpha$ denote their disjoint topological sum. Let U be any cozero set in X and let $x \in U$. Then $x \in U \cap X_\alpha$ for some $\alpha \in \Lambda$. It is easily verified that $U \cap X_\alpha$ is a cozero set in X_α. Since X_α is Z-locally connected, there exists a connected open set C in X_α containing x such that $C \subset U \cap X_\alpha$. Clearly C is a connected open set in X and so X is Z-locally connected. ∎

Theorem 4.7 *If X is a connected Z-locally connected space and if C is a component of a cozero set in X such that $X \setminus \overline{C}$ is nonempty, then $\overline{C} \setminus C$ is not empty and separates C and $X \setminus \overline{C}$ in X.*

PROOF If $\overline{C} \setminus C$ is empty, then C is closed. By Theorem 4.2 C is open and so $X \setminus \overline{C} = X \setminus C$ is a nonempty proper clopen set in X contradicting the fact that X is connected. Thus $\overline{C} \setminus C$ is nonempty. Since $X \setminus (\overline{C} \setminus C) = C \cup (X \setminus \overline{C})$, it follows that C and $X \setminus \overline{C}$ constitute a partition of $X \setminus (\overline{C} \setminus C)$. ■

5. Preservation/Interplay with mappings.

THEOREM 5.1 *Every quotient of a Z-locally connected space is Z-locally connected.*

PROOF Let $f : X \to Y$ be a quotient map from a Z-locally connected space X onto Y. To prove that Y is Z-locally connected, let V be a cozero set in Y. Then $f^{-1}(V)$ is a cozero set in X. Let C be a component of V. It suffices to show that C is open in Y or equivalently $f^{-1}(C)$ is open in X. To this end, let $x \in f^{-1}(C)$, and let C_x be the component of x in $f^{-1}(V)$. Since $f(C_x)$ is a connected set containing $f(x)$, $f(C_x) \subset C$ and so $x \in C_x \subset f^{-1}(C)$. Since X is Z-locally connected, and since $f^{-1}(V)$ is a cozero set, by Theorem 4.2 C_x is open in X and so $f^{-1}(C)$ is open being a neighbourhood of each of its points. ■

COROLLARY 5.2 *Continuous open (closed) images, adjunctions and inductive limits of Z-locally connected spaces are Z-locally connected.*

COROLLARY 5.3 *If a product space is Z-locally connected, then so is each of its factors.*

PROOF Since projection maps are continuous open maps, this is immediate in view of Corollary 5.2. ■

THEOREM 5.4 *Let $f : X \to Y$ be an open, connected, z-continuous function from X onto Y. If X is Z-locally connected, then so is Y.*

PROOF Let $y \in Y$ and let U be a cozero set in Y containing y. Since f is z-continuous, $f^{-1}(U)$ is a cozero set in X (see [32]). Again, since X is Z-locally connected, for each $x \in f^{-1}(y)$ there exists a connected open set $N(x)$ of x such that $N(x) \subset f^{-1}(U)$. Since f is a connected open map, $f(N(x))$ is an open connected set containing y which is contained in U. So Y is Z-locally connected. ■

THEOREM 5.5 *Let $f : X \to Y$ be an open, connected, cl-continuous function from X onto Y. If X is sum connected, then so is Y.*

Proof of Theorem 5.5 is similar to that of the proof of Theorem 5.4 and hence omitted.

THEOREM 5.6 *Let $f : X \to Y$ be a quotient map which is a z-supercontinuous surjection. If X is Z-locally connected,then Y is locally connected.*

Proof of Theorem 5.6 is similar to that of the proof of [20,Theorem 6.5] and hence omitted.

REMARK 5.7 The function $f : X \to Y$ in above theorem is precisely the z-quotient map [12]and Y is endowed with the z-quotient topology. Thus paraphrasing we can say that z-quotient of a Z-locally connected space is locally connected.

THEOREM 5.8 *Let $f : X \to Y$ be a quotient map which is cl-supercontinuous surjection. If X is sum connected, then Y is locally connected.*

REMARK 5.9 The quotient map f in Theorem 5.8 is a cl-quotient map and Y is endowed with cl-quotient topology (see [33]). Thus we can say that cl-quotient of a sum connected space is locally connected.

THEOREM 5.10 *Let $f : X \to Y$ be an open, connected, almost z-supercontinuous (almost D_δ-supercontinuous) function from a Z-locally connected (pseudo locally connected) space X onto Y. Then Y is almost locally connected.*

Proof of Theorem 5.10 is similar to that of the proof of [20, Theorem 6.7] and hence omitted.
Since co-products and extremal quotient objects in TOP are disjoint topological sums and quotient maps, respectively a characterization of mono-coreflective subcategories of TOP ([7, Theorem 6]) together with Theorems 4.6 and 5.1 yields the following.

