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On the degree of approximation of continuous functions by matrix means related to partial sums of a Fourier series

Abstract. In this paper we generalize some results on the degree of approximation of continuous functions by matrix means related to partial sums of a Fourier series, obtained previously by some other authors (please consult references cited in this paper).

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1. Introduction and the aim of the paper. Let $f(x)$ be a 2π- periodic continuous function. Let $S_n(f;x)$ denote the n-th partial sum of its Fourier series at x and let $\omega(\delta) = \omega(\delta, f)$ denote the modulus of continuity of f.

Let $A := (a_{n,k})$ $(k, n = 0, 1, \dots)$ be a lower triangular infinite matrix of real numbers and let the A-transform of $\{S_n(f;x)\}$ be given by

$$T_{n,A}(f) := T_{n,A}(f;x) := \sum_{k=0}^{n} a_{n,k} S_k(f;x) \quad (n = 0, 1, \dots).$$

The deviation $\|T_{n,A}(f) - f\|$ was estimated by P. Chandra [1] and [2] for monotonic sequences $\{a_{n,k}\}$, where $\|\cdot\|$ denotes the supnorm. Later on, these results are generalized by L. Leindler [3] who considered the sequences of Rest Bounded Variation and of Head Bounded Variation.

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers tending to zero is called of Rest Bounded Variation, or briefly $\mathbf{c} \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leqslant K(\mathbf{c}) c_m$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers will be called of Head Bounded Variation, or briefly $\mathbf{c} \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |c_n - c_{n+1}| \leqslant K(\mathbf{c}) c_m$$

for all natural numbers m, or only for all $m \leqslant N$ if the sequence $\mathbf{c}$ has only finite nonzero terms, and the last nonzero term is c_N.

Since Chandra's and Leindler's results are not connected directly to our results, here we shall not recall those, just for interested reader we would like to make mention that some generalizations of Leindler's results are made by present author in [4].

Very recently B. Wei and D. Yu [5] have generalized Leindler's results, and thus Chandra's results, without assumptions that $A \in RBVS$ or $A \in HBVS$. They verified there that Leindler's results are consequences of their results. Before we recall their results we shall suppose that

$$a_{n,k} \geqslant 0, \quad \sum_{k=0}^{n} a_{nk} = 1, \tag{1}$$

and $\omega(t)$ is such that

$$\int_u^\pi t^{-2}\omega(t)dt = O(H(u)), \quad (u \to 0^+), \tag{2}$$

where $H(u) \geqslant 0$, and

$$\int_0^t H(u)du = O(tH(t)), \quad (t \to 0^+). \tag{3}$$

B. Wei and D. Yu's results read as follows:

THEOREM 1.1 *Let (1) hold. Suppose that $\omega(t)$ satisfies (2). Then*

$$\|T_{n,A}(f) - f\| = O\Big(\omega(\pi/n) + \sum_{k=0}^{n} |\triangle a_{nk}| H(\pi/n)\Big).$$

If, in addition, $\omega(t)$ satisfies (3), then

$$\|T_{n,A}(f) - f\| = O\Big(\sum_{k=0}^{n} |\triangle a_{nk}| H\Big(\sum_{k=0}^{n} |\triangle a_{nk}|\Big)\Big),$$

$$\|T_{n,A}(f) - f\| = O\Big(\sum_{k=0}^{n} |\triangle a_{nk}| H(\pi/n)\Big).$$

Theorem 1.2 *Let $(a_{n,k})$ satisfies (1). Then*

$$\|T_{n,A}(f)-f\| = O\Big(\omega(\pi/n)+\sum_{k=1}^{n}k^{-1}\omega(\pi/k)\sum_{\mu=0}^{k+1}a_{n\mu}+\sum_{k=1}^{n}\omega(\pi/k)\sum_{\mu=k}^{n}|\triangle a_{n\mu}|\Big).$$

Now and further for an arbitrary sequence $\{b_n\}$ we denote

$$\triangle b_n = b_n - b_{n+1} \quad \text{and} \quad \triangle^{\nu} b_n = \triangle\left(\triangle^{\nu-1}b_n\right), \quad \nu = 2,3,\dots .$$

The aim of the present paper is to generalize above results obtaining more close estimations of the deviation $\|T_{n,A}(f)-f\|$ than those of B. Wei and D. Yu. Also, we shall show that their results are consequences of ours as a special case.

We emphasize here that throughout of this paper we write $u = O_{\beta}(v)$ if there exists a positive constant C, depending on β, such that $u \leqslant Cv$.

