COMMENTATIONES MATHEMATICAE Vol. 52, No. 2 (2012), 137-142

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Some Characterizations of Schur Matrices in Ultrametric Fields

Abstract. In this short paper, K denotes a complete, non-trivially valued, ultrametric field. Sequences and infinite matrices have entries in K. We prove a few characterizations of Schur matrices in K. We then deduce some non-inclusion theorems modelled on the results of Agnew [1] and Fridy [3] in the classical case.

2000 Mathematics Subject Classification: 40, 46.

Key words and phrases: Ultrametric field, Regular matrix, Schur matrix, Non-inclusion theorem, Steinhaus theorem.

1. Introduction and Preliminaries. In this paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field with valuation $|\cdot|$. Sequences and infinite matrices have entries in K. Given an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \dots$ and a sequence $x = \{x_k\}$, $x_k \in K$, $k = 0, 1, 2, \dots$, by the A-transform of the sequence $x = \{x_k\}$, we mean the sequence $Ax = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \dots,$$

where we suppose that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that $x = \{x_k\}$ is summable A or A-summable to ℓ. If $\lim_{n\to\infty} (Ax)_n = \ell$, whenever $\lim_{k\to\infty} x_k = \ell$, we say that A is regular. The following result is well-known.

Theorem 1.1 ([4]) *A is regular if and only if*

$$\sup_{n,k} |a_{nk}| < \infty; \tag{1}$$

$$\lim_{n\to\infty} a_{nk} = 0, \quad k = 0, 1, 2, \dots; \tag{2}$$

and

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk} = 1. \tag{3}$$

Given sequence spaces X, Y in K, we write $A = (a_{nk}) \in (X, Y)$ if $\{(Ax)_n\} \in Y$ whenever $x = \{x_k\} \in X$. We recall the following:

$$\ell_\infty = \{\,\{x_k\} : \sup_{k\geqslant 0} |x_k| < \infty\};$$
$$c = \{\,\{x_k\} : \lim_{k\to\infty} x_k = \ell, \quad \ell \in K\};$$
$$c_0 = \{\,\{x_k\} : \lim_{k\to\infty} x_k = 0\}.$$

For $x = \{x_k\} \in \ell_\infty$, define $||x|| = \sup_{k\geqslant 0} |x_k|$. We then know that ℓ_∞ is an ultrametric Banach space and c_0, c are closed subspaces of ℓ_∞.

2. Main Results. In this section, we first prove a few characterizations of Schur matrices, i.e., matrices which are in the matrix class (ℓ_∞, c).

We make use of the following definition which was introduced by Natarajan and Srinivasan in [6].

Definition 2.1 We say that $\lim_{n+k\to\infty} x_{nk} = x$, if for $\epsilon > 0$, the set $\{(n,k) \in \mathbb{N}^2 : |x_{nk} - x| \geqslant \epsilon\}$ is finite, $\mathbb{N}$ being the set of non-negative integers. In such a case, x is unique and x is called the limit of the double sequence $\{x_{nk}\}$.

It is easy to prove the following result ([6]).

Theorem 2.2 $\lim_{n+k\to\infty} x_{nk} = x$ *if and only if*

(i) $\lim_{k\to\infty} x_{nk} = x, \quad n = 0, 1, 2, \ldots;$

(ii) $\lim_{n\to\infty} x_{nk} = x, \quad k = 0, 1, 2, \ldots;$

and

(iii) $\lim_{n,k\to\infty} x_{nk} = x,$

where the double limit is taken in the Pringsheim sense, i.e., given $\epsilon > 0$, there exists a positive integer N such that $|x_{nk} - x| < \epsilon$, $n, k \geqslant N$.

