COMMENTATIONES MATHEMATICAE Vol. 52, No. 2 (2012), 153-170

D.P. Shukla , Shikha Yadav

Composition of Arithmetical functions with generalization of perfect and related numbers

Abstract. In this paper we have studied the deficient and abundent numbers connected with the composition of $\phi, \phi^*, \sigma, \sigma^*$ and ψ arithmetical functions , where ϕ is Euler totient, ϕ^* is unitary totient, σ is sum of divisor, σ^* is unitary sum of divisor and ψ is Dedekind's function. In 1988, J. Sandor conjectured that $\psi(\phi(m)) \geqslant m$, for all odd m and proved that this conjecture is equivalent to $\psi(\phi(m)) \geqslant \frac{m}{2}$ for all m , we have studied this equivalent conjecture. Further, a necessary and sufficient conditions of primitivity for unitary r-deficient numbers and unitary totient r-deficient numbers have been obtained . We have discussed the generalization of perfect numbers for an arithmetical function E_α.

2000 Mathematics Subject Classification: 11A25.

Key words and phrases: Arithmetic Functions, Abundent numbers, Deficient numbers, Inequalities, Geometric Numbers, Harmonic Numbers.

1. Numbers connected with composition of arithmetical functions. Let $\sigma(n)$ is the sum of divisors of the positive integer n, then $\sigma(n)$ is defined as

$$\sigma(n) = \sum_{d/n} d \quad \text{and} \quad \sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1}, \tag{1.1}$$

$$\sigma(n) \geqslant n, \qquad \text{with equality for } n = 1 \tag{1.2}$$

and

$$\sigma(mn) \geqslant m\sigma(n) \qquad \forall m, n \geqslant 1, \tag{1.3}$$

$$\sigma(n) \geqslant n+1, \qquad \text{with equality only for } n = \text{prime}. \tag{1.4}$$

Further, $\phi(n)$ is Euler totient function defined as

$$\phi(n) = n \prod_{p/n} \left(1 - \frac{1}{p}\right) \quad \text{and } \phi(p^\alpha) = p^\alpha - p^{\alpha-1}, \tag{1.5}$$

$$\phi(n) \leqslant n, \quad \text{with equality for } n = 1. \tag{1.6}$$

In J. Sandor [2] it has been proved that

$$\phi(mn) \leqslant m\phi(n), \quad \text{for any } m, n \geqslant 2, \tag{1.7}$$

with equality only if $pr\{m\} \subset pr\{n\}$, where $pr\{m\}$ denotes the set of distinct prime factors of m.

Now ψ is called Dedekind's function defined as

$$\psi(n) = n \prod_{p/n} \left(1 + \frac{1}{p}\right) \quad \text{and } \psi(p^\alpha) = p^\alpha + p^{\alpha-1}, \tag{1.8}$$

$$\psi(n) \geqslant n, \quad \text{with equality for } n = 1, \tag{1.9}$$

$$\psi(n) \geqslant n + 1, \quad \text{for all } n \geqslant 1, \text{ with equality only for } n = \text{prime}. \tag{1.10}$$

Further, Sandor [2], has also proved that

$$\psi(mn) \geqslant m\psi(n), \quad \text{for any } m, n \geqslant 1. \tag{1.11}$$

Let d is the divisor of n, then d is called unitary divisor of n if $d|n$ and $\left(d, \frac{n}{d}\right) = 1$. It is denoted by $d||n$. For these unitary divisors we can define unitary totient and divisor functions. Let σ^* is the sum of all unitary divisors of the positive integers defined as-

$$\sigma^*(n) = \prod_{p^\alpha || n} (p^\alpha + 1), \quad \text{where } \alpha \geqslant 1$$

or

$$\sigma^*(n) = (p_1^{\alpha_1} + 1)(p_2^{\alpha_2} + 1)...(p_r^{\alpha_r} + 1), \quad \text{for } n = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_r^{\alpha_r} \tag{1.12}$$

and

$$\sigma^*(n) \geqslant n \quad \text{with equality for } n = 1. \tag{1.13}$$

Let ϕ^* is unitary totient function defined as

$$\phi^*(n) = \prod_{p^\alpha || n} (p^\alpha - 1)$$

or

$$\phi^*(n) = (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1)...(p_r^{\alpha_r} - 1), \quad \text{for } n = p_1^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r} \tag{1.14}$$

and

$$\phi^*(n) \leqslant n, \quad \text{with equality for } n = 1. \tag{1.15}$$

All these functions mentioned above are multiplicative satisfying

$$f(mn) = f(m)f(n) \quad \forall (m, n) = 1.$$

In Mladen V. Vassilev- Missana and Krassimir T. Atanassov [4] it has been defined that a number n is called a unitary perfect number if $\sigma^*(n) = 2n$ and it has been conjectured that there are only a finite number of even ones. The first few unitary perfect numbers are 6, 60, 90, 87360.... Also, n is called unitary super perfect number if

$$\sigma^*(\sigma^*(n)) = 2n.$$

The first few unitary super perfect numbers are 2, 9, 165, 238, 1640....... Similarly a number $n \geqslant 1$ is called ϕ^*-perfect number if

$$\phi^*(n) = 2n$$

and n is called $\phi^* o \phi^*-$perfect if $\phi^*(\phi^*(n)) = 2n$.

Now some basic properties of certain arithmetical functions have been defined by the following lemmas-

Lemma 1 *Let n be a k-full number where $k \geqslant 2$. then $\phi^*(n) \geqslant n/2$.*

Proof Let $n = p_1^{\alpha_1}.p_2^{\alpha_2}....p_r^{\alpha_r}$, where $\alpha_i \geqslant k$ for all $i = \overline{1, r}$.