2. Helpful lemmas. To prove the main results we need some auxiliary statements.

Lemma 2.1 ([1]) *If (1.2) and (1.3) hold then*

$$\int_0^{\pi/n}\omega(t)dt = O\left(n^{-2}H(\pi/n)\right).$$

Lemma 2.2 ([2]) *If (1.2) and (1.3) hold then*

$$\int_0^{r}t^{-1}\omega(t)dt = O\left(rH(r)\right), \quad (r \to +0).$$

Lemma 2.3 *For any lower triangular infinite matrix $(a_{n,k})$, $k,n = 0,1,2,\dots$ of nonnegative numbers, it holds uniformly in $0 < t \leqslant \pi$, that*

$$\sum_{k=0}^{n}a_{n,k}\sin\left(k+\frac{1}{2}\right)t = O_{\beta}\Big(\sum_{k=0}^{\tau}a_{nk}+\frac{1}{t}\sum_{k=\tau}^{n}G_{nk;\beta}\Big), \tag{4}$$

where $G_{nk;\beta} := \sum_{j=1}^{\beta}|\triangle^j a_{nk}|$, $\beta \in \{1,2,\dots\}$, and τ denotes the integer part of $\frac{\pi}{t}$.

It also holds that

$$\sum_{k=0}^{n}a_{n,k}\sin\left(k+\frac{1}{2}\right)t = O_{\beta}\Big(\frac{1}{t}\sum_{k=0}^{n}G_{nk;\beta}\Big). \tag{5}$$

PROOF For arbitrary $\lambda_n \geqslant 0$ and for $n \geqslant m \geqslant 0$ we have

$$\begin{aligned}
\Lambda_{m,n} &:= \sum_{k=m}^{n} \lambda_k \sin\left(k+\frac{1}{2}\right) t \sin\frac{t}{2} \\
&= \frac{1}{2}\left[\lambda_m \cos mt - \sum_{k=m}^{n-1} \triangle\lambda_k \cos(k+1)t - \lambda_n \cos(n+1)t\right] \\
&= \frac{1}{2}\left[\lambda_m \cos mt - \sum_{k=m}^{n-1} \triangle\lambda_k \cos\left(k+\frac{1}{2}\right) t \cos\frac{t}{2}\right. \\
&\quad \left. + \sum_{k=m}^{n-1} \triangle\lambda_k \sin\left(k+\frac{1}{2}\right) t \sin\frac{t}{2} - \lambda_n \cos(n+1)t\right] \\
&= \frac{1}{2}\left[\lambda_m \cos mt - \sum_{k=m}^{n-1} \triangle\lambda_k \cos\left(k+\frac{1}{2}\right) t \cos\frac{t}{2} - \lambda_n \cos(n+1)t\right] \\
&\quad + \frac{1}{2^2}\left[\triangle\lambda_m \cos mt - \sum_{k=m}^{n-2} \triangle^2\lambda_k \cos(k+1)t - \triangle\lambda_{n-1} \cos nt\right].
\end{aligned}$$

Repeating this transformation, in the same way $\beta-$times, we easy obtain

$$\begin{aligned}
\Lambda_{m,n} &:= \frac{1}{2}\left[\lambda_m \cos mt - \sum_{k=m}^{n-1} \triangle\lambda_k \cos\left(k+\frac{1}{2}\right) t \cos\frac{t}{2} - \lambda_n \cos(n+1)t\right] \\
&\quad + \frac{1}{2^2}\left[\triangle\lambda_m \cos mt - \sum_{k=m}^{n-2} \triangle^2\lambda_k \cos(k+1)t - \triangle\lambda_{n-1} \cos nt\right] + \cdots + \\
&\quad + \frac{1}{2^\beta}\left[\triangle^{\beta-1}\lambda_m \cos mt - \sum_{k=m}^{n-\beta} \triangle^\beta\lambda_k \cos(k+1)t - \triangle^{\beta-1}\lambda_{n-\beta+1} \cos(n-\beta)t\right],
\end{aligned}$$

where $\beta \in \{1, 2, \dots\}$.

