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Weak nearly uniform smoothness of the ψ-direct sums \((X_1 \oplus \cdots \oplus X_N)_{\psi}\)

Abstract. We shall characterize the weak nearly uniform smoothness of the ψ-direct sum \((X_1 \oplus \cdots \oplus X_N)_{\psi}\) of \(N\) Banach spaces \(X_1, \ldots, X_N\), where \(\psi\) is a convex function satisfying certain conditions on the convex set \(\Delta_N = \{(s_1, \ldots, s_{N-1}) \in \mathbb{R}^{N-1}_+ : \sum_{i=1}^{N-1} s_i \leq 1\}\). To do this a class of convex functions which yield \(\ell_1\)-like norms will be introduced. We shall apply our result to the fixed point property for nonexpansive mappings (FPP). In particular an example will be presented which indicates that there are plenty of Banach spaces with FPP failing to be uniformly non-square.

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1. Introduction. Since it was introduced in Takahashi, Kato and Saito [27], the \(\psi\)-direct sum \(X \oplus_{\psi} Y\) of Banach spaces \(X\) and \(Y\) has attracted a good deal of attention and many investigations have been done, where \(\psi\) is a convex function on the unit interval satisfying some conditions (e.g., [3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 20, 21, 22, 24, 28]). The finitely many Banach spaces case is treated in Kato, Saito and Tamura [11]. The present authors [14] showed that under the condition that \(X\) or \(Y\) is of infinite dimension, \(X \oplus_{\psi} Y\) is weakly nearly uniformly smooth if and only if \(X\) and \(Y\) are weakly nearly uniformly smooth and \(\psi \neq \psi_1\), where \(\psi_1(t) = 1\) is the convex function corresponding to the \(\ell_1\)-norm.

The aim of this paper is to extend the above result for the \(\psi\)-direct sum \((X_1 \oplus \cdots \oplus X_N)_{\psi}\) of \(N\) Banach spaces \(X_1, \ldots, X_N\). The situation is much more compli-

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cated than expected in comparison with the two Banach spaces case, that is, the weak nearly uniform smoothness of $X_1, \ldots, X_N$ and the condition $\psi \neq \psi_1$ do not imply that of $(X_1 \oplus \cdots \oplus X_N)_\psi$. The point is to find out a subclass of convex functions in $\Psi_N$ including $\psi_1$ which should be excluded. As such one we shall introduce a subclass $\Psi_N^{(1)}$ consisting of the convex functions which yield $\ell^1$-like norms (Section 5; cf. [17]). In the case of $N = 2$ the subclass $\Psi_N^{(1)}$ contains only the function $\psi_1$.

As the main results we shall have the following: (i) Let $X_1, \ldots, X_N$ be all infinite dimensional spaces. Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth if and only if $X_1, \ldots, X_N$ are weakly nearly uniformly smooth and $\psi \notin \Psi_N^{(1)}$ (Theorem 6.14). (ii) In the case where some of $X_1, \ldots, X_N$ are infinite dimensional spaces and the rest finite dimensional we shall obtain a more precise result depending on which spaces $X_j$ are of infinite dimension (Theorem 6.12). This extends the previous result for the two Banach spaces case stated above (Corollary 6.13). (iii) The condition that all $X_1, \ldots, X_N$ are finite dimensional is equivalent to that $(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth for all $\psi \in \Psi_N^{(1)}$, or equivalently for $\psi_1$ (Theorem 6.17).

According to García-Falset [8] a Banach space $X$ is weakly nearly uniformly smooth if and only if $X$ is reflexive and $R(X) < 2$, where $R(X)$ is the García-Falset coefficient ([8]). Thus to obtain the above-mentioned results we shall prove that for the spaces $X_1, \ldots, X_N$, some of which do not have the Schur property, $R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2$ if and only if $R(X_j) < 2$ for all $1 \leq j \leq N$ and $\psi \notin \Psi_N^{(1)}(S)$ for all nonempty subsets $S$ of $\{1, \ldots, N - 1\}$ such that $X_j$ does not have the Schur property either for all $j \in S + 1$ or for all $j \in (S + 1)^c$, where $\Psi_N^{(1)}(S)$ is a subclass of $\Psi_N^{(1)}$ depending on the set $S$ (Theorem 6.2). In particular, if all $X_j$ do not have the Schur property, $R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2$ if and only if $R(X_j) < 2$ for all $1 \leq j \leq N$ and $\psi \notin \Psi_N^{(1)}$ (Theorem 6.4). This yields a previous result by Dhompngsa et al. [3]: If $R(X_j) < 2$ for all $1 \leq j \leq N$ and $\psi \in \Psi_N$ is strictly convex, then $R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2$ (Corollary 6.6). Also their another result ([3]) which states that, if $\psi \in \Psi_N$ is strictly convex, then $(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth if and only if $X_1, \ldots, X_N$ are so, is a corollary of Theorem 6.14 (Corollary 6.16).

In the course of doing the above we shall present a characterization for the strict monotonicity of an absolute norm on $C^n$, from which a previous result in Dowling and Turett [7] will be immediately derived (Section 3) and also we shall discuss generalized Hölder’s inequality and the dual space of $(X_1 \oplus \cdots \oplus X_N)_\psi$ (Section 4).

As an application we shall obtain that if $X_1, \ldots, X_N$ are weakly nearly uniformly smooth and $\psi \notin \Psi_N^{(1)}$, then $(X_1 \oplus \cdots \oplus X_N)_\psi$ has the fixed point property for nonexpansive mappings, FPP in short (Theorem 7.2); in particular the $l_\infty$-sum $(X_1 \oplus \cdots \oplus X_N)_{\infty}$, which is not uniformly non-square, has FPP (recall that all uniformly non-square spaces have FPP ([10])). Also another example will be constructed, which indicates that there are plenty of Banach spaces with FPP failing to be uniformly non-square (Corollary 7.6).
2. Preliminaries. A norm $\| \cdot \|$ on $\mathbb{C}^N$ is called absolute if
$$\|(z_1, \cdots, z_N)\| = \|(\lvert z_1 \rvert, \cdots, \lvert z_N \rvert)\|$$
for all $(z_1, \cdots, z_N) \in \mathbb{C}^N$ and normalized if
$$\|(1, 0, \cdots, 0)\| = \cdots = \|(0, \cdots, 0, 1)\| = 1.$$
The collection of all such norms on $\mathbb{C}^N$ is denoted by $\mathcal{A}N_N$. In case of $N = 2$ every absolute normalized norm $\| \cdot \|$ on $\mathbb{C}^2$ there corresponds a unique convex (continuous) function $\psi$ on the unit interval $[0, 1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ under the equation $\psi(t) = \|(1-t, t)\|$ ([5]; see also [25]). The $N$-dimensional version of this fact was presented by K.-S. Saito et al. [26] as follows: Let $\| \cdot \| \in \mathcal{A}N_N$, Let
$$\psi(s) = \|(1 - \sum_{i=1}^{N-1} s_i, s_1, \cdots, s_{N-1})\|$$
for $s = (s_1, \cdots, s_{N-1}) \in \Delta_N,$
where
$$\Delta_N = \{ s = (s_1, \cdots, s_{N-1}) \in \mathbb{R}^{N-1}: \sum_{i=1}^{N-1} s_i \leq 1, \ s_i \geq 0 \}. \quad (2)$$
Then $\psi$ is convex (continuous) on the convex set $\Delta_N$ and satisfies the following:
\begin{align*}
(A_0) \quad \psi(0, \cdots, 0) &= \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1, \\
(A_1) \quad \psi(s_1, \cdots, s_{N-1}) &\geq \left(\sum_{i=1}^{N-1} s_i\right) \psi\left(\frac{s_1}{\sum_{i=1}^{N-1} s_i}, \cdots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i}\right) \\
&\quad \text{if } 0 < \sum_{i=1}^{N-1} s_i \leq 1, \\
(A_2) \quad \psi(s_1, \cdots, s_{N-1}) &\geq (1 - s_1) \psi\left(0, \frac{s_2}{1 - s_1}, \cdots, \frac{s_{N-1}}{1 - s_1}\right) \\
&\quad \text{if } 0 \leq s_1 < 1, \\
\ldots \ldots \ldots \\
(A_N) \quad \psi(s_1, \cdots, s_{N-1}) &\geq (1 - s_{N-1}) \psi\left(\frac{s_1}{1 - s_{N-1}}, \cdots, \frac{s_{N-2}}{1 - s_{N-1}}, 0\right) \\
&\quad \text{if } 0 \leq s_{N-1} < 1
\end{align*}
The converse holds true: Denote by $\Psi_N$ the family of all convex functions $\psi$ on $\Delta_N$ satisfying $(A_0) - (A_N)$. For any $\psi \in \Psi_N$ define
$$\|(z_1, \cdots, z_N)\|_\psi = \begin{cases} \\
\left(\sum_{j=1}^{N} |z_j|\right) \psi\left(\frac{|z_2|}{\sum_{j=1}^{N} |z_j|}, \cdots, \frac{|z_N|}{\sum_{j=1}^{N} |z_j|}\right) & \text{if } (z_1, \cdots, z_N) \neq (0, \cdots, 0), \\
0 & \text{if } (z_1, \cdots, z_N) = (0, \cdots, 0)
\end{cases} \quad (3)$$
Then $\| \cdot \|_\psi \in AN_N$ and $\| \cdot \|_\psi$ satisfies (1). Thus we have a one-to-one correspondence between $AN_N$ and $\Psi_N$ with the equation (1). The $\ell_p$-norms

$$\|(z_1, \cdots, z_N)\|_p = \begin{cases} \left( |z_1|^p + \cdots + |z_N|^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|z_1|, \cdots, |z_N|\} & \text{if } p = \infty \end{cases}$$

are typical examples of absolute normalized norms and for any $\| \cdot \| \in AN_N$ we have

$$\| \cdot \|_\infty \leq \| \cdot \| \leq \| \cdot \|_1 \quad (4)$$

([26, Lemma 3.1], cf. [5]). The convex function corresponding to the $\ell_p$-norm, which is denoted by $\psi_p$, is

$$\psi_p(s_1, \cdots, s_{N-1}) = \begin{cases} \left( \left(1 - \sum_{i=1}^{N-1} s_i\right)^p + s_1^p + \cdots + s_{N-1}^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1 - \sum_{i=1}^{N-1} s_i, s_1, \cdots, s_{N-1}\} & \text{if } p = \infty \end{cases}$$

and for any $\psi \in \Psi_N$ we have

$$\frac{1}{N} \leq \psi_\infty(\cdot) \leq \psi(\cdot) \leq \psi_1(\cdot) \quad (5)$$

([26, Lemma 3.2]). A function $\psi \in \Psi_N$ is called strictly convex provided, if $s, t \in \Delta_N$, $s \neq t$, one has $\psi((1-\alpha)s + \alpha t) < (1-\alpha)\psi(s) + \alpha\psi(t)$ for all $0 < \alpha < 1$.

Let $X_1, \ldots, X_N$ be Banach spaces and let $\psi \in \Psi_N$. Let $(X_1 \oplus \cdots \oplus X_N)_\psi$ be the direct sum of $X_1, \ldots, X_N$ equipped with the norm

$$\|(x_1, \cdots, x_N)\|_\psi := \|(x_1, \cdots, x_N)\| \quad \text{for } (x_1, \cdots, x_N) \in X_1 \oplus \cdots \oplus X_N \quad (6)$$

([11, 27]). Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ is a Banach space, which extends the notion of $\ell_p$-sum.