Theorem 2.3 *The following statements are equivalent:*

(a) $A = (a_{nk}) \in (\ell_\infty, c_0)$;

(b) *(i)*

$$\lim_{k\to\infty} a_{nk} = 0, n = 0, 1, 2, \ldots; \tag{4}$$

and

(ii)

$$\lim_{n\to\infty} \sup_{k\geqslant 0} |a_{nk}| = 0. \tag{5}$$

(c) *(i)* *(2) holds*

and

(ii)

$$\lim_{k\to\infty} \sup_{n\geqslant 0} |a_{nk}| = 0. \tag{6}$$

(d)

$$\lim_{n+k\to\infty} a_{nk} = 0. \tag{7}$$

Proof In [5, p. 422], the author proved that (a) and (b) are equivalent. We now prove that (b) and (c) are equivalent. Let (b) hold. In view of (5), given $\epsilon > 0$, there exists a positive integer N such that

$$\sup_{k\geqslant 0} |a_{nk}| < \epsilon, \quad n \geqslant N. \tag{8}$$

Consequently, for fixed $k = 0, 1, 2, \ldots,$

$$|a_{nk}| < \epsilon, \quad n \geqslant N$$

so that

$$\lim_{n\to\infty} a_{nk} = 0, \quad k = 0, 1, 2, \ldots,$$

i.e., (2) holds. Now, for $n = 0, 1, 2, \ldots, N-1$, using (4), we can choose a positive integer U such that

$$\max_{0\leqslant n\leqslant N-1} |a_{nk}| < \epsilon, \quad k \geqslant U. \tag{9}$$

Already,

$$\sup_{n\geqslant N} |a_{nk}| < \epsilon, \quad k \geqslant U, \tag{10}$$

using (8). Combining (9) and (10), we have,

$$\sup_{n\geqslant 0} |a_{nk}| < \epsilon, \quad k \geqslant U,$$

so that (6) holds. Thus (c) holds, i.e., (b) implies (c). The reverse implication is similarly established.

We shall now prove that (c) and (d) are equivalent. It is clear that (c) implies (d), using Theorem 2.2. Conversely, let (d) hold. Using Theorem 2.2 again, (2) and (4) hold and

$$\lim_{n,k\to\infty} a_{nk} = 0. \tag{11}$$

(4) along with (11) imply (6). Thus (c) holds, i.e., (d) implies (c). This completes the proof of the theorem. ∎

THEOREM 2.4 *The following statements are equivalent:*

(a) $A = (a_{nk}) \in (\ell_\infty, c)$;

(b) *(i)* *(4) holds;*

and

(ii)

$$\lim_{n\to\infty} \sup_{k\geqslant 0} |a_{n+1,k} - a_{nk}| = 0; \tag{12}$$

(c) *(i)* *(4) holds;*

(ii)

$$\lim_{n\to\infty} a_{nk} \text{ exists}, \quad k = 0, 1, 2, \ldots;$$

and

(iii)

$$\lim_{k\to\infty} \sup_{n\geqslant 0} |a_{n+1,k} - a_{nk}| = 0. \tag{13}$$

(d) *(i)* *(4) holds;*

and

(ii)

$$\lim_{n+k\to\infty} (a_{n+1,k} - a_{nk}) = 0. \tag{14}$$

PROOF In [5, pp. 418–421], the author proved the equivalence of (a) and (b). The remaining part of the proof is an easy consequence of Theorem 2.3. For instance we shall prove the equivalence of (b) and (d). Both in (b) and (d), (4) ensures that $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$, $n = 0, 1, 2, \ldots$ is defined. Let (b) hold. Let $B = (b_{nk})$, $b_{nk} = a_{n+1,k} - a_{nk}$. Then $B = (b_{nk})$ satisfies (4) and (5) with a_{nk} replaced by b_{nk}. In view of Theorem 2.3,

$$\lim_{n+k\to\infty} b_{nk} = 0,$$

$$\textit{i.e.,} \quad \lim_{n+k\to\infty} (a_{n+1,k} - a_{nk}) = 0.$$

Thus (b) implies (d). Converse is similarly proved using Theorem 2.3. The proof of the theorem is now complete. ■

As an application of Theorem 2.3 and Theorem 2.4, we shall now prove some non-inclusion theorems in K modelled on the results of Agnew [1] and Fridy [3] in the classical case.