Thus,

$$\begin{aligned}
|\Lambda_{m,n}| &\leqslant \frac{1}{2}\left(\lambda_m + \sum_{k=m}^{n-1} |\triangle\lambda_k| + \lambda_n\right) \\
&\quad + \frac{1}{2^2}\left(|\triangle\lambda_m| + \sum_{k=m}^{n-2} |\triangle^2\lambda_k| + |\triangle\lambda_{n-1}|\right) + \cdots + \\
&\quad + \frac{1}{2^\beta}\left(|\triangle^{\beta-1}\lambda_m| + \sum_{k=m}^{n-\beta} |\triangle^\beta\lambda_k| + |\triangle^{\beta-1}\lambda_{n-\beta+1}|\right), \qquad (6)
\end{aligned}$$

where $\beta \in \{1, 2, \dots\}$.

But since (a_{nk}) is a lower triangular matrix, then

$$a_{nm} \leqslant \sum_{k=m}^{n} |\triangle a_{nk}|,\ |\triangle a_{nm}| \leqslant \sum_{k=m}^{n} |\triangle^2 a_{nk}|, \dots, |\triangle^{\beta-1} a_{nm}| \leqslant \sum_{k=m}^{n} |\triangle^\beta a_{nk}| \qquad (7)$$

hold for $n-\beta+1\geqslant m\geqslant 0$.

Now by (6) and (7), supposing that $n-\beta+1\geqslant\tau$, we have

$$\begin{aligned}\left|\sum_{k=0}^{n}a_{nk}\sin\left(k+\frac{1}{2}\right)t\right| &\leqslant \sum_{k=0}^{\tau}a_{nk}+\left|\sum_{k=\tau}^{n}a_{nk}\sin\left(k+\frac{1}{2}\right)t\right|\\ &\leqslant \sum_{k=0}^{\tau}a_{nk}+O_\beta\Bigg(\frac{1}{t}\Bigg(\Bigg(a_{n0}+\sum_{k=\tau}^{n-1}|\triangle a_{nk}|+a_{nn}\Bigg)\\ &\quad+\Bigg(|\triangle a_{n0}|+\sum_{k=\tau}^{n-2}|\triangle^2 a_{nk}|+|\triangle a_{nn-1}|\Bigg)\\ &\quad+\cdots+\Bigg(|\triangle^{\beta-1}a_{n0}|+\sum_{k=\tau}^{n-\beta}|\triangle^{\beta}a_{nk}|+|\triangle^{\beta-1}a_{nn-\beta+1}|\Bigg)\Bigg)\Bigg)\\ &= O_\beta\Bigg(\sum_{k=0}^{\tau}a_{nk}+\frac{1}{t}\sum_{k=\tau}^{n}\big(|\triangle a_{nk}|+|\triangle^2 a_{nk}|+\cdots+|\triangle^{\beta}a_{nk}|\big)\Bigg),\end{aligned}$$

where O_β depends on β. This completely proves (4).

By a similar technique we have

$$\begin{aligned}\left|\sum_{k=0}^{n}a_{nk}\sin\left(k+\frac{1}{2}\right)t\right| &= O_\beta\Bigg(\frac{1}{t}\Bigg(\Bigg(a_{n0}+\sum_{k=0}^{n-1}|\triangle a_{nk}|+a_{nn}\Bigg)\\ &\quad+\Bigg(|\triangle a_{n0}|+\sum_{k=0}^{n-2}|\triangle^2 a_{nk}|+|\triangle a_{nn-1}|\Bigg)\\ &\quad+\cdots+\Bigg(|\triangle^{\beta-1}a_{n0}|+\sum_{k=0}^{n-\beta}|\triangle^{\beta}a_{nk}|+|\triangle^{\beta-1}a_{nn-\beta+1}|\Bigg)\Bigg)\Bigg)\\ &= O_\beta\Bigg(\frac{1}{t}\sum_{k=0}^{n}\big(|\triangle a_{nk}|+|\triangle^2 a_{nk}|+\cdots+|\triangle^{\beta}a_{nk}|\big)\Bigg),\end{aligned}$$

which completes (5), and with this the proof of the lemma. ■

3. Main Results. We establish the following.

Theorem 3.1 *Let* $(a_{n,k})$ *satisfies conditions*

$$a_{n,k}\geqslant 0 \quad \textit{and} \quad \sum_{k=0}^{n}a_{nk}=\frac{1}{2^{\beta}-1},\quad \beta\in\{1,2,\dots\}. \tag{8}$$

Assume that $\omega(t)$ *satisfies condition (2). Then*

$$\|T_{n,A}(f)-f\|=O_\beta\Bigg(\omega(\pi/n)+\sum_{k=0}^{n}G_{nk;\beta}H(\pi/n)\Bigg). \tag{9}$$