As usual $B_X$ and $S_X$ stand for the closed unit ball and unit sphere of a Banach space $X$, respectively. A sequence $\{x_n\}$ in $X$ is called a basic sequence if it is a Schauder basis for its closed linear span, that is, if every $x$ in the span of $\{x_n\}$ has a unique representation of the form $x = \sum_{n=1}^\infty x_n$. A Banach space $X$ is said to be weakly nearly uniformly smooth ([18]) provided there exist $0 < \epsilon < 1$ and $\nu > 0$ such that for any basic sequence $\{x_n\}$ in $B_X$ and any $0 < t < \nu$ there is $k > 0$ so that $\|x_1 + tx_k\| \leq 1 + \epsilon t$. The constant $R(X)$ ([8]), which is referred to as García-Falset coefficient in [2], is defined by

$$R(X) = \sup \{ \liminf_{n \to \infty} \|x_n + x\| \}, \quad (7)$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in $B_X$ and all $x \in B_X$. Clearly the formula (7) can be rewritten as $R(X) = \sup \{ \liminf_{n \to \infty} \|x_n - x\| \}$. As is readily seen, $1 \leq R(X) \leq 2$ and it is known that $R(\ell_0) = R(\ell_1) = 1$, $R(\ell_p) = 2^{1/p}$ ($1 < p < \infty$), $R(\ell_1) = 2$ (cf. [19, p.165]). It is known
that uniformly convex, resp., uniformly smooth spaces are weakly nearly uniformly smooth (cf. [19, p.165], [23, p.508]), and $X$ is weakly nearly uniformly smooth if and only if $X$ is reflexive and $R(X) < 2$ (García-Falset [8]). A Banach space $X$ is said to have the Schur property if every weakly convergent sequence in $X$ converges strongly. It is clear that $R(X) = 1$ if $X$ has the Schur property. $X$ is called strictly convex provided, whenever $x, y \in S_X$, $x \neq y$, one has $\|x + y\|/2 < 1$. $X$ is said to have the fixed point property (resp. weak fixed point property) for nonexpansive mappings if every nonexpansive self-mapping $T$ of any nonempty bounded closed (resp. weakly compact) convex subset $C$ of $X$ has a fixed point ($T$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). We say the former as FPP in short. It is well known that, if $R(X) < 2$, $X$ has the weak fixed point property (García-Falset [9]) and hence weakly nearly uniformly smooth spaces have FPP.

3. Strict monotonicity of absolute norms. We begin with recalling a fundamental fact.

**Lemma 3.1** ([26]; cf. [5]) Let $\|\cdot\| \in AN_N$.

(i) If $|p_j| \leq |q_j|$ for all $1 \leq j \leq N$, then $\|(p_1, \ldots, p_N)\| \leq \|(q_1, \ldots, q_N)\|$.

(ii) If $|p_j| < |q_j|$ for all $1 \leq j \leq N$, then $\|(p_1, \ldots, p_N)\| < \|(q_1, \ldots, q_N)\|$.

A norm $\|\cdot\|$ in $AN_N$ is called strictly monotone if $|p_j| \leq |q_j|$ for all $1 \leq j \leq N$ and $|p_{j_0}| < |q_{j_0}|$ for some $1 \leq j_0 \leq N$, then $\|(p_1, \ldots, p_N)\| < \|(q_1, \ldots, q_N)\|$. As is readily seen, the $\ell_\infty$-norm does not have this property. The next result is useful in our later discussion.

**Proposition 3.2** Let $\psi \in \Psi_N$. Let $(p_1, \ldots, p_N) \in C^N$ and $0 < |p_{j_0}| < |q_{j_0}|$ for some $1 \leq j_0 \leq N$. Then the following are equivalent.

(i) $\|(p_1, \ldots, p_{j_0}, \ldots, p_N)\|_\psi < \|(p_1, \ldots, q_{j_0}, \ldots, p_N)\|_\psi$

(ii) $\||(p_1, \ldots, 0, \ldots, p_N)\|_\psi < \||(p_1, \ldots, q_{j_0}, \ldots, p_N)\|_\psi$

**Proof** The implication (i) $\Rightarrow$ (ii) is clear. Assume (ii) to be true. Let $|p_{j_0}| = \alpha |q_{j_0}|$ with some $0 < \alpha < 1$. Then we have

$$\|(p_1, \ldots, p_{j_0}, \ldots, p_N)\|_\psi$$

$$= \|(p_1, \ldots, \alpha q_{j_0}, \ldots, p_N)\|_\psi$$

$$= \|\alpha (p_1, \ldots, q_{j_0}, \ldots, p_N) + (1 - \alpha)(p_1, \ldots, 0, \ldots, p_N)\|_\psi$$

$$\leq \alpha \|(p_1, \ldots, q_{j_0}, \ldots, p_N)\|_\psi + (1 - \alpha)\|(p_1, \ldots, 0, \ldots, p_N)\|_\psi$$

$$< \|(p_1, \ldots, q_{j_0}, \ldots, p_N)\|_\psi,$$

or (i).

Now, we note that according to the formulae (1) and (3) the properties $(A_0)$ – $(AN)$ for a function $\psi \in \Psi_N$ are rewritten in words of the corresponding norm $\|\cdot\|_\psi$.
as follows:

\((A^0)\) \( \| (1, 0, \cdots, 0) \| = \cdots = \| (0, \cdots, 0, 1) \| = 1, \)

\((A^1)\) \( \| (1 - \sum_{i=1}^{N-1} s_i, s_1, \cdots, s_{N-1}) \| \geq \| (0, s_1, \cdots, s_{N-1}) \|, \)

\((A^2)\) \( \| (1 - \sum_{i=1}^{N-1} s_i, s_1, \cdots, s_{N-1}) \| \geq \| (1 - \sum_{i=1}^{N-1} s_i, 0, s_2, \cdots, s_{N-1} - 0) \|, \)

\[ \vdots \]

\((A^N)\) \( \| (1 - \sum_{i=1}^{N-1} s_i, s_1, \cdots, s_{N-1}) \| \geq \| (1 - \sum_{i=1}^{N-1} s_i, s_1, \cdots, s_{N-2}, 0) \| \)

As is readily seen from the above observation, the next result in Dowling and Turett \([7]\) is an immediate consequence of Proposition 3.2.

**Corollary 3.3** (\([7]\)) Let \( \psi \in \Psi_N \). Then \( \| \cdot \| \) is strictly monotone if and only if \( \psi \) satisfies the following conditions:

\((sA_1)\) \( \psi(s_1, \cdots, s_{N-1}) > \left( \sum_{i=1}^{N-1} s_i \right) \psi\left( \frac{s_1}{\sum_{i=1}^{N-1} s_i}, \cdots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i} \right) \)

\[ \text{if } 0 < \sum_{i=1}^{N-1} s_i < 1, \]

\((sA_2)\) \( \psi(s_1, \cdots, s_{N-1}) > (1 - s_1) \psi\left( 0, \frac{s_2}{1 - s_1}, \cdots, \frac{s_{N-1}}{1 - s_1} \right) \)

\[ \text{if } 0 < s_1 < 1, \]

\[ \vdots \]

\((sA_N)\) \( \psi(s_1, \cdots, s_{N-1}) > (1 - s_{N-1}) \psi\left( \frac{s_1}{1 - s_{N-1}}, \cdots, \frac{s_{N-2}}{1 - s_{N-1}}, 0 \right) \)

\[ \text{if } 0 < s_{N-1} < 1. \]

### 4. Generalized Hölder’s inequality and dual space.

In this section we shall discuss generalized Hölder’s inequality for the \( \psi \)-direct sum \( (X_1 \oplus \cdots \oplus X_N)_\psi \) and its dual space (cf. \([4, 20, 14]\)); for completeness we shall present our proofs. In the following \( X^* \) stands for the dual space of \( X \).

**Theorem 4.1** (Generalized Hölder’s inequality; cf. \([20, 14]\)) Let \( X_1, \cdots, X_N \) be Banach spaces and let \( \psi \in \Psi_N \). Let \( \psi^* \) be the function on \( \Delta_N \) defined by

\[ \psi^*(s_1, \cdots, s_{N-1}) = \sup_{(t_1, \cdots, t_{N-1}) \in \Delta_N} \frac{(1 - \sum_{i=1}^{N-1} s_i)(1 - \sum_{i=1}^{N-1} t_i) + \sum_{i=1}^{N-1} s_i t_i}{\psi(t_1, \cdots, t_{N-1})} \]

\[ \text{for } \psi(t_1, \cdots, t_{N-1}) = \psi(t_1, \cdots, t_{N-1}) = \psi(0, \cdots, 0) = 1. \]
Then \( \psi^* \in \Psi_N \) and
\[
\left| \sum_{j=1}^{N} \psi \left( x_j \right) \right| \leq \| (x_1^*, \cdots, x_N^*) \| \psi \| (x_1, \cdots, x_N) \| \psi
\]
for all \( (x_1, \cdots, x_N) \in (X_1 \oplus \cdots \oplus X_N) \psi \) and \((x_1^*, \cdots, x_N^*) \in (X_1^* \oplus \cdots \oplus X_N^*) \psi^* \).

**Proof** It is straightforward to see that \( \psi^* \) is convex and the proof is omitted. Since \( t_i \leq \max\{t_1, \cdots, t_{N-1}\} \leq \psi(t_1, \cdots, t_{N-1}) \) and \( \psi(0, 0, 0, 1, 0, \cdots, 0) = 1 \), we have
\[
\psi^*(0, 0, 0, 1, 0, \cdots, 0) = \sup_{(t_1, \cdots, t_{N-1}) \in \Delta_N} \frac{t_i}{\psi(t_1, \cdots, t_{N-1})} = 1,
\]
or \((A_0)\). Let \( 0 < \sum_{i=1}^{N-1} s_i \leq 1 \). Then we have
\[
\left( \sum_{i=1}^{N-1} s_i \right) \psi^* \left( \frac{s_1}{\sum_{i=1}^{N-1} s_i}, \cdots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i} \right) = \sup_{(t_1, \cdots, t_{N-1}) \in \Delta_N} \frac{\sum_{i=1}^{N-1} s_i t_i}{\psi(t_1, \cdots, t_{N-1})} \leq \sup_{(t_1, \cdots, t_{N-1}) \in \Delta_N} \frac{(1 - \sum_{i=1}^{N-1} s_i)(1 - \sum_{i=1}^{N-1} t_i) + \sum_{i=1}^{N-1} s_i t_i}{\psi(t_1, \cdots, t_{N-1})} = \psi^*(s_1, \cdots, s_{N-1}),
\]
or \((A_1)\). Let \( 0 \leq s < 1 \). Then
\[
(1 - s_1) \psi^* \left( 0, \frac{s_2}{1 - s_1}, \cdots, \frac{s_{N-1}}{1 - s_1} \right) = \sup_{(t_1, \cdots, t_{N-1}) \in \Delta_N} \frac{(1 - \sum_{i=1}^{N-1} s_i)(1 - \sum_{i=1}^{N-1} t_i) + \sum_{i=2}^{N-1} s_i t_i}{\psi(t_1, \cdots, t_{N-1})} \leq \sup_{(t_1, \cdots, t_{N-1}) \in \Delta_N} \frac{(1 - \sum_{i=1}^{N-1} s_i)(1 - \sum_{i=1}^{N-1} t_i) + \sum_{i=1}^{N-1} s_i t_i}{\psi(t_1, \cdots, t_{N-1})} = \psi^*(s_1, \cdots, s_{N-1}),
\]
or we have \((A_2)\). The rest properties \((A_3)-(A_N)\) are similarly shown. Thus we have \( \psi^* \in \Psi_N \). Next, let \((x_1, \cdots, x_N)\) and \((x_1^*, \cdots, x_N^*)\) be arbitrary nonzero elements in \((X_1 \oplus \cdots \oplus X_N) \psi\) and \((X_1^* \oplus \cdots \oplus X_N^*) \psi^*\), respectively. Let
\[
s_j = \frac{\| x_{j+1} \|}{\sum_{j=1}^{N} \| x_j^* \|} \quad \text{and} \quad t_j = \frac{\| x_{j+1} \|}{\sum_{j=1}^{N} \| x_j \|} \quad \text{for} \quad 1 \leq j \leq N - 1
\]
Then \((s_1, \ldots, s_{N-1}), (t_1, \ldots, t_{N-1}) \in \Delta_N\), and by (8) we have

\[
\psi^*(s_1, \ldots, s_{N-1})\psi(t_1, \ldots, t_{N-1}) \geq \left( 1 - \sum_{i=1}^{N-1} s_i \right) \left( 1 - \sum_{i=1}^{N-1} t_i \right) + \sum_{i=1}^{N-1} s_i t_i
\]

\[= \frac{\|x_1^j\|}{\sum_{j=1}^{N} \|x_j^*\|} \cdot \frac{\|x_1\|}{\sum_{j=1}^{N} \|x_j\|} + \frac{\sum_{j=1}^{N-1} \|x_{j+1}^*\|}{\sum_{j=1}^{N} \|x_j^*\|} \frac{\|x_{j+1}\|}{\sum_{j=1}^{N} \|x_j\|}
\]

\[= \frac{1}{\sum_{j=1}^{N} \|x_j^*\|} \cdot \frac{\sum_{j=1}^{N} \|x_j\|}{\sum_{j=1}^{N} \|x_j\|} \sum_{j=1}^{N} x_j^*(x_j),
\]

from which it follows that \(\|(x_1^*, \ldots, x_N^*)\|_{\psi^*} \cdot \|(x_1, \ldots, x_N)\|_{\psi} \geq \sum_{j=1}^{N} x_j^*(x_j)\), or (9). This completes the proof.