THEOREM 5.11 *The full subcategory of Z-locally connected spaces is a mono-coreflective subcategory of TOP containing the full subcategory of almost locally connected spaces*

6. Function spaces and Z-locally connected spaces. It is well known that in general, $C(X, Y)$ the set of all continuous functions from a space X into a uniform space Y is closed in Y^X in the topology of uniform convergence; however, it is not closed in the topology of pointwise convergence.It is of fundamental importance in topology, analysis and other branches of mathematics to know that whether a given function space is closed / compact in the topology of pointwise convergence / uniform convergence. So it is of considerable significance from intrinsic considerations as well as from applications view point to formulate conditions on the spaces X, Y and subsets of $C(X, Y)$ to be closed / compact in the topology of pointwise convergence / uniform convergence. Results of this nature and Ascoli type theorems abound in the literature (see [1] [10]). Furthermore, in this direction, Naimpally [26] showed that if X is locally connected and Y is Hausdorff, then the set $S(X, Y)$ of all strongly continuous functions from X into Y is closed in in the topology of pointwise convergence. It seems natural to ask: Is Naimpally's result still true if X is Z-locally connected, instead of locally connected? The answer is in the affirmative as will become clear in the sequel. Recently, Kohli and Singh [15] extended Naimpally's result to a larger framework, wherein it is shown that if X is sum connected (in particular if X is connected or locally connected) and Y is Hausdorff, then the function space $P(X,Y)$ of all perfectly continuous functions as well as the function space $L(X,Y)$ of all cl-supercontinuous functions is closed in Y^X in the topology of pointwise convergence. In view of Proposition 3.2 every Z-locally connected space is sum connected. So by [15, Theorem 3.7 and Proposition 3.8] we conclude with the following.

Corollary 6.1 *If X is a Z-locally connected space and Y is Hausdorff, then $S(X,Y) = P(X,Y) = L(X,Y)$ is closed in Y^X in the topology of pointwise convergence. Further, if in addition Y is compact, then $S = P = L$ is a compact Hausdorff subspace of Y^X in the topology of pointwise convergence.*

Remark 6.2 Since every quasi (pseudo) locally connected space is Z-locally connected, Corollary 6.1 includes [20, Corollary 7.1].

7. Change of topology. In this section we show that if the topology of a Z-locally connected space is changed in an appropriate way then it is simply a locally connected space.

7.1 Let (X,τ) be a topological space. Let τ_θ denote the collection of all θ-open subsets of the space (X,τ). Since arbitrary union and finite intersections of θ-open sets are θ-open (see [38]), the collection τ_θ is a topology for X. The topology τ_θ has been extensively referred to in the literature (see [4] [16][22]).

7.2 Let (X,τ) be a topological space and let β_{d_δ} denote the collection of all regular F_σ-subsets of (X,τ). Since the intersection of two regular F_σ-sets is a regular F_σ set, the collection β_{d_δ} is a base for a topology τ_* on X. The topology τ_* has been described and used in ([13] [16]) and therein it has been denoted by τ_{d_δ}.

7.3 Let (X,τ) be a topological space and let β_z denote the collection of all cozero subsets of (X,τ). Since the intersection of two cozero sets is a cozero set, the collection β_z is a base for a topology τ_z on X. The topology τ_z has been extensively referred to in the literature (see [3] [5] [37]).

In general $\tau_z \subset \tau_* \subset \tau_\theta \subset \tau$. However, in general none of the above inclusions is reversible (see [16]).

Observations

7.4 The spaces (X,τ), (X,τ_θ), (X,τ_*) and (X,τ_z) have same classes of clopen sets.

PROOF clopen $\Rightarrow$ cozero $\Rightarrow$ regular F_σ $\Rightarrow$ θ-open $\Rightarrow$ open. ■

7.5 Either all the four spaces (X,τ), (X,τ_θ) (X,τ_*) and (X,τ_z) are connected or all four are disconnected.

LEMMA 7.6 *If a cozero set in X is a disjoint union of sets open in X, then each open set in the union is a cozero set in X.*

PROOF Let U be a cozero set in X of a continuous mapping f and let $U = \cup V_i$ where each V_i is open and $V_i \cap V_j = \emptyset$ for $i \neq j$. Let us define $g : X \to R$ as follows

$$g(x) = \begin{cases} f(x) & \text{if } x \in V_i \\ 0 & \text{if } x \notin V_i \end{cases}$$