In this context, we recall the following:

$$\ell_1 = \{ \{x_k\} : \sum_{k=0}^{\infty} |x_k| < \infty\};$$

$$c_A = \{ \{x_k\} : \{(Ax)_n\} \in c\};$$

and

$$(c_0)_A = \{ \{x_k\} : \{(Ax)_n\} \in c_0\}.$$

Agnew [1] proved the following result in the classical case.

THEOREM 2.5 *If A is regular and satisfies (11), then* $c \subsetneqq c_A$.

Fridy [3] proved a non-inclusion theorem in the following form.

THEOREM 2.6 *If* A, B *are regular and if* A *satisfies (11) and* B *does not, i.e.,* $\lim_{n,k\to\infty} b_{nk} \neq 0$*, then* $c_A \not\subseteq c_B$.

We have the following results in the ultrametric set up.

THEOREM 2.7 *If* $A = (a_{nk})$ *satisfies (4),* $\lim_{n\to\infty} a_{nk}$ *exists,* $k = 0, 1, 2, \ldots$ *and*

$$\lim_{n,k\to\infty} (a_{n+1,k} - a_{nk}) = 0, \tag{15}$$

then $c \subsetneqq c_A$.

PROOF Since A satisfies (4), $(Ax)_n$ is defined, $n = 0, 1, 2, \ldots$ for any $x = \{x_k\} \in \ell_\infty$. Again, in view of (4), we have,

$$\lim_{k\to\infty} (a_{n+1,k} - a_{nk}) = 0, \quad n = 0, 1, 2, \ldots. \tag{16}$$

(15) and (16) together imply that (13) holds.
Since $\lim_{n\to\infty} a_{nk}$ exists, $k = 0, 1, 2, \ldots$,

$$\lim_{n\to\infty} (a_{n+1,k} - a_{nk}) = 0, \quad k = 0, 1, 2, \ldots. \tag{17}$$

Appealing to (13), (17) and Theorem 2.3, we have,

$$\lim_{n\to\infty} \{(Ax)_{n+1} - (Ax)_n\} = 0, \quad x = \{x_k\} \in \ell_\infty,$$

i.e., $\{(Ax)_n\}$ is a Cauchy sequence in K. Since K is complete, $\{(Ax)_n\} \in c$, $x = \{x_k\} \in \ell_\infty$, i.e., A is a Schur matrix. Consequently $c \subsetneqq c_A$. ■

THEOREM 2.8 *If* $A = (a_{nk})$*,* $B = (b_{nk})$ *are matrices such that each row of* A *and* B *is a null sequence, each column of* A *and* B *is a convergent sequence and* A *satisfies (15), while* B *does not satisfy (15), then* $c_A \not\subseteq c_B$.

PROOF Under the hypotheses of the theorem $A \in (\ell_\infty, c)$, while $B \notin (\ell_\infty, c)$. Consequently $c_A \not\subseteq c_B$. ■

REMARK 2.9 In Theorem 2.7 and Theorem 2.8, A cannot be regular in view of Steinhaus theorem (see [5]).

The effective analogue of the classical space ℓ_1 in the ultrametric set up seems to be the space c_0. Consequently, we have the following results modelled on those of Fridy (see [3], Theorem 3.1, Theorem 3.2), which are easy consequences of Theorem 2.3.

THEOREM 2.10 *If* $A = (a_{nk})$ *satisfies (7), then* $c_0 \subsetneqq (c_0)_A$.

THEOREM 2.11 *If* $A = (a_{nk})$ *satisfies (7) and* $B = (b_{nk})$ *does not satisfy (7), then* $(c_0)_A \nsubseteq (c_0)_B$.

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(Received: 8.01.2012)