**Theorem 4.2** (cf. [4, 20, 14]) Let \(X_1, \ldots, X_N\) be Banach spaces and \(\psi \in \Psi_N\). Then

\[(X_1 \oplus \cdots \oplus X_N)^\psi = (X_1^* \oplus \cdots \oplus X_N^*)_{\psi^*}.
\]

**Proof** Let \((x_1^*, \ldots, x_N^*) \in (X_1^* \oplus \cdots \oplus X_N^*)_{\psi^*}\). Let

\[f(x_1, \ldots, x_N) = \sum_{j=1}^{N} x_j^*(x_j) \quad \text{for} \quad (x_1, \ldots, x_N) \in (X_1 \oplus \cdots \oplus X_N)_{\psi}
\]

Then by Theorem 4.1 we have \(|f(x_1, \ldots, x_N)| \leq \|(x_1^*, \ldots, x_N^*)\|_{\psi^*} \cdot \|(x_1, \ldots, x_N)\|_{\psi}\), from which it follows that \(f \in (X_1 \oplus \cdots \oplus X_N)^\psi\) and \(|f| \leq \|(x_1^*, \ldots, x_N^*)\|_{\psi^*}\).

Conversely, take an arbitrary \(f \in (X_1 \oplus \cdots \oplus X_N)^\psi\). For each \(1 \leq j \leq N\) let

\[x_j^*(x_j) = f(0, \ldots, 0, x_j, 0, \ldots) \quad \text{for} \quad x_j \in X_j
\]

Then we have \(x_j^* \in X_j^*\) and the formula (11). Next we see \(\|(x_1^*, \ldots, x_N^*)\|_{\psi^*} \leq \|f\|\).

Assume that \((x_1^*, \ldots, x_N^*) \neq (0, \ldots, 0)\). For any \(\varepsilon > 0\) take \(u_j \in S_{X_j}\), the unit sphere of \(X_j\), so that \(\|x_j\| \leq x_j^*(u_j) + \varepsilon\). Let

\[s_j = \frac{\|x_{j+1}^*\|}{\sum_{j=1}^{N} \|x_j^*\|} \quad \text{for} \quad 1 \leq j \leq N - 1
\]

Then, noting that \(\|(x_1^*, \ldots, x_N^*)\|_{\psi^*} = (\sum_{j=1}^{N} \|x_j^*\|)\psi^*(s_1, \ldots, s_{N-1})\), we have for
any \((t_1, \ldots, t_{N-1}) \in \Delta_N\)
\[
\left( \sum_{j=1}^{N} \|x_j^*\| \right) \left( 1 - \sum_{j=1}^{N-1} s_j \right) \left( 1 - \sum_{j=1}^{N-1} t_j \right) + \sum_{j=1}^{N-1} s_j t_j
\]
\[
\psi(t_1, \ldots, t_{N-1})
\]
\[
= \frac{1}{\psi(t_1, \ldots, t_{N-1})} \left[ \|x_1^*\| \left( 1 - \sum_{j=1}^{N-1} t_j \right) + \sum_{j=1}^{N-1} \|x_j^*\| t_j \right]
\]
\[
\leq \frac{1}{\psi(t_1, \ldots, t_{N-1})} \left[ (x_1^*(u_1) + \varepsilon) \left( 1 - \sum_{j=1}^{N-1} t_j \right) + \sum_{j=1}^{N-1} (x_{j+1}^*(u_{j+1}) + \varepsilon) t_j \right]
\]
\[
= \frac{1}{\psi(t_1, \ldots, t_{N-1})} \left[ x_1^* \left( 1 - \sum_{j=1}^{N-1} t_j \right) + \sum_{j=2}^{N} x_j^* \left( \frac{t_{j-1}}{\psi(t_1, \ldots, t_{N-1})} u_j \right) + N \varepsilon \right]
\]
\[
\leq f \left( \frac{1 - \sum_{j=1}^{N-1} t_j}{\psi(t_1, \ldots, t_{N-1})} u_1, \frac{t_1}{\psi(t_1, \ldots, t_{N-1})} u_2, \ldots, \frac{t_{N-1}}{\psi(t_1, \ldots, t_{N-1})} u_N \right) + N \varepsilon
\]
\[
\leq \|f\| + N \varepsilon,
\]
where one should note that
\[
\left\| \left( \frac{1 - \sum_{j=1}^{N-1} t_j}{\psi(t_1, \ldots, t_{N-1})} u_1, \frac{t_1}{\psi(t_1, \ldots, t_{N-1})} u_2, \ldots, \frac{t_{N-1}}{\psi(t_1, \ldots, t_{N-1})} u_N \right) \right\| = \frac{1}{\psi(t_1, \ldots, t_{N-1})} \left\| (1 - \sum_{j=1}^{N-1} t_j, t_1, \ldots, t_{N-1}) \right\| = 1.
\]

Therefore \(\|(x_1^*, \ldots, x_N^*)\|_{\psi^*} < \|f\| + N \varepsilon\). Since \(\varepsilon > 0\) is arbitrary, we have \(\|(x_1^*, \ldots, x_N^*)\|_{\psi^*} < \|f\|\), which completes the proof.

By Theorem 4.2 we have the next result.

**Proposition 4.3** Let \(X_1, \ldots, X_N\) be Banach spaces and \(\psi \in \Psi_N\). Then the following are equivalent.

(i) \(\{(x_1^{(k)}, \ldots, x_N^{(k)})\}_k\) is a weakly null sequence in \((X_1 \ominus \cdots \ominus X_N)_\psi\).

(ii) \(\{x_j^{(k)}\}_k\) is a weakly null sequence in \(X_j\) for each \(1 \leq j \leq N\).

**Proof** (i) \(\Rightarrow\) (ii). Let \(\{(x_1^{(k)}, \ldots, x_N^{(k)})\}_k\) tend weakly to 0 in \((X_1 \ominus \cdots \ominus X_N)_\psi\). For each \(1 \leq j \leq N\) take arbitrary \(x_j^* \in X_j^*\). Since \((0, \ldots, x_j^*, 0, \ldots, 0) \in (X_1 \ominus \cdots \ominus X_N)_\psi^*\),
we have
\[ \lim_{k \to \infty} x^*_j(x^{(k)}) = \lim_{k \to \infty} \langle (0, \ldots, x^*_j, 0, \ldots, 0), (x^{(k)}_1, \ldots, x^{(k)}_N) \rangle = 0 \]

(ii) \( \Rightarrow \) (i). Let \( \{x^{(k)}_1\}, \ldots, \{x^{(k)}_N\} \) be weakly null sequences in \( X_1, \ldots, X_N \) respectively. By Theorem 4.2 for any \( f \in (X_1 \oplus \cdots \oplus X_N)^* \), there exists a unique \( (x^*_1, \ldots, x^*_N) \in (X^*_1 \oplus \cdots \oplus X^*_N)^* \) such that
\[ f(x_1, \ldots, x_N) = \sum_{j=1}^N x^*_j(x_j) \quad \text{for all } (x_1, \ldots, x_N) \text{ in } (X_1 \oplus \cdots \oplus X_N)^* \]

Therefore we have \( \lim_{k \to \infty} f(x^{(k)}_1, \ldots, x^{(k)}_N) = 0 \), which completes the proof. 

By Proposition 4.3 we have the following.

Proposition 4.4. Let \( X_1, \ldots, X_N \) be Banach spaces and \( \psi \in \Psi_N \). Then the following are equivalent.

(i) \( (X_1 \oplus \cdots \oplus X_N)^\psi \) has the Schur property.

(ii) All \( X_1, \ldots, X_N \) have the Schur property.

5. A class of convex functions \( \Psi_{(1)}^N \). We shall introduce a subclass \( \Psi_{(1)}^N \) of \( \Psi_N \).

Definition 5.1. Let \( \psi \in \Psi_N \). Let \( S \) be a nonempty subset of \( \{1, \ldots, N-1\} \) and \( \chi_S \) its characteristic function.

(i) We say \( \psi \in \Psi_{(1)}^N(S) \) if there exists an element \( (s_1, \ldots, s_{N-1}) \in \Delta_N \) with
\[ 0 < \sum_{i=1}^{N-1} \chi_S(i)s_i < 1 \] (12)

such that, letting \( M = \sum_{i=1}^{N-1} \chi_S(i)s_i \),
\[ \psi(s_1, \ldots, s_{N-1}) = M\psi\left( \frac{\chi_S(1)s_1}{M}, \ldots, \frac{\chi_S(N-1)s_{N-1}}{M} \right) + (1 - M)\psi\left( \frac{\chi_S(1)s_1}{1 - M}, \ldots, \frac{\chi_S(N-1)s_{N-1}}{1 - M} \right) \] (13)

(ii) We say \( \psi \in \Psi_{(1)}^N \) when \( \psi \in \Psi_{(1)}^N(S) \) for some nonempty subset \( S \) of \( \{1, \ldots, N-1\} \).
According to (1) and (3) the equation (14) is interpreted as
\[
\| (1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1} ) \|_\psi \leq \|(0, \chi_S(1)s_1, \ldots, \chi_S(N-1)s_{N-1})\|_\psi
\]
\[
+ \|(1 - \sum_{i=1}^{N-1} s_i, \chi_S(1)s_1, \ldots, \chi_S(N-1)s_{N-1})\|_\psi
\]
This indicates that the norm \( \| \cdot \|_\psi \) for \( \psi \in \Psi^{(1)}_N \) has an \( \ell_1 \)-norm like property, which will be reformulated in Theorems 5.5 and 5.8 below.