If, in addition, $\omega(t)$ satisfies (3), then

$$\|T_{n,A}(f)-f\| = O_\beta\Bigg(\sum_{k=0}^{n} G_{nk;\beta} H\Bigg(\sum_{k=0}^{n} G_{nk;\beta}\Bigg)\Bigg), \tag{10}$$

$$\|T_{n,A}(f)-f\| = O_\beta\Bigg(\sum_{k=0}^{n} G_{nk;\beta} H(\pi/n)\Bigg). \tag{11}$$

Proof First we set the notation

$$\phi_{x;\beta}(t) := \frac{f(x+t)+f(x-t)-2(2^\beta-1)f(x)}{2},$$

and throughout we shall keep in mind that $\beta \in \{1,2,\dots\}$.

Then we easy obtain

$$T_{n,A}(f;x)-f(x) = \frac{2}{\pi}\int_0^\pi \phi_{x;\beta}(t)\left(2\sin\frac{t}{2}\right)^{-1}\sum_{k=0}^{n} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\,dt. \tag{12}$$

By (12) we have

$$\|T_{n,A}(f;x)-f(x)\| \leqslant \frac{2}{\pi}\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right) := J_1(n)+J_2(n). \tag{13}$$

According to (8) and the inequality $|\sin t| \leqslant t$ for $0 \leqslant t \leqslant \pi/n$, we have

$$\left|\sum_{k=0}^{n} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| \leqslant \frac{2nt}{2^\beta-1}.$$

Thus,

$$J_1(n) = O_\beta(n)\int_0^{\pi/n}\omega(t)dt = O_\beta(\omega(\pi/n)). \tag{14}$$

Also, by (5) and (2) we have

$$J_2(n) = O_\beta\Bigg(\sum_{k=0}^{n} G_{nk;\beta}\Bigg)\int_{\pi/n}^{\pi} t^{-2}\omega(t)dt = O_\beta\Bigg(\sum_{k=0}^{n} G_{nk;\beta} H(\pi/n)\Bigg). \tag{15}$$

Now (9) follows from (13)–(15).

Then according to (8), and

$$\sum_{k=0}^{n} G_{nk;\beta} = \sum_{k=0}^{n}\sum_{j=1}^{\beta} |\triangle^j a_{nk}| \leqslant (2+2^2+\cdots+2^\beta)\sum_{k=0}^{n} a_{nk} = 2 < \pi,$$

we get

$$\|T_{n,A}(f;x)-f(x)\| \leqslant \frac{2}{\pi}\left(\int_0^{\sum_{k=0}^{n} G_{nk;\beta}}+\int_{\sum_{k=0}^{n} G_{nk;\beta}}^{\pi}\right) := R_1(n)+R_2(n). \tag{16}$$

It is obvious from (8) that

$$\left|\sum_{k=0}^{n} a_{n,k} \sin\left(k+\frac{1}{2}\right)t\right| \leqslant \frac{1}{2^{\beta}-1}.$$

Thus, by Lemma 2.1 we have

$$R_1(n) = O_{\beta}(1)\int_0^{\sum_{k=0}^n G_{nk;\beta}} t^{-1}\omega(t)dt = O_{\beta}\bigg(\sum_{k=0}^{n} G_{nk;\beta} H\bigg(\sum_{k=0}^{n} G_{nk;\beta}\bigg)\bigg). \quad (17)$$

Using (5) and (2) we obtain

$$\begin{aligned} R_2(n) &= O_{\beta}\bigg(\sum_{k=0}^{n} G_{nk;\beta}\int_{\sum_{k=0}^n G_{nk;\beta}}^{\pi} t^{-2}\omega(t)dt\bigg) \\ &= O_{\beta}\bigg(\sum_{k=0}^{n} G_{nk;\beta} H\bigg(\sum_{k=0}^{n} G_{nk;\beta}\bigg)\bigg). \quad (18) \end{aligned}$$

From (16), (17), and (18) follows (10).