We show that g is continuous. Let x be any point in V_i. Then f(x) = g(x). So for any neighbourhood N of f(x), there exists a neighbourhood W of x such that $f(W) \subset N$. Now $W \cap V_i$ is a neighbourhood of x and $g(W \cap V_i) = f(W \cap V_i) \subset f(W) \subset N$. So g is continuous at x. Now let $x \in X \setminus U$, then g(x) = f(x) = 0. Since f is continuous at x, for any neighbourhood I of $g(x) = f(x) = 0$, there exists a neighbourhood W_1 of x such that $f(W_1) \subset I$. Now $g(W_1 \cap V_i) = f(W_1 \cap V_i) \subset I$ and $g(W_1 \setminus V_i) = \{0\} \subset I$. Hence $g(W_1) \subset I$ and so g is continuous at x. Finally, let $x \in V_j$, $j \neq i$. Then g(x) = 0. Let G be a neighbourhood of $g(x) = 0$. Then V_j is a neighbourhood of $x \in X$ and $g(V_j) = \{0\} \subset G$. Hence g is continuous on X and V_i is a cozero set of g in X. ■

LEMMA 7.7 *In a topological space (X,τ) if a set U is a cozero set and τ-disconnected, then U is τ_z-disconnected.*

PROOF Let U be a cozero set which is a τ-disconnected subset of (X,τ). Let $U = G \cup H$, where G and H are nonempty disjoint τ-open subsets of X. Then G and H are cozero sets in (X,τ) by Lemma 7.6. Hence U is τ_z-disconnected. ■

THEOREM 7.8 *The space (X,τ) is Z-locally connected if and only if the space (X,τ_z) is locally connected.*

PROOF Suppose (X,τ) is a Z-locally connected space. Let $x \in X$ and let U be a τ_z-open set containing x. Let C_1 be the τ_z-component of U containing x. Since U is a τ_z-open set, there exists a cozero set W in (X,τ) such that $x \in W \subset U$. Let C_2 be the component of W in (X,τ_z) containing x. Then $C_2 \subset C_1$. Again, since (X,τ) is Z-locally connected, there exists a τ-open, τ-connected set V such that $x \in V \subset W$. Since V is τ_z-connected and so $V \subset C_2$. Therefore, C_2 is τ-open. Hence C_2 is a cozero set by Lemma 7.6 in (X,τ). So C_1 is τ_z-open.
To prove sufficiency, suppose that the space (X,τ_z) is locally connected. Let $x \in X$ and let U be a cozero set in (X,τ) containing x. Then U is τ_z-open. By local

connectedness of (X, τ_z), each τ_z component of U is open in (X, τ_z). Therefore U is the disjoint union of τ_z-open sets and hence τ-open sets. So each component of U is a cozero set in (X, τ) by Lemma 7.6. Let C be a component of U in (X, τ_z) containing x. Since C is a cozero set and connected in (X, τ_z), it is connected in (X, τ) by Lemma 6.7. Therefore, C is a τ-open connected set in (X, τ) such that $x \in C \subset U$. Thus (X, τ) is Z-locally connected. ■

8. Open questions. The following two problems are raised in [20] and still remain open.

8.1 Give an example of a pseudo locally connected space which is not quasi locally connected.

8.2 Formulate and prove a product theorem for pseudo locally connected spaces similar to [20, Theorem 4.7].

To this list we now add the following two problems.

8.3 Give an example of a Z-locally connected space which is not pseudo locally connected.

8.4 Formulate and prove a product theorem for Z-locally connected spaces.

9. Products.

9.1 **Conjecture**: let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be any collection of Z-locally connected spaces. If in addition all except finitely many spaces are connected, then their product $X = \prod X_\alpha$ is Z-locally connected.

We are unable to prove this conjecture. However, a word about a possible line of attempt to prove it is in order.

9.2 For a topological space (X, τ) the topology τ_z discussed in the preceding section is a completely regular topology and it is the largest completely regular topology contained in τ.

9.3 Since the product of completely regular topologies is a completely regular topology, $\prod \tau_{\alpha z}$ is a completely regular topology on the product space X and is contained in $\prod \tau_\alpha$. Again, since in view of 9.2, $(\prod \tau_\alpha)_z$ is the largest completely regular topology contained in $\prod \tau_\alpha$, $\prod \tau_{\alpha z} \subset (\prod \tau_\alpha)_z$.
In case the equality $\prod \tau_{\alpha z} = (\prod \tau_\alpha)_z$ is attained, the conjecture will be proved in view of Theorem 7.8 and the standard product theorem for locally connected spaces.

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