Example 5.2 We see that \( \psi_1 \in \Psi^{(1)}_N \). In the case \( N = 2 \) the class \( \Psi^{(1)}_2 \) consists of only the function \( \psi_1 \). Indeed, according to the formula (15), \( \psi \in \Psi^{(1)}_2 \) if and only if there exists \( s \in (0, 1) \) such that \( \psi(s) = \| (1 - s, s) \|_\psi = \| (0, s) \|_\psi + \| (1 - s, 0) \|_\psi = 1 \), which is possible only when \( \psi = \psi_1 \). If \( N \geq 3 \), let \( S = \{ 1 \} \) and \( (s_1, \ldots, s_{N-1}) = (1/N, \ldots, 1/N) \). Then
\[
\| (1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1} ) \|_{\psi_1} = \| (1/N, 1/N, \ldots, 1/N) \|_1
\]
\[
= \| (0, 1/N, 0, \ldots, 0) \|_1 + \| (1/N, 0, 1/N, \ldots, 1/N) \|_1
\]
\[
= \| (0, \chi_S(1)s_1, \ldots, \chi_S(N-1)s_{N-1})\|_{\psi_1}
\]
\[
+ \| (1 - \sum_{i=1}^{N-1} s_i, \chi_S(1)s_1, \ldots, \chi_S(N-1)s_{N-1})\|_{\psi_1},
\]
which implies \( \psi_1 \in \Psi^{(1)}_N \).

Example 5.3 Let \( \alpha + \beta = 1, \ \alpha, \beta > 0 \) and let
\[
\psi(s_1, s_2) = \max \{ 1 - \alpha s_1, 1 - \alpha s_2, \alpha s_1 + \beta, \alpha s_2 + \beta \} \quad \text{for } (s_1, s_2) \in \Delta_3
\]
Then \( \psi \in \Psi^{(1)}_3 \). Indeed it is obvious that \( \psi \) is convex on \( \Delta_3 \) and \( \psi(0, 0) = \psi(1, 0) = \psi(0, 1) = 1 \). Also we have
\[
(s_1 + s_2) \psi \left( \frac{s_1}{s_1 + s_2}, \frac{s_2}{s_1 + s_2} \right)
\]
\[
= \max \left\{ (s_1 + s_2) - \alpha s_1, (s_1 + s_2) - \alpha s_2, \alpha s_1 + \beta (s_1 + s_2), \alpha s_2 + \beta (s_1 + s_2) \right\}
\]
\[
\leq \psi(s_1, s_2),
\]
\[
(1 - s_1) \psi \left( 0, \frac{s_2}{1 - s_1} \right) = \max \left\{ 1 - s_1, (1 - s_1) - \alpha s_2, \beta (1 - s_1), \alpha s_2 + \beta (1 - s_1) \right\}
\]
\[
\leq \psi(s_1, s_2).
\]
and in the same way, \((1 - s_2)\psi(s_1/(1 - s_2), 0) \leq \psi(s_1, s_2)\). Thus we have \(\psi \in \Psi_3\).

Let next \((s_1, s_2) = (\alpha/2, \beta/2)\) and \(S = \{1, 2\}\). Then, letting \(M = \chi_S(1)s_1 + \chi_S(2)s_2\), we have

\[
M\psi\left(\frac{\chi_S(1)s_1}{M}, \frac{\chi_S(2)s_2}{M}\right) + (1 - M)\psi\left(\frac{\chi_S(1)s_1}{1 - M}, \frac{\chi_S(2)s_2}{1 - M}\right)
\]

\[
= \frac{1}{2} \psi(\alpha, \beta) + \frac{1}{2} \psi(0, 0) = \max\left\{ \frac{1 - \alpha^2}{2}, \frac{1 - \alpha\beta}{2}, \frac{\alpha^2 + \beta}{2}, \frac{\alpha\beta + \beta}{2} \right\} + \frac{1}{2}
\]

\[
= \max\left\{ 1 - \alpha \cdot \frac{\alpha}{2}, 1 - \alpha \cdot \frac{\beta}{2} \right\} = \psi(s_1, s_2),
\]

where one should note that \(\alpha s_1 + \beta \leq (\alpha^2 + \beta + 1)/2 = 1 - \alpha/2\) and \(\alpha s_2 + \beta \leq (\alpha\beta + \beta + 1)/2 = 1 - \alpha^2/2\). Therefore we have \(\psi \in \Psi_3^{(1)}\).

The above function \(\psi\) yields the following norm on \(C^3\):

\[
\|(z_1, z_2, z_3)\|_\psi = \max\{|z_1| + \beta|z_2| + |z_3|, |z_1| + |z_2| + \beta|z_3|, \beta|z_1| + |z_2| + \beta|z_3|, \beta|z_1| + |z_2| + |z_3|\}
\]

The next lemma is useful in Theorem 5.5.

**Lemma 5.4 ([12])** For nonzero elements \(x\) and \(y\) in a Banach space \(X\) the following are equivalent.

(i) \(\|x + y\| = \|x\| + \|y\|\)

(ii) \(\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = 2\)

For a nonempty subset \(S\) of \(\{1, \ldots, N - 1\}\) let \(S + 1 = \{i + 1 : i \in S\}\).

**Theorem 5.5** Let \(\psi \in \Psi_N\) and let \(S\) be a nonempty subset of \(\{1, \ldots, N - 1\}\). Then the following are equivalent.

(i) \(\psi \in \Psi_N^{(1)}(S)\)

(ii) There exists \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) (that is, \(a_1, \ldots, a_n \geq 0\)) such that

\[
\|(a_1, \ldots, a_N)\|_\psi = \|(0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\|_\psi
\]

\[
= \|(a_1, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\|_\psi,
\]

where \((0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\) and \((a_1, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\) are nonzero.

(iii) There exists \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) such that the formula (16) holds true with

\[
\|(0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\|_\psi = \|(a_1, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\|_\psi = 1
\]

(iv) There exists \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) such that for all \(\alpha, \beta \in \mathbb{R}\)

\[
\|\alpha(0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N) + \beta(a_1, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\|_\psi
\]

\[
= |\alpha|\|(0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\|_\psi + |\beta|\|(a_1, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\|_\psi.
\]
where \((0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\) and \((a_1, \chi_{(S+1)^c}(2)a_2, \ldots, \chi_{(S+1)^c}(N)a_N)\) are nonzero.

(v) There exists \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) such that for all \(\alpha, \beta \in \mathbb{R}\) the formula (18) holds true with (17).

**Proof** Owing to Lemma 5.4 the assertions (ii) and (iii) are equivalent, and so are (iv) and (v). By (15) we have the implication (i) \(\Rightarrow\) (ii), and the implication (iv) \(\Rightarrow\) (ii) is clear. Therefore it is sufficient to show (ii) \(\Rightarrow\) (iv). Let \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) satisfy (16) with nonzero \((0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\) and \((a_1, \chi_{(S+1)^c}(2)a_2, \ldots, \chi_{(S+1)^c}(N)a_N)\) are nonzero. Let \(M = \sum_{j=1}^N a_j\) and let

\[
\begin{align*}
s_i &= \frac{a_{i+1}}{M} & \text{for } 1 \leq i \leq N - 1
\end{align*}
\]

We first see that

\[
0 < \sum_{i=1}^{N-1} \chi_S(i)s_i < 1
\]

Noting that

\[
\sum_{i=1}^{N-1} \chi_S(i)s_i = \frac{1}{M} \sum_{i=1}^{N-1} \chi_S(i)a_{i+1} = \frac{1}{M} \sum_{i=1}^{N-1} \chi_{S+1}(i + 1)a_{i+1} = \frac{1}{M} \sum_{i=2}^N \chi_{S+1}(i)a_i,
\]

we have \(\sum_{i=1}^{N-1} \chi_S(i)s_i > 0\). As \((a_1, \chi_{(S+1)^c}(2)a_2, \ldots, \chi_{(S+1)^c}(N)a_N)\) is nonzero, we have \(a_1 \neq 0\) or \(\chi_{(S+1)^c}(i)a_i \neq 0\) for some \(2 \leq i \leq N\). This implies that \(\sum_{i=2}^N \chi_{S+1}(i)a_i < M\), and hence \(\sum_{i=1}^{N-1} \chi_S(i)s_i < 1\). Next, by (16) we have

\[
\begin{align*}
\|1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1}\| &= \frac{1}{M} \|(a_1, a_2, \ldots, a_N)\| \\
&= \frac{1}{M} \|(0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\| \\
&= \frac{1}{M} \||(a_1, \chi_{(S+1)^c}(2)a_2, \ldots, \chi_{(S+1)^c}(N)a_N)\| \\
&= \|(0, \chi_{S}(1)s_1, \ldots, \chi_{S}(N-1)s_{N-1})\| \\
&= \|(1 - \sum_{i=1}^{N} s_i, \chi_{S}(1)s_1, \ldots, \chi_{S}(N-1)s_{N-1})\|,
\end{align*}
\]

or the formula (15). Thus \(\psi \in \Psi_{N+1}^\alpha(S)\).

(ii) \(\Rightarrow\) (iv). Let \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) satisfy (16) with nonzero \((0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\) and \((a_1, \chi_{(S+1)^c}(2)a_2, \ldots, \chi_{(S+1)^c}(N)a_N)\).
Then for any $\alpha, \beta$ with $|\alpha| \geq |\beta|$

$$
\|\alpha(0, x_{S+1}(2)a_2, \ldots, x_{S+1}(N)a_N) + \beta(\chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N)\|_{\psi} =
$$

$$
\|[(\alpha a_1, \chi_{S+1}(2)a_2 + \beta(x_{S+1}(2)a_2, \ldots, \alpha\chi_{S+1}(N)a_N + \beta\chi_{S+1}(N)a_N)\|_{\psi} =
$$

$$
\|\alpha(0, x_{S+1}(2)a_2, \ldots, x_{S+1}(N)a_N) + \beta(0, x_{S+1}(2)a_2, \ldots, x_{S+1}(N)a_N)\|_{\psi}
$$

which implies (18). This completes the proof.

The classes $\Psi^1_N(S)$ with $S = \{i\}$, $1 \leq i \leq N - 1$ and $S = \{1, \ldots, N - 1\}$ will play an important role in our later discussion. By definition with the formula (15) and Theorem 5.5 we have the following.

**Corollary 5.6** (a) Let $1 \leq i \leq N - 1$. Then the following are equivalent.

(i) $\psi \in \Psi^1_N(\{i\})$

(ii) There exists $(s_1, \ldots, s_{N-1}) \in \Delta_N$ with $0 < s_i < 1$ such that

$$
\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1})\|_{\psi} =
$$

$$
\|((0, \ldots, 0, s_i, 0, \ldots, 0))\|_{\psi} + \|(1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_{N-1})\|_{\psi}
$$

(iii) There exists $(a_1, \ldots, a_N) \in \mathbb{R}^N_+$ such that

$$
\|(a_1, \ldots, a_N)\|_{\psi} =
$$

$$
\max_{i+1 \leq i} \|a_{i+1}\|_{\psi} + \|(a_1, \ldots, a_i, 0, a_{i+2}, \ldots, a_N)\|_{\psi},
$$

where $a_{i+1} \neq 0$ and $(a_1, \ldots, a_i, 0, a_{i+2}, \ldots, a_N)$ is nonzero.

(iv) There exists $(a_1, \ldots, a_N) \in \mathbb{R}^N_+$ such that

$$
\|(a_1, \ldots, a_N)\|_{\psi} =
$$

$$
\max_{i+1 \leq i} \|a_{i+1}\|_{\psi} + \|(a_1, \ldots, a_i, 0, a_{i+2}, \ldots, a_N)\|_{\psi},
$$

where $a_{i+1} = \|(a_1, \ldots, a_i, 0, a_{i+2}, \ldots, a_N)\|_{\psi} = 1$.

(b) The following are equivalent.