Now we turn back to prove (11). Since $a_{nk}=0$ for $k>n$, we deduce that

$$a_{n\ell} \leqslant \sum_{k=\ell}^{n} |\triangle a_{nk}| \leqslant \sum_{k=0}^{n}\big(|\triangle a_{nk}| + |\triangle^2 a_{nk}| + \cdots + |\triangle^{\beta} a_{nk}|\big)$$

for $\ell = 0,1,2,\ldots,n$, which implies

$$\frac{1}{2^{\beta}-1} = \sum_{\ell=0}^{n} a_{n\ell} \leqslant (n+1)\sum_{k=0}^{n}\big(|\triangle a_{nk}| + |\triangle^2 a_{nk}| + \cdots + |\triangle^{\beta} a_{nk}|\big),$$

i.e.

$$\sum_{k=0}^{n}\big(|\triangle a_{nk}| + |\triangle^2 a_{nk}| + \cdots + |\triangle^{\beta} a_{nk}|\big) \geqslant \frac{1}{2(2^{\beta}-1)n}.$$

Whence, according to Lemma 2.2 we obtain

$$J_1(n) = O_{\beta}\bigg(\frac{1}{n}H(\pi/n)\bigg) = O_{\beta}\bigg(\sum_{k=0}^{n} G_{nk;\beta} H(\pi/n)\bigg). \quad (19)$$

Therefore, by (13), (15), and (19), (11) is proved. ■

Example 3.2 We give an example of a lower triangular infinite matrix $A := (a_{nk})$, $k,n = 0,1,2,\ldots$, that satisfies conditions (1.5). Namely, it is clear that the matrix

$$A = \begin{pmatrix} \frac{1}{2^{\beta}} & 0 & 0 & 0 & \cdots & 0 & \ldots \\ \frac{1}{2^{\beta}} & \frac{1}{4^{\beta}} & 0 & 0 & \cdots & 0 & \ldots \\ \frac{1}{2^{\beta}} & \frac{1}{4^{\beta}} & \frac{1}{8^{\beta}} & 0 & \cdots & 0 & \ldots \\ \frac{1}{2^{\beta}} & \frac{1}{4^{\beta}} & \frac{1}{8^{\beta}} & \frac{1}{16^{\beta}} & \cdots & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \ddots \end{pmatrix}$$

satisfies conditions $a_{nk} \geqslant 0$ and for n large enough holds

$$\sum_{k=0}^{n} a_{nk} = \frac{1}{2^{\beta}-1}, \quad \beta \in \{1,2,\dots\}.$$

Also note that in particular case $\beta = 1$ and n large enough holds $\sum_{k=0}^{n} a_{nk} = 1$.

THEOREM 3.3 *Let $(a_{n,k})$ satisfies (8). Then*

$$\|T_{n,A}(f) - f\| = O_{\beta}\left(\omega(\pi/n) + \sum_{k=1}^{n} k^{-1}\omega(\pi/k)\sum_{\mu=0}^{k+1} a_{n\mu} + \sum_{k=1}^{n}\omega(\pi/k)\sum_{\mu=k}^{n} G_{n\mu;\beta}\right). \tag{20}$$

PROOF According to (4) and the property of the monotonicity of $\omega(t)$, we have

$$\begin{aligned}
J_2(n) &= \frac{2}{\pi}\int_{\pi/n}^{\pi} \phi_{x;\beta}(t)\left(2\sin\frac{t}{2}\right)^{-1}\sum_{k=0}^{n} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\,dt \\
&= O_{\beta}\left(\int_{\pi/n}^{\pi} t^{-1}\omega(t)\left(\sum_{\mu=0}^{\tau} a_{n\mu} + \frac{1}{t}\sum_{\mu=\tau}^{n} G_{n\mu;\beta}\right)dt\right) \\
&= O_{\beta}\left(\sum_{k=1}^{n-1}\int_{\pi/(k+1)}^{\pi/k} t^{-1}\omega(t)\left(\sum_{\mu=0}^{\tau} a_{n\mu} + \frac{1}{t}\sum_{\mu=\tau}^{n} G_{n\mu;\beta}\right)\right)dt \\
&= O_{\beta}\left(\sum_{k=1}^{n} k^{-1}\omega(\pi/k)\sum_{\mu=0}^{k+1} a_{n\mu} + \sum_{k=1}^{n}\omega(\pi/k)\sum_{\mu=k}^{n} G_{n\mu;\beta}\right). \qquad (21)
\end{aligned}$$

Combining (13), (14), and (21), we immediately obtain (20). The proof of the theorem is completed. ∎

REMARK 3.4 Note that if we take $s = 1$ in our results we exactly obtain Theorems 1.1-1.2.

REMARK 3.5 Also, if we take $s = 1$ and suppose that $\{a_{nk}\} \in HBVS$ or $\{a_{nk}\} \in RBVS$, then our results imply Leindler's results, and thus also results of P. Chandra [1]–[2] (see discussions in [5]).

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