Now using (1.14), we have

$$\phi^*(n) = (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1)....(p_r^{\alpha_r} - 1),$$

so,

$$\begin{aligned} \phi^*(n) &= p_1^{\alpha_1}.p_2^{\alpha_2}....p_r^{\alpha_r}\left(1 - \frac{1}{p_1^{\alpha_1}}\right)\left(1 - \frac{1}{p_2^{\alpha_2}}\right)....\left(1 - \frac{1}{p_r^{\alpha_r}}\right) \\ &\geqslant p_1^{\alpha_1}.p_2^{\alpha_2}....p_r^{\alpha_r}\left(1 - \frac{1}{p_1^{\alpha_1}}\right)\left(1 - \frac{1}{p_2^{\alpha_2}}\right)....\left(1 - \frac{1}{p_r^{\alpha_r}}\right) \\ &> n \prod_{p\,prime}\left(1 - \frac{1}{p^k}\right) = \frac{n}{\zeta(k)}. \end{aligned}$$

Hence

$$\phi^*(n) \geqslant \frac{n}{\zeta(k)} > \frac{n}{2},$$

since

$$\frac{1}{\zeta(k)} > \frac{n}{2},$$

as for $n = 2$, $\frac{1}{\zeta(2)} = \frac{6}{\pi^2} = 0.60792......$ ■

Lemma 2 *Let n be a k-full number where $k \geqslant 2$, then $\sigma^*(n) < 2n$.*

Proof Let $n = p_1^{\alpha_1}.p_2^{\alpha_2}....p_r^{\alpha_r}$, where $\alpha_i \geqslant k$ for all $i = \overline{1, r}$.

Now using (1.12), we have

$$\begin{aligned}
\sigma^*(n) &= (p_1^{\alpha_1}+1)(p_2^{\alpha_2}+1)....(p_r^{\alpha_r}+1) \\
&= p_1^{\alpha_1}.p_2^{\alpha_2}....p_r^{\alpha_r}\left(1+\frac{1}{p_1^{\alpha_1}}\right)\left(1+\frac{1}{p_2^{\alpha_2}}\right)....\left(1+\frac{1}{p_r^{\alpha_r}}\right) \\
&\leqslant n\left(1+\frac{1}{p_1^{\alpha_1}}\right)\left(1+\frac{1}{p_2^{\alpha_2}}\right)....\left(1+\frac{1}{p_r^{\alpha_r}}\right) \\
&< n\prod_p\left(1+\frac{1}{p^k}\right) = \frac{n\prod_p\left(1-\frac{1}{p^{2k}}\right)}{\prod_p\left(1-\frac{1}{p^k}\right)} \\
&= \frac{n.\zeta(k)}{\zeta(2k)} < 2n,
\end{aligned}$$

■

since,

$$\frac{n.\zeta(k)}{\zeta(2k)} < 2,$$

as for

$$k=2, \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2} = 1.51859504....$$

Theorem 1.1 *Let n is k-full number where $k \geqslant 2$, then*
(i) $\sigma^*(\phi^*(n)) > n/2$
(ii) $\phi^*(\sigma^*(n)) < 2n$

Proof (i) Let n is k-full number, where $k \geqslant 2$ using (1.13) and lemma 1, we have

$$\sigma^*(\phi^*(n)) \geqslant \phi^*(n) > n/2.$$

(ii) Using (1.15) and lemma 2, we have

$$\phi^*(\sigma^*(n)) \leqslant \sigma^*(n) < 2n.$$

n is called f-abundent and f-deficient if $f(n) > 2n$ and $f(n) < 2n$, where f is an arithmetic function. Further, we have discussed about f-deficient and f-abundent numbers, where $f = hog$ and $h, g : N \to N$ be two arithmetical functions.

Theorem 1.2 *Let n be a positive integer and $h, g : N \to N$ be two arithmetical functions such that $h(g(n)) \leqslant n$, for any $n \geqslant 1$, then n is a hog-deficient number.*

Proof Let n be a positive integer such that $h(g(n)) \leqslant n < 2n$. So,

$$h(g(n)) < 2n.$$

Hence n is *hog*-deficient number. ■

Corollary 1.2.1 *There are infinitely many $\phi o\psi-$deficient numbers.*

Let $n = 3.2^\alpha$ for any $\alpha \geqslant 1$, then

$$\psi(3.2^\alpha) = 3.2^{\alpha+1}$$

and

$$\phi(2^{\alpha+1}.3) = 2^{\alpha+1},$$

so,

$$\phi(\psi(3.2^\alpha)) = \frac{2}{3}n < n.$$

Put $n = 2^\alpha.5^\beta (\beta \geqslant 2)$, then

$$\phi(\psi(2^\alpha.5^\beta)) = 2^{\alpha+2}.3.5^{\beta-2} = \frac{12}{25}n < n.$$

Put $n = 3^\alpha (\alpha \geqslant 1)$, then

$$\phi(\psi(3^\alpha)) = \phi(3^{\alpha-1}2^2) = \frac{4}{9}n < n.$$

Let $n = 2^\alpha.7^\beta (\beta \geqslant 2)$, then

$$\phi(\psi(n)) = \frac{24}{49}n < n.$$

In J. Sandor [2] it has been proved that for n is a squarefull number $\phi(\psi(n)) \leqslant n$ and if n is a product of Mersenne primes then also $\phi(\psi(n)) \leqslant n$.