(i) $\psi \in \Psi^1_N(\{1, \ldots, N - 1\})$
(ii) There exists \( (s_1, \ldots, s_{N-1}) \in \Delta_N \) with \( 0 < \sum_{i=1}^{N-1} s_i < 1 \) such that
\[
\| (1 - \sum_{i=1}^{N-1} s_i, \ldots, 0) \|_\psi = \| (0, s_1, 0, \ldots, 0) \|_\psi + \| (0, s_1, \ldots, s_{N-1}) \|_\psi
\]

(iii) There exists \( (a_1, \ldots, a_N) \in \mathbb{R}^N_+ \) such that
\[
\| (a_1, \ldots, a_N) \|_\psi = \| (a_1, 0, \ldots, 0) \|_\psi + \| (0, a_2, \ldots, a_N) \|_\psi,
\]
where \( a_1 \neq 0 \) and \( (0, a_2, \ldots, a_N) \) is nonzero.

(iv) There exists \( (a_1, \ldots, a_N) \in \mathbb{R}^N_+ \) such that
\[
\| (a_1, \ldots, a_N) \|_\psi = \| (a_1, 0, \ldots, 0) \|_\psi + \| (0, a_2, \ldots, a_N) \|_\psi,
\]
where \( a_1 = \| (0, a_2, \ldots, a_N) \|_\psi = 1 \).

Example 5.7 In Example 5.2 we saw that \( \psi_1 \in \Psi_N^{(1)} \), in fact \( \psi_1 \in \Psi_N^{(1)}(\{1\}) \). In the same way we have
\[
\psi_1 \in \Psi_N^{(1)}(\{1, \ldots, N-1\}) \cap \Psi_N^{(1)}(\{1\}) \cap \cdots \cap \Psi_N^{(1)}(\{N-1\})
\]
If \( N = 2, \Psi_2^{(1)}(\{1\}) = \Psi_2^{(1)}(\{1\}) = \{\psi_1\} \). We shall see that if \( N \geq 3 \)
\[
\{\psi_1\} \subseteq \Psi_N^{(1)}(\{1, \ldots, N-1\}) \cap \Psi_N^{(1)}(\{1\}) \cap \cdots \cap \Psi_N^{(1)}(\{N-1\})
\]
(19)

Let us construct a convex function \( \psi \) in \( \Psi_N^{(1)}(\{1, \ldots, N-1\}) \cap (\cap_{i=1}^{N-1} \Psi_N^{(1)}(\{i\})) \) with \( \psi \neq \psi_1 \). Indeed, in the case that \( N \) is an odd integer, let
\[
\psi(s_1, \ldots, s_{N-1}) = \left( 1 - \sum_{i=1}^{N-1} s_i \right) + \sqrt{s_1^2 + s_2^2} + \sqrt{s_3^2 + s_4^2} + \cdots + \sqrt{s_{N-2}^2 + s_{N-1}^2}
\]
for \( (s_1, \ldots, s_{N-1}) \in \Delta_N \)

(20)

It is obvious that \( \psi \) is convex on \( \Delta_N \) and satisfies (A0). If \( 0 < \sum_{i=1}^{N-1} s_i \leq 1 \), we have
\[
\psi(s_1, \ldots, s_{N-1}) \geq \sqrt{s_1^2 + s_2^2} + \sqrt{s_3^2 + s_4^2} + \cdots + \sqrt{s_{N-2}^2 + s_{N-1}^2}
\]
\[
= \left( \sum_{i=1}^{N-1} s_i \right)^\frac{1}{\left( \frac{s_1}{\sum_{i=1}^{N-1} s_i}, \ldots, \frac{s_{N-1}}{\sum_{i=1}^{N-1} s_i} \right)}
\]
for \( (s_1, \ldots, s_{N-1}) \in \Delta_N \)

or (A1). When \( 0 \leq s_1 < 1 \),
\[
\psi(s_1, \ldots, s_{N-1}) \geq \left( 1 - \sum_{i=1}^{N-1} s_i \right) + \sqrt{s_1^2 + s_2^2} + \sqrt{s_3^2 + s_4^2} + \cdots + \sqrt{s_{N-2}^2 + s_{N-1}^2}
\]
\[
= (1 - s_1) \psi\left( 0, \frac{s_2}{1 - s_1}, \ldots, \frac{s_{N-1}}{1 - s_1} \right),
\]
or we have \( (A_2) \). The properties \((A_3) - (A_N)\) are shown in the same way and we have \( \psi \in \Psi_N \). By (3) the corresponding norm \( \| \cdot \|_\psi \) is given by the formula
\[
\|(a_1, a_2, \ldots, a_N)\|_\psi = |a_1| + \|(a_2, a_3)\|_2 + \cdots + \|(a_{N-1}, a_N)\|_2
\]  
(21)

Let \((a_1, a_2, \ldots, a_N) = (1, 0, 1, 0, \ldots, 1, 0, 1)\). Since
\[
\|(a_1, \ldots, a_N)\|_\psi = \|(1, 0, 1, 0, \ldots, 1, 0, 1)\|_\psi = \|(1, 0, 1, 0, \ldots, 1, 0, 1)\|_1
= \|(1, 0, \ldots, 0)\|_1 + \|(0, 0, 1, 0, \ldots, 1, 0, 1)\|_1
= \|(1, 0, \ldots, 0)\|_\psi + \|(0, 0, 1, 0, \ldots, 1, 0, 1)\|_\psi,
\]
we have \( \psi \in \Psi_N^{(1)}(\{1, \ldots, N - 1\}) \) by Corollary 5.6(b). In the same way \( \psi \in \Psi_N^{(1)}(\{i\}), i = 2, 4, \ldots, N - 1 \). Next let \((a_1, a_2, \ldots, a_N) = (0, 1, 0, \ldots, 0, 1, 0)\). Since
\[
\|(a_1, \ldots, a_N)\|_\psi = \|(0, 1, 0, 1, \ldots, 0, 1, 0)\|_\psi = \|(0, 1, 0, 1, \ldots, 0, 1, 0)\|_1
= \|(0, 1, 0, 0, \ldots, 0)\|_1 + \|(0, 0, 0, 1, \ldots, 0, 1, 0)\|_1
= \|(0, 1, 0, 0, \ldots, 0)\|_\psi + \|(0, 0, 0, 1, \ldots, 0, 1, 0)\|_\psi,
\]
we have \( \psi \in \Psi_N^{(1)}(\{1\}) \) by Corollary 5.6(a). In the same way \( \psi \in \Psi_N^{(1)}(\{i\}), i = 3, 5, \ldots, N - 2 \). Therefore we obtain
\[
\psi \in \Psi_N^{(1)}(\{1, \ldots, N - 1\}) \cap (\cap_{i=1}^{N-1} \Psi_N^{(1)}(\{i\}))
\]  
(22)

In the case that \( N \) is an even integer, let
\[
\psi(s_1, \ldots, s_{N-1}) = \sqrt{\left(1 - \sum_{i=1}^{N-1} s_i^2\right)^2 + s_2^2 + s_3^2 + \cdots + s_{N-2}^2 + s_{N-1}^2}
\]
for \((s_1, \ldots, s_{N-1}) \in \Delta_N\)
(23)

Then we have \( \psi \in \Psi_N \) and
\[
\|(a_1, a_2, \ldots, a_N)\|_\psi = \|(a_1, a_2)\|_2 + \|(a_3, a_4)\|_2 + \cdots + \|(a_{N-1}, a_N)\|_2
\]  
(24)

By the same discussion as the foregoing case we have the formula (22).

Now, from Theorem 5.5 we have the following characterizations of the class \( \Psi_N^{(1)} \).

**Theorem 5.8** Let \( \psi \in \Psi_N \). Then the following are equivalent.
\( \psi \in \Psi_N^{(1)} \)
\( \psi \in \Psi_N^{(1)} \)
(25)
where \((\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\) and \((\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\) are nonzero.

(iii) There exists \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) such that with some nonempty proper subset \(T\) of \(\{1, \ldots, N\}\) the formula (26) holds true with
\[
\| (\chi_T(1)a_1, \ldots, \chi_T(N)a_N) \|_\psi = \| (\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N) \|_\psi = 1 \tag{26}
\]

(iv) There exists \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) such that with some nonempty proper subset \(T\) of \(\{1, \ldots, N\}\)
\[
\| \alpha(\chi_T(1)a_1, \ldots, \chi_T(N)a_N) + \beta(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N) \|_\psi = \| \alpha \| (\chi_T(1)a_1, \ldots, \chi_T(N)a_N) \|_\psi + |\beta| \| (\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N) \|_\psi
\]
for all \(\alpha, \beta \in \mathbb{R}\), where \((\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\) and \((\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\) are nonzero.

(v) There exists \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) such that with some nonempty proper subset \(T\) of \(\{1, \ldots, N\}\), the formula (28) holds for all \(\alpha, \beta \in \mathbb{R}\) with (26).

**Proposition 5.9** Let \(\psi \in \Psi_N\) be strictly convex. Then \(\psi \not\in \Psi_N^{(1)}\).

**Proof** Take an arbitrary \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) and assume that
\[
\| (\chi_T(1)a_1, \ldots, \chi_T(N)a_N) \|_\psi = \| (\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N) \|_\psi = 1
\]
for some nonempty proper subset \(T\) of \(\{1, \ldots, N\}\). Then
\[
\| (a_1, \ldots, a_N) \|_\psi < \| (\chi_T(1)a_1, \ldots, \chi_T(N)a_N) \|_\psi + \| (\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N) \|_\psi
\]
as the norm \(\| \cdot \|_\psi\) is strictly convex. (Note that \(\| \cdot \|_\psi\) is strictly convex if and only if \(\psi\) is strictly convex ([26, Theorem 4.2]). Hence we have \(\psi \not\in \Psi_N^{(1)}\) by Theorem 5.8.

**Example 5.10** Let \(1 < p \leq \infty\) and \(\psi_p\) the corresponding convex function to the \(\ell_p\)-norm. Then \(\psi_p \not\in \Psi_N^{(1)}\). Indeed if \(1 < p < \infty\), the \(\ell_p\)-norm and hence \(\psi_p\) is strictly convex. Therefore we have \(\psi_p \not\in \Psi_N^{(1)}\). Let \(p = \infty\). Take any \((a_1, \ldots, a_N) \in \mathbb{R}_+^N\) and assume that \((\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\) and \((\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\) are nonzero for some nonempty proper subset \(T\) of \(\{1, \ldots, N\}\). Then, since \(\|(a_1, \ldots, a_N)\|_\infty = \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\infty \) or \(\|(a_1, \ldots, a_N)\|_\infty = \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\infty\), we have
\[
\|(a_1, \ldots, a_N)\|_\infty < \|(\chi_T(1)a_1, \ldots, \chi_T(N)a_N)\|_\infty + \|(\chi_{T^c}(1)a_1, \ldots, \chi_{T^c}(N)a_N)\|_\infty
\]
which implies that \(\psi_\infty \not\in \Psi_N^{(1)}\) by Theorem 5.8.

6. **Weak nearly uniform smoothness.** We need the following lemma.
Lemma 6.1 ([14]) Let \( \{x^{(k,n)}\}_{k,n} \) and \( \{y^{(k,n)}\}_{k,n} \) be nonzero bounded double sequences in a Banach space \( X \) with \( \lim_{k \to \infty} \|x^{(k,n)}\| > 0 \) and \( \lim_{k \to \infty} \|y^{(k,n)}\| > 0 \) for each \( n \). Then the following are equivalent.

(i) \( \lim_{n \to \infty} \liminf_{k \to \infty} \|x^{(k,n)} + y^{(k,n)}\| = \lim_{n \to \infty} \liminf_{k \to \infty} \left\{ \|x^{(k,n)}\| + \|y^{(k,n)}\| \right\} \).

(ii) \( \lim_{n \to \infty} \liminf_{k \to \infty} \frac{\|x^{(k,n)}\| + \|y^{(k,n)}\|}{\|x^{(k,n)}\|} = 2. \)

The next theorem is the heart of this paper.

Theorem 6.2 Let \( X_1, \ldots, X_N \) be Banach spaces and let \( \psi \in \Psi_N \). Assume that some \( X_j \) does not have the Schur property. Let \( U \) be the set of all indices \( j, 1 \leq j \leq N \), for which \( X_j \) does not have the Schur property. Then the following are equivalent.

(i) \( R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2. \)

(ii) \( R(X_j) < 2 \) for all \( 1 \leq j \leq N \) and \( \psi \not\in \Psi_N^{(1)}(S) \) for all nonempty subsets \( S \) of \( \{1, \ldots, N-1\} \) with \( S + 1 \subset U \) or \((S + 1)^c \subset U\).

Proof (i) \( \Rightarrow \) (ii). Let \( R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2. \) As \( X_j \) is regarded as a subspace of \((X_1 \oplus \cdots \oplus X_N)_\psi\), we have \( R(X_j) < 2 \) for all \( 1 \leq j \leq N \) by Proposition 4.3. Suppose that \( \psi \in \Psi_N^{(1)}(S) \) for some nonempty subset \( S \) of \( \{1, \ldots, N-1\} \) with \( S + 1 \subset U \) or \((S + 1)^c \subset U\). Then by Theorem 5.5 (iii) there exists \( (a_1, \ldots, a_N) \in \mathbb{R}_+^N \) such that

\[
\|(a_1, \ldots, a_N)\|_{\psi} = \|0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N\|_{\psi} + \|(a_1, \chi_{(S+1)^c}(2)a_2, \ldots, \chi_{(S+1)^c}(N)a_N)\|_{\psi},
\]

where

\[
\|0, \chi_{S+1}(2)a_2, \ldots, \chi_{S+1}(N)a_N\|_{\psi} = \|(a_1, \chi_{(S+1)^c}(2)a_2, \ldots, \chi_{(S+1)^c}(N)a_N)\|_{\psi} = 1
\]

Let \( S + 1 \subset U \). Without loss of generality we may assume that \((S + 1)^c = \{1, \ldots, r\}\) with some \( 1 \leq r < N \). Then

\[
\|(a_1, \ldots, a_N)\|_{\psi} = \|(a_1, \ldots, a_r, 0, \ldots, 0)\|_{\psi} + \|(0, \ldots, 0, a_{r+1}, \ldots, a_N)\|_{\psi} \quad (28)
\]

and

\[
\|(a_1, \ldots, a_r, 0, \ldots, 0)\|_{\psi} = \|(0, \ldots, 0, a_{r+1}, \ldots, a_N)\|_{\psi} = 1 \quad (29)
\]

Since \( X_j \) does not have the Schur property for each \( j, r + 1 \leq j \leq N \), there exists a weakly null sequence \( \{x^{(j,k)}\}_k \) in the unit sphere of \( X_j \). Take arbitrary \( x_1, \ldots, x_r \) from the unit spheres of \( X_1, \ldots, X_r \), respectively. Let

\[
u = (a_1x_1, \ldots, a_rx_r, 0, \ldots, 0) \in (X_1 \oplus \cdots \oplus X_N)_{\psi}
\]

and

\[
u^{(k)} = (0, \ldots, 0, a_{r+1}x^{(k)}_{r+1}, \ldots, a_Nx^{(k)}_N) \in (X_1 \oplus \cdots \oplus X_N)_{\psi} \quad (k = 1, 2, \ldots)
\]
Then \( \|u(k)\|_\psi = \|u\|_\psi = 1 \) by (29) and \( \{u(k)\}_k \) is a weakly null sequence in \((X_1 \oplus \cdots \oplus X_N)_\psi\) by Proposition 4.3. Consequently we have

\[
R((X_1 \oplus \cdots \oplus X_N)_\psi) \geq \liminf_{k \to \infty} \|u(k) + u\|_\psi = \|(a_1, \ldots, a_N)\|_\psi = 2
\]

by (28), which is a contradiction. For the case of \((S + 1)^c \subset U\) a parallel argument works. Thus we have the assertion (ii).

(ii) \(\Rightarrow\) (i). Assume that the assertion (ii) holds. Suppose that \(R((X_1 \oplus \cdots \oplus X_N)_\psi) = 2\). We shall construct a nonempty subset \(S\) of \(\{1, \ldots, N-1\}\) with \(S + 1 \subset U\) or \((S + 1)^c \subset U\) such that \(\psi \in \Psi_N^{(1)}(S)\). For each \(n \in \mathbb{N}\) there exist a weakly null sequence \(\{(x_1^{(k,n)}, \ldots, x_N^{(k,n)})\}_k\) and an element \((x_1^{(n)}, \ldots, x_N^{(n)})\) in the unit ball of \((X_1 \oplus \cdots \oplus X_N)_\psi\) such that

\[
2 - \frac{1}{n} \leq \liminf_{k \to \infty} \||x_1^{(k,n)} + x_1^{(n)}||, \ldots, ||x_N^{(k,n)} + x_N^{(n)}||\|_\psi \leq 2
\]

By choosing subsequences if necessary, we may assume that for all \(1 \leq j \leq N\)

\[
\lim_{n \to \infty} \lim_{k \to \infty} ||x_j^{(k,n)} + x_j^{(n)}|| = \alpha_j, \quad \lim_{n \to \infty} \lim_{k \to \infty} ||x_j^{(k,n)}|| = \mu_j, \quad \lim_{n \to \infty} ||x_j^{(n)}|| = \nu_j
\]

Then

\[
\|(\mu_1, \ldots, \mu_N)\|_\psi = \lim_{n \to \infty} \lim_{k \to \infty} \||x_1^{(k,n)}||, \ldots, ||x_N^{(k,n)}||\|_\psi \leq 1
\]

and

\[
\|(\nu_1, \ldots, \nu_N)\|_\psi \leq 1
\]

By (30)

\[
2 - \frac{1}{n} \leq \liminf_{k \to \infty} \||x_1^{(k,n)} + x_1^{(n)}||, \ldots, ||x_N^{(k,n)} + x_N^{(n)}||\|_\psi
\]

\[
\leq \|\lim_{k \to \infty} ||x_1^{(k,n)}|| + ||x_1^{(n)}||, \ldots, \lim_{k \to \infty} ||x_N^{(k,n)}|| + ||x_N^{(n)}||\|_\psi
\]

\[
\leq \lim_{k \to \infty} \||x_1^{(n)}||, \ldots, ||x_N^{(n)}||\|_\psi + \||x_1^{(n)}||, \ldots, ||x_N^{(n)}||\|_\psi \leq 2
\]

for all \(n \in \mathbb{N}\). Letting \(n \to \infty\), we have

\[
\|(\alpha_1, \ldots, \alpha_N)\|_\psi = \|(\mu_1 + \nu_1, \ldots, \mu_N + \nu_N)\|_\psi
\]

\[
= \|(\mu_1, \ldots, \mu_N)\|_\psi + \|(\nu_1, \ldots, \nu_N)\|_\psi = 2
\]

Let

\[
\hat{T} = \{j : \alpha_j = \mu_j + \nu_j, \; 1 \leq j \leq N\}
\]

Then \(\hat{T} \neq \emptyset\) by Lemma 3.1 (ii). We shall see that

\[
\min\{\mu_j, \nu_j\} = 0 \quad \text{and hence} \quad \alpha_j = \max\{\mu_j, \nu_j\} \quad \text{for all} \; j \in \hat{T}
\]

Indeed, suppose that \(\min\{\mu_{j_0}, \nu_{j_0}\} > 0\) with some \(j_0 \in \hat{T}\). Then

\[
\lim_{n \to \infty} \lim_{k \to \infty} ||x_j^{(k,n)} + x_j^{(n)}|| = \alpha_{j_0} = \mu_{j_0} + \nu_{j_0} = \lim_{n \to \infty} \lim_{k \to \infty} ||x_j^{(k,n)}|| + \lim_{n \to \infty} ||x_j^{(n)}||.
\]
from which it follows that
\[
\lim_{n \to \infty} \lim_{k \to \infty} \left\| \frac{x_{j0}^{(k,n)}}{\|x_{j0}^{(k,n)}\|} + \frac{x_{j0}^{(n)}}{\|x_{j0}^{(n)}\|} \right\| = 2
\]
by Lemma 6.1. Since \(\{x_{j0}^{(k,n)}/\|x_{j0}^{(k,n)}\|\}_k\) is a weakly null sequence in \(X_{j0}\) for every \(n\), we have
\[
R(X_{j0}) \geq \lim_{n \to \infty} \lim_{k \to \infty} \left\| \frac{x_{j0}^{(k,n)}}{\|x_{j0}^{(k,n)}\|} + \frac{x_{j0}^{(n)}}{\|x_{j0}^{(n)}\|} \right\| = 2
\]
This contradicts the assumption that \(R(X_j) < 2\) for all \(1 \leq j \leq N\). Thus we have (34). Next let
\[
\beta_j = \chi_T(j)\alpha_j, \sigma_j = \chi_T(j)\mu_j \text{ and } \tau_j = \chi_T(j)\nu_j \quad (1 \leq j \leq N)
\]
(35)
Then by (34)
\[
\min\{\sigma_j, \tau_j\} = 0 \quad \text{and} \quad \max\{\sigma_j, \tau_j\} = \beta_j \quad \forall 1 \leq j \leq N
\]
(36)
Owing to Proposition 3.2 we have
\[
\|\langle \beta_1, \ldots, \beta_N \rangle\| = \|\langle \alpha_1, \ldots, \alpha_N \rangle\| = 2
\]
(37)
Let
\[
T = \{j \in \tilde{T} : \beta_j = \sigma_j > 0\}
\]
(38)
Then we have
\[
T \subset U
\]
(39)
Indeed, if \(j \in T\), we have \(\lim_{n \to \infty} \lim_{k \to \infty} \|x_{j0}^{(k,n)}\| = \mu_j = \sigma_j > 0\). Then there exists \(n_0\) such that \(\lim_{k \to \infty} \|x_{j0}^{(k,n_0)}\| > 0\). Since \(\{x_{j0}^{(k,n_0)}\}_k\) is a weakly null sequence in \(X_j\), \(X_j\) does not have the Schur property, that is, \(j \in U\). Further \(T\) is a nonempty proper subset of \(\{1, \ldots, N\}\). Indeed, suppose that \(T = \{1, \ldots, N\}\). Then we have
\[
2 = \|\langle \beta_1, \ldots, \beta_N \rangle\|_\psi = \|\langle \sigma_1, \ldots, \sigma_N \rangle\|_\psi = \|\langle \mu_1, \ldots, \mu_N \rangle\|_\psi \leq 1,
\]
which is a contradiction. Suppose that \(T\) is empty, or \(T^c = \{1, \ldots, N\}\). Then we have
\[
\beta_j = 0 \quad \text{or} \quad \beta_j = \tau_j > 0 \quad \forall 1 \leq j \leq N.
\]
(40)
In fact, if \(\beta_j > 0\), then \(\beta_j > \sigma_j\), whence we have \(\beta_j = \tau_j\) by (36). Therefore
\[
2 = \|\langle \beta_1, \ldots, \beta_N \rangle\|_\psi \leq \|\langle \tau_1, \ldots, \tau_N \rangle\|_\psi \leq \|\langle \nu_1, \ldots, \nu_N \rangle\|_\psi \leq 1,
\]
which is a contradiction.
Now we shall define the desired subset \(S\).
\textbf{Case 1:} \(T \neq 1\). Let \(S = T - 1 = \{i - 1 : i \in T\}\). Then \(S + 1 = T \subset U\) by (39). Let us see that \(\psi \in \Psi^1_N(S)\). Without loss of generality we may assume that
$T = \{s + 1, \ldots, N\}$, that is, $S = \{s, \ldots, N - 1\}$ for some $1 \leq s < N$. Since (40) is valid for all $j \in T^c$, we have

$$
2 = \|\{\beta_1, \ldots, \beta_N\}\|_\psi
\leq \|\{\beta_1, \ldots, \beta_s, 0, \ldots, 0\}\|_\psi + \|\{0, \ldots, 0, \beta_{s+1}, \ldots, \beta_N\}\|_\psi
\leq \|(\tau_1, \ldots, \tau_s, 0, \ldots, 0)\|_\psi + \|\{0, \ldots, 0, \sigma_{s+1}, \ldots, \sigma_N\}\|_\psi
\leq \|(\nu_1, \ldots, \nu_s, 0, \ldots, 0)\|_\psi + \|\{0, \ldots, 0, \mu_{s+1}, \ldots, \mu_N\}\|_\psi < 2,
$$

from which it follows that

$$
\|(\beta_1, \ldots, \beta_N)\|_\psi = \|(\beta_1, \ldots, \beta_s, 0, \ldots, 0)\|_\psi + \|\{0, \ldots, 0, \beta_{s+1}, \ldots, \beta_N\}\|_\psi \leq (41)
$$

and

$$
\|(\beta_1, \ldots, \beta_s, 0, \ldots, 0)\|_\psi = \|(0, \ldots, 0, \beta_{s+1}, \ldots, \beta_N)\|_\psi = 1 \quad (42)
$$

Consequently we obtain that $\psi \in \Psi^1(S)$ by Theorem 5.5, which is a contradiction.