Hence by theorem 1.2, there are infinitely many $\phi o\psi-$deficient numbers.

Proposition 1.2.2 *All k-full ($k \geqslant 2$) positive integers are $\phi^* o\sigma^*-$deficient numbers.*

Proof It can be proved easily by theorem- 1.1(ii)

Theorem 1.3 Let $h.g : N \to N$ be two arithmetical functions and $n \geqslant 1$ such that n is $g-$deficient i.e. $g(n) < 2n$ and $h(m) \leqslant m$ for all $m \geqslant 1$, with equality only for $m = 1$ then n is $hog-$deficient number.

Proof Let n is a g-deficient number. Since, $h(m) \leqslant m$ and $g(n) \geqslant 1$, for all $n \geqslant 1$ therefore, $h(g(n)) \leqslant g(n) < 2n$. Hence n is hog-deficient number.

Corollary 1.3.1 *All $\psi-$deficient positive integers which are greater than equal to 3 are $\phi o\psi-$deficient number.*

Let $n \geqslant 3$ and $\psi(n) < 2n$. In J. Sandor [2] it has been proved by Theorem 3.7 that $\phi(\psi(n)) < n$. Now by theorem 1.2, n is a $\phi o\psi-$deficient number.

Theorem 1.4 *Let $h, g : N \to N$ be two arithmetic functions such that*

(i) $g(mn) = g(m)g(n) \quad \forall (m,n) = 1$

(ii) $h(ab) \leqslant ah(b)$ *for all* $a, b \geqslant 1$

(iii) $g(m) \leqslant m$ *with equality for* $m = 1$

(iv) $h(g(2^{\alpha})) < 2^{\alpha+1}$

then for all even n $h(g(n)) < 2n$.

Proof Let $n = 2^{\alpha}m$, with m =odd and $\alpha \geqslant 1$. Now, using properties (i), (ii), (iii) and (iv) we have

$$h(g(2^{\alpha}m)) = h(g(2^{\alpha}).g(m)) \leqslant g(m)h(g(2^{\alpha})) < m.2^{\alpha+1} = 2n.$$

Hence $h(g(n)) < 2n$. ■

Corollary 1.4.1 *All even numbers are $\phi o \phi^*-$deficient.*

Let $n = 2^{\alpha}.m$, with $m = odd$ and $\alpha \geqslant 1$. Since ϕ^* is multiplicative function, $\phi(ab) \leqslant a\phi(b)$ for any $m, n \geqslant 2$, also $\phi^*(m) \leqslant m$ with equality only for $m = 1$. Further, using (1.14) and (1.6)

$$\phi(\phi^*(2^{\alpha})) = \phi(2^{\alpha} - 1) \leqslant 2^{\alpha} - 1 < 2^{\alpha+1} \qquad \forall \alpha \geqslant 1.$$

Hence by theorem 1.4, n is a $\phi o \phi^*-$deficient number.

Corollary 1.4.2 *All even $\psi o \sigma-$abundent numbers n have the form $n = 2^{\alpha}.m$ with $m = odd$ and where $m > 1, \alpha \geqslant 1$ such that $\alpha + 1$ is not prime.*

On putting $h(m) = \psi(m)$ and $g(n) = \sigma(n)$ in theorem 1.4 and using (1.11) and (1.2), properties (ii) and (iii) are followed. Since, $\sigma(n)$ is a multiplicative function therefore, property (i) is known. Further, using (1.1) and (1.10), we have

$$\psi(\sigma(2^{\alpha})) = \psi(2^{\alpha+1} - 1) \geqslant 2^{\alpha+1},$$

with equality for $2^{\alpha+1} - 1$ =prime. Using property (i) and equation (1.11), (1.2), (1.1) and (1.10), we have

$$\begin{aligned} \psi(\sigma(2^{\alpha}m)) &= \psi(\sigma(2^{\alpha})\sigma(m)) \geqslant \sigma(m)\psi(\sigma(2^{\alpha})) \\ &\geqslant m\psi(2^{\alpha+1} - 1) \geqslant m.2^{\alpha+1} = 2n, \end{aligned}$$

with equality for $m = 1$ and $2^{\alpha+1} - 1$ =prime.

Since, $2^{\alpha+1} - 1$ =prime if $\alpha + 1$ =prime therefore, $\psi(\sigma(2^{\alpha}m)) > 2n$ for $m > 1$ and $\alpha \geqslant 1$ such that $\alpha + 1$ is not prime.

Corollary 1.4.3 *Set of all even numbers except 2 are $\sigma o \psi-$abundent numbers.*

Let $n = 2^{\alpha}.m$, with $m = odd$ and $\alpha \geqslant 1$, put $h(n) = \sigma(n)$ and $g(n) = \psi(n)$ in theorem 1.4. Now, $\psi(n)$ is multiplicative and using (1.3), (1.9), (ii) and (iii) are followed. Further,

$$\sigma(\psi(2^{\alpha})) = \sigma(2^{\alpha-1}.3) = (2^{\alpha} - 1).4 \geqslant 2^{\alpha+1},$$

using (1.8), (1.1) and (1.2), we have

$$\begin{aligned}\sigma(\psi(2^{\alpha}.m)) &= \sigma(2^{\alpha-1}.3.m) \geqslant \sigma(m)\sigma(2^{\alpha-1}.3)\\ &\geqslant m(2^{\alpha}-1).4 \geqslant m.2^{\alpha+1},\end{aligned}$$

with equality for only $m = 1$ and $\alpha = 1$.

Hence $n = 2^{\alpha}.m$, for $\alpha > 1$ and $m > 1$ is $\sigma o\psi-$abundent number.

2. Study of Conjecture. In 1988, J. Sandor [3] conjectured that

$$\phi(\psi(n)) \leqslant n, \quad \text{for any } n \geqslant 2. \tag{2.1}$$

In 2005, Sandor [2] also studied the conjecture (2.1) and its properties.

Beside these Sandor [1], [3] also conjectured that

$$\psi(\phi(n)) \geqslant n, \quad \text{for all odd } n \tag{2.2}$$

and showed that this is equivalent to

$$\psi(\phi(n)) \geqslant \frac{n}{2}, \quad \text{for all } n. \tag{2.3}$$

Now we will study the conjecture (2.3).

Theorem 2.1 *Let n is squarefull. Then inequality (2.3) holds ture.*

Proof Let $n = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_r^{\alpha_r}$, where $p_i \geqslant 2$ and $\alpha_i \geqslant 2$ for $i = \overline{1,r}$ Now using (1.5)

$$\psi(\phi(n)) = \psi\left(p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)....\left(1-\frac{1}{p_r}\right)\right)$$

using (1.9) and (1.8), we have

$$\psi\left(p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_r^{\alpha_r}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)....\left(1-\frac{1}{p_r}\right)\right) \geqslant$$

$$\begin{aligned}&\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)....\left(1-\frac{1}{p_r}\right)\psi\left(p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}...p_r^{\alpha_r}\right)\\ =\ &\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)....\left(1-\frac{1}{p_r}\right)p_1^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r}\\ &\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)....\left(1+\frac{1}{p_r}\right)\\ =\ &p_1^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r}\left(1-\frac{1}{p_1^2}\right)\left(1-\frac{1}{p_2^2}\right)....\left(1-\frac{1}{p_r^2}\right)\\ =\ &n\prod_{i=1}^{r}\left(1-\frac{1}{p_i^2}\right) > n\prod_{p\ prime}\left(1-\frac{1}{p^2}\right) = \frac{n}{\zeta(2)} = \frac{n.6}{\pi^2} > \frac{n}{2},\end{aligned}$$

∎

since,

$$\prod_{p\ prime}\left(1-\frac{1}{p^2}\right)=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}=0.60792.....$$

Hence

$$\psi(\phi(n))\geqslant\frac{n}{2}.$$

THEOREM 2.2 *If* $\phi(n)=\frac{\psi(n)}{2}$, *then inequality (2.3) holds true.*

PROOF Since, $\phi(n)=\frac{\psi(n)}{2}$ therefore using (1.9)

$$\psi(\phi(n))=\psi\left(\frac{\psi(n)}{2}\right)\geqslant\frac{\psi(n)}{2}\geqslant\frac{n}{2}.$$

Hence inequality (2..3) holds true.

THEOREM 2.3 *There are infinitely many* n *such that* $\psi\left(\phi(n)\right)\geqslant\frac{n}{2}$

PROOF Let $n=2^a.3^b$, for any $a\geqslant 1$ and $b\geqslant 2$

$$\phi(n)=\phi(2^a.3^b)=2^{a-1}3^{b-1}2=2^a3^{b-1}.$$

now

$$\begin{aligned}\psi(\phi(n)) &= \psi(2^a.3^{b-1})=2^{a-1}3.3^{b-2}2^2\\ &= 2^{a+1}3^{b-1}\\ &= \frac{2}{3}n\geqslant\frac{n}{2}.\end{aligned}$$

Let $n=2^a.5^b$, for any $a\geqslant 1$ and $b\geqslant 2$

$$\phi(n)=\phi(2^a.5^b)=2^{a-1}5^{b-1}2^2=2^{a+1}5^{b-1}.$$

Now

$$\begin{aligned}\psi(\phi(n)) &= 2^a.3.5^{b-2}.2.3\\ &= 2^{a+1}5^{b-2}.3^2\\ &= \frac{18}{25}.n\geqslant\frac{n}{2}.\end{aligned}$$

■

We can see that for $n=3^k$, where $k\geqslant 1$

$$\phi(n)=\frac{\psi(n)}{2}.$$

So,

$$\psi\left(\phi(n)\right) \geqslant \frac{n}{2}.$$

Further, let $n = 2^a.7^b$ for $a \geqslant 1$ and $b \geqslant 2$

$$\psi\left(\phi(n)\right) = \frac{48}{49}.\ n \geqslant \frac{n}{2}.$$

For $n = 5^a.2.3.7$, with $a \geqslant 2$

$$\phi(n) = 2^4.3.5^{a-1},$$

now,

$$\psi(\phi(n)) = \psi(2^4.3.5^{a-1}) = \frac{96}{175}.n \geqslant \frac{n}{2}.$$

For $n = 7^a.2.3.5$, with $a \geqslant 2$

$$\phi(n) = 2^4.3.7^{a-1},$$

$$\psi(\phi(n)) = \frac{128}{245}.n \geqslant \frac{n}{2}.$$

Theorem 2.4 *There exist so many n for which inequality (2.3) does not hold, true.*

Proof For many value of n we have found that inequality (2.3) does not hold true. Let $n = 2^a.3.5$, with $a \geqslant 1$

$$\psi(\phi(n)) = \frac{2}{5}.n < \frac{n}{2}.$$

Let

$$n = 3^a.2.5.7, \quad \text{with } a \geqslant 1, \psi(\phi(n)) = \frac{16}{35}.n < \frac{n}{2}.$$

Let

$$n = 3^a.2.5.7.11, \quad \text{with } a \geqslant 1,\ \psi(\phi(n)) = \frac{192}{385}.n < \frac{n}{2}.$$

Now for $n = 39270$, $n = 82110$ or $n = 2.3.5.7.17.23.M$, where M is a Mersenne prime greater than or equal to 31, then (2.3) is not true.