**Case 2:** $T \ni 1$. Let $S = \{i - 1 : i \in T\}$. Then $(S + 1)^c = (T^c)^c = T \subset U$. Without loss of generality we may assume that $T^c = \{s + 1, \ldots, N\}$ for some $1 \leq s < N$, that is, $S = \{s, \ldots, N - 1\}$ and $T = \{1, \ldots, s\}$. Since

$$
\|(\beta_1, \ldots, \beta_s, 0, \ldots, 0)\|_\psi = \|(\sigma_1, \ldots, \sigma_s, 0, \ldots, 0)\|_\psi
$$

and

$$
\|(0, \ldots, 0, \beta_{s+1}, \ldots, \beta_N)\|_\psi = \|(0, \ldots, 0, \tau_{s+1}, \ldots, \tau_N)\|_\psi
$$

we have (41) and (42) by the same argument as before, we obtain that $\psi \in \Psi^1(S)$, a contradiction. This completes the proof. \[\blacksquare\]

In the case of $N = 2$ (then $S = \{1\}$) we have

$$
S + 1 \subset U \text{ if and only if } Y \text{ does not have the Schur property, and } (S + 1)^c \subset U \text{ if and only if } X \text{ does not have the Schur property.} \quad (43)
$$

Thus the next result is a direct consequence of Theorem 6.2.

**Corollary 6.3 ([14])** Let $X$ and $Y$ be Banach spaces and let $\psi \in \Psi_2$. Assume that either $X$ or $Y$ does not have the Schur property. Then the following are equivalent.

(i) $R(X \oplus_\psi Y) < 2$.

(ii) $R(X) < 2$, $R(Y) < 2$ and $\psi \neq \psi_1$.

In the case that each of $X_1, \ldots, X_N$ does not have the Schur property we have the following.

**Theorem 6.4** Let $X_1, \ldots, X_N$ be Banach spaces and let $\psi \in \Psi_N$. Assume that all $X_1, \ldots, X_N$ do not have the Schur property. Then the following are equivalent.

(i) $R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2$.

(ii) $R(X_j) < 2$ for all $1 \leq j \leq N$ and $\psi \notin \Psi^1_N$. 

Remark 6.5 The implication (ii) ⇒ (i) in Theorem 6.4 holds true without the assumption of the Schur property on X’s. (See the proof of (ii) ⇒ (i) of Theorem 6.2.)

Since strictly convex functions \( \psi \in \Psi_N \) do not belong to the class \( \Psi_N^{(1)} \) by Proposition 5.9, the following result by Dhompongsa et al. [3] is a direct consequence of Theorem 6.4 and Remark 6.5.

Corollary 6.6 ([3]) Let \( \psi \in \Psi_N \) be strictly convex. Let \( R(X_j) < 2 \) for all \( 1 \leq j \leq N \). Then \( R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2 \).

Our next concern is the case that all \( X_1, \ldots, X_N \) have the Schur property, where we have \( R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2 \) with some \( \psi \in \Psi_N^{(1)} \).

Proposition 6.7 Let \( X_1, \ldots, X_N \) be Banach spaces. Then the following are equivalent.

(i) \( R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2 \) for all \( \psi \in \Psi_N^{(1)} \).

(ii) \( R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2 \) for all \( \psi \in \Psi_N^{(1)}(\{1, \ldots, N\}) \cap (\bigcap_{i=1}^{N-1} \Psi_N^{(1)}(\{i\})) \).

(iii) \( R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2 \) for some \( \psi \in \Psi_N^{(1)}(\{1, \ldots, N\}) \cap (\bigcap_{i=1}^{N-1} \Psi_N^{(1)}(\{i\})) \).

(iv) \( R((X_1 \oplus \cdots \oplus X_N)_1) < 2 \).

(v) All \( X_1, \ldots, X_N \) have the Schur property.

Proof By Proposition 4.4, if all \( X_1, \ldots, X_N \) have the Schur property, then \( (X_1 \oplus \cdots \oplus X_N)_\psi \) does, and hence we have \( R((X_1 \oplus \cdots \oplus X_N)_\psi) = 1 \) for all \( \psi \in \Psi_N \). (Recall that \( R(X) = 1 \) if \( X \) has the Schur property.) Therefore the implications (v) ⇒ (i) ⇒ (ii) ⇒ (iv) ⇒ (iii) are valid. We show the implication (iii) ⇒ (v). Let \( R((X_1 \oplus \cdots \oplus X_N)_\psi) < 2 \) with some \( \psi \in \Psi_N^{(1)}(\{1, \ldots, N\}) \cap (\bigcap_{i=1}^{N-1} \Psi_N^{(1)}(\{i\})) \). Suppose that some \( X_{j_0} \) does not have the Schur property. Then there exists a weakly null sequence \( \{x_{j_0}^{(k)}\}_k \) in the unit sphere of \( X_{j_0} \). Let first \( j_0 = 1 \). Then, since \( \psi \in \Psi_N^{(1)}(\{1, \ldots, N-1\}) \), there exists \( (a_1, a_2, \ldots, a_N) \in \mathbb{R}_+^N \) with \( a_1 = \|0, a_2, \ldots, a_N\|_\psi = 1 \) such that

\[
\|(a_1, a_2, \ldots, a_N)\|_\psi = \|(a_1, 0, \ldots, 0)\|_\psi + \|(0, a_2, \ldots, a_N)\|_\psi
\]

by Corollary 5.6 (b). Take arbitrary \( x_j \in S_{X_j} \), \( j = 2, \ldots, N \). Let

\[
u^{(k)} = (x_1^{(k)}, 0, \ldots, 0) \quad \text{and} \quad u = (0, a_2 x_2, \ldots, a_N x_N)
\]

Then \( \{u^{(k)}\}_k \) is a weakly null sequence in \( S_{(X_1 \oplus \cdots \oplus X_N)_\psi} \) by Proposition 4.3, and \( u \in S_{(X_1 \oplus \cdots \oplus X_N)_\psi} \). Since

\[
R((X_1 \oplus \cdots \oplus X_N)_\psi) \geq \liminf_{k \to \infty} \|u^{(k)} + u\|_\psi
\]

\[
= \liminf_{k \to \infty} \|(x_1^{(k)}, a_2 x_2, \ldots, a_N x_N)\|_\psi
\]

\[
= \|(1, a_2, \ldots, a_N)\|_\psi
\]

\[
= \|(1, 0, \ldots, 0)\| + \|(0, a_2, \ldots, a_N)\| = 2,
\]

contradicting the given condition.

we have \( R((X_1 \oplus \cdots \oplus X_N)_{\psi}) = 2 \), which is a contradiction. Therefore \( X_1 \) has the Schur property. Let next \( j_0 > 1 \). Then, since \( \psi \in \Psi_N^{(1)}(\{j_0 - 1\}) \), there exists \((a_1, \ldots, a_{j_0-1}, j_0, a_{j_0+1}, \ldots, a_N) \in \mathbb{R}^N_N \) with \( a_{j_0} = \| (a_1, \ldots, a_{j_0-1}, 0, a_{j_0+1}, \ldots, a_N) \|_{\psi} = 1 \) such that

\[
\|(a_1, a_2, \ldots, a_N)\|_{\psi} = \|(0, \ldots, 0, a_{j_0}, 0, \ldots, 0)\|_{\psi} + \|(a_1, \ldots, a_{j_0-1}, 0, a_{j_0+1}, \ldots, a_N)\|_{\psi}
\]

by Corollary 5.6 (a). Then the same discussion as above works to obtain that \( X_{j_0} \) has the Schur property. This completes the proof.

\[\blacksquare\]

\textbf{Remark 6.8} As we shall see in the Example 6.9 below, we cannot add in Proposition 6.7 the statement,

\[ (1') \ R((X_1 \oplus \cdots \oplus X_N)_{\psi}) < 2 \text{ for some } \psi \in \Psi_N^{(1)} \]

In other words the following assertion is not valid: "If \( R((X_1 \oplus \cdots \oplus X_N)_{\psi}) < 2 \) for some \( \psi \in \Psi_N^{(1)} \), all \( X_1, \ldots, X_N \) have the Schur property", which is restated as follows: "For any \( \psi \in \Psi_N^{(1)} \) the condition \( R((X_1 \oplus \cdots \oplus X_N)_{\psi}) < 2 \) implies that all \( X_1, \ldots, X_N \) have the Schur property."

\textbf{Example 6.9} Let \( X_1 \) and \( X_2 \) be Banach spaces with the Schur property and let \( X_3 \) be a Banach space with \( R(X_3) < 2 \) which does not have the Schur property (for example, \( X_3 = \ell_2 \)). Define the convex function \( \psi \in \Psi^3_3 \) by

\[
\psi(s_1, s_2) = \max\{1 - s_1, s_2\} \quad \text{for } (s_1, s_2) \in \Delta_3
\]

Then we have \( \psi \in \Psi^{(1)}_3 \setminus (\Psi^{(1)}_3(\{1, 2\}) \cap \Psi^{(1)}_3(\{1\}) \cap \Psi^{(1)}_3(\{2\})) \) and \( R((X_1 \oplus X_2 \oplus X_3)_{\psi}) < 2 \). Indeed, the corresponding norm to \( \psi \) is given by

\[
\|(a_1, a_2, a_3)\|_{\psi} = \max\{|a_1| + |a_2|, |a_3|\} \quad \text{for } (a_1, a_2, a_3) \in \mathbb{C}^3
\]

Since \( \|(1, 1, 0)\|_{\psi} = 2 = \|(1, 0, 0)\|_{\psi} + \|(0, 1, 0)\|_{\psi} \), we have \( \psi \in \Psi^{(1)}_3(\{1, 2\}) \cap \Psi^{(1)}_3(\{1\}) \) by Corollary 5.6. Suppose that \( \psi \in \Psi^{(1)}_3(\{2\}) \). Then by Corollary 5.6 again there exists \((a_1, a_2, a_3) \in \mathbb{R}^N_N \) with \( a_1 + a_2 = a_3 = 1 \) such that

\[
\|(a_1, a_2, a_3)\|_{\psi} = \|(a_1, a_2, 0)\|_{\psi} + \|(0, 0, a_3)\|_{\psi}
\]

Hence we have \( 2 = \|(a_1, a_2, a_3)\|_{\psi} = \max\{a_1 + a_2, a_3\} = 1 \), a contradiction. Thus we have \( \psi \not\in \Psi^{(1)}_3(\{2\}) \). Next, take a weakly null sequence \( \{(x_{k}^{(1)}, x_{k}^{(2)}, x_{k}^{(3)})\} \) and an element \((x_1, x_2, x_3)\) arbitrarily in the unit ball of \((X_1 \oplus X_2 \oplus X_3)_{\psi}\). Then each \( \{x_{j}^{(k)}\} \) is a weakly null sequence in \( X_j \). Since the sequences \( \{x_{1}^{(k)}\} \) and \( \{x_{2}^{(k)}\} \) converge strongly to 0 in \( X_1 \) and \( X_2 \), respectively, we have

\[
\liminf_{k \to \infty} \|(x_{1}^{(k)}, x_{2}^{(k)}, x_{3}^{(k)}) + (x_1, x_2, x_3)\|_{\psi}
\]

\[
= \liminf_{k \to \infty} \|x_{1}^{(k)} + x_1\|_{\psi} + \liminf_{k \to \infty} \|x_{2}^{(k)} + x_2\|_{\psi} + \liminf_{k \to \infty} \|x_{3}^{(k)} + x_3\|_{\psi}
\]

\[
\leq \|\|(x_1, x_2, R(X_3))\|_{\psi}
\]

\[
= \max\{|x_1| + |x_2|, R(X_3)| = R(X_3) < 2
\]
(as \(\|x_1\| + \|x_2\| \leq \|(x_1, x_2, x_3)\|_\psi \leq 1\)), and hence \(R((X_1 \oplus X_2 \oplus X_3)_\psi) = R(X_3) < 2\).