So many other counter examples of n are found for which inequality (2.3) is not true, given as following -

71610, 78540, 117810, 143220, 157080, 164010, 164220, 214830, 235620, 246330, 286440, 314160, 328440, 353430. ∎

Remark For $m = 217140$ and $n = 214830$, $\psi(\phi(m)) = \psi(\phi(n))$ but $\psi(\phi(m)) \geqslant \frac{m}{2}$.

Theorem 2.5 *Let $n = 2^k m$, where m is odd and $(2^k, m) = 1$ with $k \geqslant 1$ such that inequality (2.2) is true for m.Then inequality (2.3) holds true for n.*

PROOF Let $n = 2^k m$, where m is odd and $(2^k, m) = 1$ with $k \geqslant 1$, then

$$\phi(n) = \phi(2^k m) = \phi(2^k)\phi(m).$$

Using (1.11), we have

$$\begin{aligned}\psi(\phi(n)) &= \psi(2^{k-1}\phi(m)) \geqslant 2^{k-1}\psi(\phi(m)) \\ &\geqslant 2^{k-1}m = \frac{n}{2},\end{aligned} \qquad \blacksquare$$

since, inequality (2.2) is true for m.

Hence

$$\psi(\phi(n)) \geqslant \frac{n}{2}.$$

THEOREM 2.6 *Let $n \geqslant 2$, then inequality (2.3) is true for n if it is true for squarefree part of $n \geqslant 2$.*

PROOF Let $n = p_1^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r}$ and square free part of n is m. Then, $m = p_1.p_2...p_r$ Using (1.5), (1.11), we have

$$\begin{aligned}\psi(\phi(n)) &= \psi(\phi(p_1^{\alpha_1}.p_2^{\alpha_2}...p_r^{\alpha_r})) \\ &= \psi(p_1^{\alpha-1}.p_2^{\alpha_2-1}...p_r^{\alpha_r-1}(p_1-1)(p_2-1)........(p_r-1)) \\ &\geqslant p_1^{\alpha-1}.p_2^{\alpha_2-1}...p_r^{\alpha_r-1}\psi((p_1-1)(p_2-1)........(p_r-1)) \\ &= \frac{n}{m}\psi(\phi(m)).\end{aligned} \qquad \blacksquare$$

So,

$$\psi(\phi(n)) \geqslant \frac{n}{m}\psi(\phi(m)).$$

Thus,

$$\frac{\psi(\phi(n))}{n} \geqslant \frac{\psi(\phi(m))}{m}.$$

Since, inequality (2.3) is true for square free part m of n therefore,

$$\psi(\phi(n)) \geqslant \frac{n}{2}.$$

3. Primitive Unitary Deficient Numbers. In V.Siva Rama Prasad and D. Ram Reddy [5] primitive abundent numbers have been defined as-

An abundent number $n \geqslant 1$ is said to be primitive if each of its divisor, other than n is deficient.

A positive integer n is called unitary abundent or deficient if $\sigma^*(n) > 2n$ or $\sigma^*(n) < 2n$. Further, n is called unitary r-abundent or unitary r-deficient for a real number $r \geqslant 2$ if

$$\sigma^*(n) > rn \quad \text{or } \sigma^*(n) < rn.$$

For a real number $r \geqslant 2$ an unitary r-abundent number is said to be primitive if each of its unitary divisor other than n is unitary r-deficient.

V. Siva Rama Prasad and D. Ram Reddy [5] have given a neccessary and sufficient condition for a unitary r-abundent number to be primitive. Further, primitive unitary r-deficient numbers have been defined as-

An unitary r-deficient number is said to be primitive if each of its unitary divisor other than n is unitary r-abundent.

Now, for a real number $r \geqslant 2$ primitive unitary totient r-abundent and primitive unitary totient r-deficient numbers have been defined as-

A positive integer n is called unitary totient r-abundent if

$$\phi^*(n) > rn, \tag{3.1}$$

n is called unitary totient r-deficient if

$$\phi^*(n) < rn. \tag{3.2}$$

An unitary totient r-abundent number is called primitive if all of its unitary divisors other than n are unitary totient r-deficient.

An unitary totient r-deficient number is called primitive if all of its unitary divisors other than n are unitary totient r-abundent.

A necessary sufficient condition of primitivity for unitary r-deficient and unitary totient r-deficient numbers have been given by the following theorems-

Let p^α be a unitary divisor of $n \geqslant 1$ then it can be proved very easily that a unitary r-deficient number n is primitive if and only if

$$\frac{\sigma^*\left(\frac{n}{p^\alpha}\right)}{\frac{n}{p^\alpha}} > r,$$

p being a prime and $\alpha \geqslant 1$ is an integer.

THEOREM 3.1: *If a natural number $n \geqslant 1$ is a unitary r-deficient number. Then n is primitive if and only if*

$$\sigma^*(n) - rn > \frac{rn}{q^\gamma},$$

where $r \geqslant 2$ is a real number and q^γ is the largest prime power such that $q^\gamma || n$.

PROOF If a natural number $n \geqslant 1$ is a unitary r-deficient number then n is primitive if and only if

$$\frac{\sigma^*\left(\frac{n}{p^\alpha}\right)}{\frac{n}{p^\alpha}} > r \ , \ \text{ for } p^\alpha || n, \tag{3.3}$$

with p being a prime and $\alpha \geqslant 1$ an integer.

As $\sigma^*(n)$ is a multiplicative function so, using (3.3), we have

$$\frac{\sigma^*(n)}{\sigma^*(q^\gamma)} = \sigma^*\left(\frac{n}{q^\gamma}\right) > r\frac{n}{q^\gamma},$$

for $q^\gamma||n$, with q being a prime and $\gamma \geqslant 1$

Further, using (1.12), we have

$$\sigma^*(n)q^\gamma > rn(q^\gamma + 1),$$

which implies.

$$\sigma^*(n) - rn > \frac{rn}{q^\gamma}.$$

Now conversely for a real number $r \geqslant 2$ and natural number $n \geqslant 1$.

Let

$$\sigma^*(n) - rn > \frac{rn}{q^\gamma},$$

this implies that

$$\sigma^*(n)q^\gamma > rn(q^\gamma + 1),$$

using (1.12), we have

$$\frac{\sigma^*(n)}{\sigma^*(q^\gamma)} = \sigma^*\left(\frac{n}{q^\gamma}\right) > \frac{rn}{q^\gamma},$$

where $q^\gamma||n$ with q prime and $r \geqslant 1$ a integer.

Hence the theorem. ∎

Theorem 3.2 *Let n is unitary totient r-deficient number. Then n is primitive if and only if*

$$\phi^*(n) + \frac{rn}{q^\gamma} > rn$$

where $r \geqslant 2$ be a real number and q^γ is the largest prime power such that $q^\gamma||n$.

Proof Let n is primitive unitary totient r-deficient number then we have

$$\frac{\phi^*\left(rn/p^\alpha\right)}{\frac{n}{p^\alpha}} > r, \qquad \text{for each } p^\alpha||n \tag{3.4}$$

using (3.4) and (1.14), we have

$$\frac{\phi^*(n)}{\phi^*(q^\gamma)} = \phi^*\left(\frac{n}{q^\gamma}\right) > \frac{rn}{q^\gamma},$$

this implies that

$$\phi^*(n)q^\gamma > rn(q^\gamma - 1),$$

so,

$$\phi^*(n) + \frac{rn}{q^\gamma} > rn.$$

Conversely, for $r \geqslant 2$, being a real number and $n \geqslant 1$, be a natural number let $\phi^*(n) + \frac{rn}{q^\gamma} > rn$ which implies that $\phi^*\left(\frac{n}{q^\gamma}\right) > r\frac{n}{q^\gamma}$, where $q^\gamma||n$.

Hence n is an unitary totient r-deficient number. ∎

THEOREM 3.3

$$\lim_{\substack{n\to\infty\\ n\in D_r}} \frac{\phi^*(n)}{n} = r,$$

where D_r denotes the set of all primitive unitary totient r-deficient numbers.

PROOF Since $\phi^*(n) \leqslant rn$ for $n \in D_r$, we have

$$\lim_{\substack{n\to\infty\\ n\in D_r}} \frac{\phi^*(n)}{n} \leqslant r \tag{3.5}$$

If strict inequality holds in (3.5), then we have

$$\lim_{\substack{n\to\infty\\ n\in D_r}} \frac{\phi^*(n)}{n} = r - \epsilon \quad \text{for } \epsilon > 0.$$

Now, by the definition of limit infimum, there are infinitely many n in D_r for which

$$\frac{\phi^*(n)}{n} \leqslant r + \frac{\epsilon}{2}. \tag{3.6}$$

But for every $n\epsilon D_r$ by theorem 3.2, we have

$$\frac{\phi^*(n)}{n} > r - \frac{r}{q^\gamma}, \quad \text{where } q^\gamma || n, \tag{3.7}$$

using (3.6) and (3.7), we have

$$r - \frac{r}{q^\gamma} < r + \frac{\epsilon}{2},$$

which gives $q^\gamma < \frac{2r}{\epsilon}$ which shows that possible values of n are finite which contradicts (3.6). Hence

$$\lim_{n\to\infty} \frac{\phi^*(n)}{n} = r.$$

4. Generalization of Perfect Numbers for Certain Arithmetical Functions. Here we will discuss some arithmetical functions and their properties given by Mladen V. Vassilev Missana and Krassimir T. Atanassov [4] as following:-

For $n \in N$ and $n \geqslant 1$, we define a function $\sigma_\alpha(n)$ is the sum of $\alpha-$th power of all divisors of $n \in N$, where α is a real number and

$$\sigma_\alpha(1) = 1 \qquad \text{i.e.}$$

$$\sigma_\alpha(n) = \sum_{d/n} d^\alpha, \tag{4.1}$$

$\sigma_\alpha(n)$ is multiplicative and σ_0 is denoted by d.

$d(n)$ is the number of all different divisors of $n \in N$ and

$$d(n) = \underset{i=1}{\overset{k}{\pi}}(\alpha_i + 1) \qquad \text{if } n = \underset{i=1}{\overset{k}{\pi}} p_i^{\alpha_i}$$

also, $$d(n) = \sum_{d/n} 1.$$

Now $d(1) = 1$. As usually σ_1 is denoted by σ. Where $\sigma(n)$ is the sum of all divisors of n.

$$d(n) \geqslant 2 \qquad \text{if } n > 1, \tag{4.2}$$

with equality holds for only $n = p$, where p is prime.