Proposition 6.7 includes the previous result in [14]:

**Corollary 6.10 ([14])** Let \(X\) and \(Y\) be Banach spaces. The following are equivalent.

(i) \(R(X \oplus_1 Y) < 2\).

(ii) \(X\) and \(Y\) have the Schur property.

Now we are in a position to discuss the weak nearly uniform smoothness, where our discussion is based on the next result.

**Theorem 6.11 (García-Falset [8])** Let \(X\) be a Banach space. Then the following are equivalent.

(i) \(X\) is weakly nearly uniformly smooth.

(ii) \(X\) is reflexive and \(R(X) < 2\).

The space \((X_1 \oplus \cdots \oplus X_N)_\psi\) is reflexive if and only if all \(X_j\) are reflexive and a reflexive space \(X\) has the Schur property if and only if \(X\) is of finite dimension. Therefore by Theorems 6.2 and 6.11 we obtain a sequence of main theorems.

**Theorem 6.12** Let \(X_1, \ldots, X_N\) be Banach spaces and let \(\psi \in \Psi^N\). Assume that some \(X_j\) is of infinite dimension. Let \(U\) be the set of all indices \(j, 1 \leq j \leq N\), for which \(X_j\) are of infinite dimension. Then the following are equivalent.

(i) \((X_1 \oplus \cdots \oplus X_N)_\psi\) is weakly nearly uniformly smooth.

(ii) All \(X_1, \ldots, X_N\) are weakly nearly uniformly smooth and \(\psi \notin \Psi^N(1)(S)\) for all nonempty subsets \(S\) of \(\{1, \ldots, N - 1\}\) with \((S + 1) \subset U\) or \((S + 1)^c \subset U\).

As the case \(N = 2\) Theorem 6.12 includes the following result in Kato and Tamura [14] (recall (43) before Corollary 6.3).

**Corollary 6.13 ([14])** Let \(X\) and \(Y\) be Banach spaces and \(\psi \in \Psi^2\). Assume that \(X\) or \(Y\) is of infinite dimension. Then the following are equivalent.

(i) \(X \oplus_\psi Y\) is weakly nearly uniformly smooth.

(ii) \(X\) and \(Y\) are weakly nearly uniformly smooth and \(\psi \neq \psi_1\).

In the case where all \(X_1, \ldots, X_N\) are of infinite dimension we have the following.

**Theorem 6.14** Let \(X_1, \ldots, X_N\) be infinite dimensional Banach spaces and let \(\psi \in \Psi^N\). Then the following are equivalent.

(i) \((X_1 \oplus \cdots \oplus X_N)_\psi\) is weakly nearly uniformly smooth.

(ii) All \(X_1, \ldots, X_N\) are weakly nearly uniformly smooth and \(\psi \notin \Psi^N(1)\)

**Remark 6.15** By Remark 6.5 the implication (ii) \(\Rightarrow\) (i) in Theorem 6.14 is true without the assumption on the dimension of \(X_j\)’s.
The next result of Dhompongsa et al. [3] is an immediate consequence of Theorem 6.14 and Remark 6.15.

**Corollary 6.16 ([3])** Let $\psi \in \Psi_N$ be strictly convex. Then the following are equivalent.

(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth.
(ii) All $X_1, \ldots, X_N$ are weakly nearly uniformly smooth.

We shall close this section with mentioning the case that all $X_1, \ldots, X_N$ are of finite dimension. Owing to Proposition 6.7 we have the following:

**Theorem 6.17** Let $X_1, \ldots, X_N$ be Banach spaces. Then the following are equivalent.

(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth for all $\psi \in \Psi_N^{(1)}$.
(ii) $(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth for all $\psi \in \Psi_N^{(1)}(\{1, \ldots, N-1\}) \cap \left( \bigcap_{i=1}^{N-1} \Psi_N^{(1)}(\{i\}) \right)$.
(iii) $(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth for some $\psi \in \Psi_N^{(1)}(\{1, \ldots, N-1\}) \cap \left( \bigcap_{i=1}^{N-1} \Psi_N^{(1)}(\{i\}) \right)$.
(iv) $(X_1 \oplus \cdots \oplus X_N)_1$ is weakly nearly uniformly smooth.
(v) All $X_1, \ldots, X_N$ are reflexive and have the Schur property.
(vi) All $X_1, \ldots, X_N$ are of finite dimension.

**Remark 6.18** According to Remark 6.8, we cannot add in Theorem 6.17 the statement,

(i') $(X_1 \oplus \cdots \oplus X_N)_\psi$ is weakly nearly uniformly smooth for some $\psi \in \Psi_N^{(1)}$.

7. **Fixed point property.** As an application we shall discuss FPP for $\psi$-direct sums. As is well known, all uniformly non-square spaces have FPP (García-Falset et al. [10]). We shall construct (infinitely many) Banach spaces with FPP which fail to be uniformly non-square. Recall that a Banach space $X$ is called uniformly non-square if there exists $\varepsilon$ ($0 < \varepsilon < 1$) such that $\min\{|x+y||x-y|\} \leq 2(1-\varepsilon)$ for all $x, y \in S_X$. The next result is well known.

**Theorem 7.1** (García-Falset [9]) Let $X$ be a Banach space and let $R(X) < 2$. Then $X$ has the weak fixed point property for nonexpansive mappings.

Combining Theorem 6.14 and 7.1 with Remark 6.15, we have

**Theorem 7.2** Let $X_1, \ldots, X_N$ be weakly nearly uniformly smooth Banach spaces. Let $\psi \in \Psi_N$ with $\psi \notin \Psi_N^{(1)}$. Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ has the fixed point property for nonexpansive mappings.
COROLLARY 7.3 (cf. [3]) Let $X_1, \ldots, X_N$ be weakly nearly uniformly smooth Banach spaces. Let $\psi \in \Psi_N$ be strictly convex. Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ has FPP.

The $\ell_\infty$-sum $(X_1 \oplus \cdots \oplus X_N)_\infty$ cannot be uniformly non-square, whereas, since the function $\psi_\infty$ is allowed in Theorem 7.2, we obtain the following.

COROLLARY 7.4 Let $X_1, \ldots, X_N$ be weakly nearly uniformly smooth Banach spaces. Then $(X_1 \oplus \cdots \oplus X_N)_\infty$ has FPP.

For some other recent results on FPP for the $\ell_1$- and $\ell_\infty$-sums of Banach spaces we refer the reader to [15, 16].

Now we shall construct (infinitely many) convex functions $\psi \in \Psi_N$ with $\psi \notin \Psi_N^{(1)}$, $N \geq 3$, so that the corresponding norms $\| \cdot \|_\psi$ on $C^N$ are not uniformly non-square.

EXAMPLE 7.5 Let $N \geq 3$ and let $\varphi \in \Psi_2$, $\varphi \neq \psi_1$. Let

$$\psi(s_1, \ldots, s_{N-1}) = \max \left\{ \| (1 - \sum_{i=1}^{N-1} s_i, s_1) \|_\varphi, \| (s_1, s_2) \|_\varphi, \| (s_2, s_3) \|_\varphi, \ldots, \| (s_{N-2}, s_{N-1}) \|_\varphi \right\}$$

for $(s_1, \ldots, s_{N-1}) \in \Delta_N$.

Then, as in Example 5.7, $\psi \in \Psi_N$ and

$$\|(a_1, a_2, \ldots, a_n)\|_\psi = \max \{ \|(a_1, a_2)\|_\varphi, \|(a_2, a_3)\|_\varphi, \ldots, \|(a_{N-1}, a_N)\|_\varphi \}$$

for $(a_1, \ldots, a_N) \in C^N$.

We shall see that $\psi \notin \Psi_N^{(1)}$ and the norm $\| \cdot \|_\psi$ is not uniformly non-square. Suppose that $\psi \in \Psi_N^{(1)}$. Then, by Theorem 4 there exist $(a_1, \ldots, a_N) \in R_+^N$ and a nonempty proper subset $T$ of $\{1, \ldots, N\}$ such that

$$\|(a_1, \ldots, a_N)\|_\psi = \|(x_T(1)a_1, \ldots, x_T(N)a_N)\|_\psi + \|(x_T^c(1)a_1, \ldots, x_T^c(N)a_N)\|_\psi$$

and $\|(x_T(1)a_1, \ldots, x_T(N)a_N)\|_\psi = \|(x_T^c(1)a_1, \ldots, x_T^c(N)a_N)\|_\psi = 1$. Then $|a_j| \leq 1$ for all $1 \leq j \leq N$. Hence we have

$$\|(a_1, \ldots, a_N)\|_\psi = \max \{ \|(a_1, a_2)\|_\varphi, \|(a_2, a_3)\|_\varphi, \ldots, \|(a_{N-1}, a_N)\|_\varphi \} \leq \|(1, 1)\|_\varphi < 2$$

since $\varphi \neq \psi_1$, which is a contradiction (as $\|(a_1, a_N)\|_\psi = 2$). Therefore $\psi \notin \Psi_N^{(1)}$. Next let $a = (1, 0, 1, 0, \ldots, 0)$, $b = (1, 0, -1, 0, \ldots, 0) \in C^N$. Then we have $\|a\|_\psi = \|b\|_\psi = 1$ and $\|a \pm b\|_\psi = 2$, which implies that the norm $\| \cdot \|_\psi$ is not uniformly non-square.

By Theorem 7.2 with Example 7.5 we have the following result (note that if $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly non-square, so is the norm $\| \cdot \|_\psi$ on $C^N$). This indicates the existence of plenty of Banach spaces with FPP which are not uniformly non-square.
Corollary 7.6 Let $X_1, \ldots, X_N$, $N \geq 3$, be weakly nearly uniformly smooth Banach spaces. Let $\varphi \in \Psi_2$, $\varphi \neq \psi_1$ and let

$$
\psi(s_1, \ldots, s_{N-1}) = \max \left\{ \| (1 - \sum_{i=1}^{N-1} s_i, s_1) \varphi, \| (s_1, s_2) \varphi, \| (s_2, s_3) \varphi, \cdots, \| (s_{N-2}, s_{N-1}) \varphi \right\}
$$

for $(s_1, \ldots, s_{N-1}) \in \Delta_N$

Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ has FPP, whereas it is not uniformly non-square.

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