E_α is the completely multiplicative function given by

$$E_\alpha(n) = n^\alpha, \qquad \text{for } n \in N, \tag{4.3}$$

where α is a fixed real number

$$E_\alpha(n) = \begin{Bmatrix} 1 & \text{if } \alpha = 0 \\ n & \text{if } \alpha = 1 \end{Bmatrix}, \tag{4.4}$$

e is the unit function defined by $e(n) = 1$ for all $n \in N$,

now we obtain

$$\sigma_\alpha = E_\alpha * e. \tag{4.5}$$

Mladen V. Vassilevi and Krassimir T. Atanassov [4] have defined generalization of perfect numbers as follows:-

Perfect Numbers- A number $n \in N$ is called perfect number if $\sigma(n) = 2n$, where $\sigma(n)$ denotes the sum of all divisors of n.

f-perfect numbers- Let f be a fixed arithmetic function. A number $n \in N$ is said to be f-perfect iff

$$\sum_{d/n} f(d) = 2f(n). \tag{4.6}$$

fg- perfect numbers- Let f and g are two arithmetic functions then a number $n \in N$ is said to be fg-perfect number iff

$$(g * f)(n) = 2f(n), \tag{4.7}$$

where $(g * f)(n) = \sum_{d/n} g(d) f\left(\frac{n}{d}\right)$.

Ω_G-perfect numbers- For a given arithmetic function G, $n \in N$ is said to be Ω_G-perfect number if

$$G(n) = n. \tag{4.8}$$

If $G = \frac{1}{2}\sigma$ i.e. for each $n \in N$

$$G(n) = \frac{1}{2}\sigma(n),$$

then Ω_G-perfect numbers coincide with ordinary perfect numbers.

It is seen that if $f(n) > 0$ for each $n \in N$, then f-perfect numbers are Ω_G-perfect numbers too, because we can write for each $n \in N$

$$G(n) = \frac{n\sum_{d/n} f(d)}{2f(n)}.$$

Also, it is seen that if $G(n) > 0$ for each $n \in N$, then fg-perfect numbers are Ω_G-perfect numbers too for G that is given for each $n \in N$ by

$$G(n) = \frac{n(g * f)(n)}{2f(n)}.$$

$[F, S]$ **perfect numbers-** Let F be a given arithmetic function and $S \neq 0$ be a real number. A number $n \in N$ is said to be $[F, S]$ perfect number iff n satisfies

$$F(n) = S.n. \tag{4.9}$$

It is seen that if n is $[F, S]$ perfect number then n is Ω_G-perfect number too, for G that is given for each $n \in N$ by

$$G(n) = \frac{1}{S}F(n). \tag{4.10}$$

It is seen that if n is Ω_G-perfect number then n is a $[F, S]$- perfect number too because we can write for each $n \in N$

$$F(n) = S.G(n). \tag{4.11}$$

It is seen that if $f(n) > 0$ for each $n \in N$ then $[F, 2]$ perfect numbers coincide with f-perfect numbers.

Where F is the function given by

$$F(n) = \frac{n\sum_{d/n} f(d)}{f(n)}. \tag{4.12}$$

$[F, 2]$- perfect numbers coincide with fg-perfect numbers, where F is the function given for each $n \in N$ by

$$F(n) = \frac{n(g * f)(n)}{f(n)}. \tag{4.13}$$

Here we have generalized all the perfect numbers given above for arithmetic function E_α.

Proposition 4.1. *For $\alpha = \beta$, $E_{\beta-\alpha}-$perfect numbers coincide with the set of all prime numbers.*

Proof Since,

$$\sum_{d/n} E_{\beta-\alpha}(d) = \sum_{d/n} E_0(d), \quad \text{for } \alpha = \beta.$$

Therefore, using (4.4) and (4.2), we get

$$\sum_{d/n} E_0(d) = \sum_{d/n} 1 = d(n) \geqslant 2E_0(n), \quad \text{for} \quad n > 1,$$

since, equality holds for $n = p$, where p is a prime number.
Therefore, $\sum_{d/n} E_0(d) = 2E_0(n)$ only for $n = p$, where p is a prime number.

Hence, for $\alpha = \beta$, $E_{\beta-\alpha}-$perfect numbers coincide with the set of all prime numbers. ■

THEOREM 4.2. *Let $\alpha = \beta - 1$ and $\beta \geqslant 1$ be a fixed real number. Then $E_{\beta-\alpha}$ perfect numbers coincide with ordinary perfect numbers.*

PROOF Let n is a $E_{\beta-\alpha}$ perfect number. Using (4.3) and (4.1), we get

$$\sum_{d/n} E_{\beta-\alpha}(d) = \sum_{d/n} d^{\beta-\alpha} = \sum_{d/n} d = \sigma(n), \qquad \text{for } \alpha = \beta - 1 \text{ and } \beta \geqslant 1.$$

Using (4.6), we have

$$\sum_{d/n} E_{\beta-\alpha}(d) = 2E_{\beta-\alpha}(n) = 2n^{\beta-\alpha} = 2n, \quad \text{for } \alpha = \beta - 1 \text{ and } \beta \geqslant 1.$$

So, $\sigma(n) = 2n$. Hence n is a perfect number. ■

PROPOSITION 4.3. *If $G(n) = E_{\beta-\alpha}$, where $\alpha = \beta - 1$ and $\beta \geqslant 1$ then n is Ω_G-perfect number.*

PROOF Using (4.8) and (4.4)

$$E(n) = n \qquad \text{for } \alpha = \beta - 1.$$

Hence n is a Ω_G-perfect number. ■

THEOREM 4.4. *If $F(n) = E_\alpha \sigma_{\beta-\alpha}$, where $\alpha = \beta - 1$ and $\beta \geqslant 1$ be a fixed real number then for a real number $S = 2n^\alpha$, $[F, S]$- pefect numbers coincide with $E_{\beta-\alpha}$ perfect numbers.*

PROOF Let n is a $[F, S]$-perfect number, using (4.9), $E_\alpha \sigma_{\beta-\alpha}(n) = S.n = 2n^\alpha n = 2n^\beta$, where $S = 2n^\alpha$ and $\alpha = \beta - 1$. Using (4.3) and (4.1),

$$E_\alpha \sigma_{\beta-\alpha}(n) = n^\alpha \sum_{d/n} d^{\beta-\alpha} = 2n^\beta,$$

so, $\sum_{d/n} d^{\beta-\alpha} = 2n^{\beta-\alpha}$. Thus, $\sum_{d/n} E_{\beta-\alpha}(d) = 2E_{\beta-\alpha}(n)$. Hence n is a $E_{\beta-\alpha}-$ perfect number. ■

THEOREM 4.5. *If $F = E_\alpha \sigma_{\beta-\alpha}$, where $\alpha = \beta - 1$ and $\beta \geqslant 1$ be fixed real number then for a real number $S = 2n^\alpha, [F, S]$ perfect numbers coincide with the set of perfect numbers.*

PROOF Let n is a $[F, S]$- perfect number. Using (4.9), we get

$$E_\alpha \sigma_{\beta-\alpha}(n) = 2n^\beta,$$

using (4.3)

$$\sigma_{\beta-\alpha}(n) = 2n^{\beta-\alpha}.$$

Since, $\alpha = \beta - 1$ therefore, $\sigma(n) = 2n$. Hence n is a perfect number. ■

THEOREM 4.6. *$[E_\alpha \sigma_{\beta-\alpha}, S]-$perfect numbers coincide with $(E_{\beta-\alpha})e-$perfect numbers, where $S = 2n^\alpha, \alpha = \beta - 1$ and $\beta \geqslant 1$.*

PROOF Let n is $[E_\alpha \sigma_{\beta-\alpha}, S]-$perfect number, using (4.9), we get

$$E_\alpha \sigma_{\beta-\alpha}(n) = S.n = 2n^\alpha.n.$$

Since, $E_\alpha \sigma_{\beta-\alpha}(n) = E_\alpha(n)\sigma_{\beta-\alpha}(n) = n^\alpha \sigma_{\beta-\alpha}(n)$ therefore, $\sigma_{\beta-\alpha}(n) = 2n$. Since, $\sigma_{\beta-\alpha}(n) = (E_{\beta-\alpha} * e)(n) = (e * E_{\beta-\alpha})(n) = 2n$ (using, (4.5)) therefore,

$(E_{\beta-\alpha} * e)(n) = 2n^{\beta-\alpha} = 2E_{\beta-\alpha}$, for $\alpha = \beta - 1$. Hence n is $(E_{\beta-\alpha})e-$perfect number. ■

THEOREM 4.7. *For $\alpha = \beta - 1$, $(E_{\beta-\alpha})e-$perfect numbers coincide with the set of all perfect numbers.*

PROOF Let n is a $(E_{\beta-\alpha})e-$perfect number. Then, $(e * E_{\beta-\alpha})(n) = 2E_{\beta-\alpha}(n) = 2n^{\beta-\alpha}$. Since, $(e * E_{\beta-\alpha})(n) = (E_{\beta-\alpha} * e)(n) = \sigma_{\beta-\alpha}(n)$ (using, (4.5)) therefore, $\sigma_{\beta-\alpha}(n) = 2n^{\beta-\alpha}$. For $\alpha = \beta - 1$, we can write $\sigma(n) = 2n$. Hence n is a perfect number. ■

We have concluded that if $F(n) = E_\alpha \sigma_{\beta-\alpha}, f(n) = E_{\beta-\alpha}$ and $g(n) = e(n)$ then for $\alpha = \beta$, $f-$perfect numbers coincide with the set of all prime numbers. If $\alpha = \beta - 1$ and $\beta \geqslant 1$ and $S = 2n^\alpha$ be a fixed real number then f-perfect numbers, $[F, S]-$perfect numbers and fg-perfect numbers coincide with ordinary perfect numbers.

REFERENCES

[1] J. Sandor, *On Dedekind's arithmetical function*, Seminarul de t. Structurilor No. **5**, Univ. Timisoara, Romania, (1988), 1-15.

[2] J. Sandor, *On the composition of some arithmetic functions*, II, Journal of inequalities in pure and applied mathematics, Vol.**6**, issue **3**, article **73**, (2005), 1-37.

[3] J. Sandor, *Notes on the inequality $\phi(\psi(n)) < n$*, (1988), unpublished manuscript.

[4] M.V. Vassilev-Missana and K.T. Atanassov, *A new point of view on perfect and other similar numbers*, Advanced studies in contemporary mathematics **15** No.**2**, (2007), 153-169.

[5] V. Siva Rama Prasad and D. Ram Reddy, *On primitive unitary abundent numbers*, Indian J. Pure Appl. Math. **21(1)**, (1990), 40-44.

D.P. Shukla
Department of Mathematics & Astronomy, Lucknow University
Lucknow 226007

Shikha Yadav
Department of Mathematics & Astronomy, Lucknow University
Lucknow 226007

(Received: 18.04.2012)