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On the construction of common fixed points for semigroups of nonlinear mappings in uniformly convex and uniformly smooth Banach spaces

Abstract. *Let C be a bounded, closed, convex subset of a uniformly convex and uniformly smooth Banach space X. We investigate the weak convergence of the generalized Krasnosel'skii-Mann and Ishikawa iteration processes to common fixed points of semigroups of nonlinear mappings $T_t : C \to C$. Each of T_t is assumed to be pointwise Lipschitzian, that is, there exists a family of functions $\alpha_t : C \to [0, \infty)$ such that $\|T_t(x) - T_t(y)\| \leqslant \alpha_t(x)\|x - y\|$ for $x, y \in C$. The paper demonstrates how the weak compactness of C plays an essential role in proving the weak convergence of these processes to common fixed points.*

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1. Introduction. Let C be a bounded, closed, convex subset of a Banach space X. Let us consider a pointwise Lipschitzian semigroup of nonlinear mappings, that is, a family of mappings $T_t : C \to C$, where $t \in [0, \infty)$, satisfying the following conditions: $T_0(x) = x$, $T_{s+t}(x) = T_s(T_t(x))$, $t \mapsto T_t(x)$ is strong continuous for each $x \in C$, and each T_t is pointwise Lipschitzian. The latter means that there exists a family of functions $\alpha_t : C \to [0, \infty)$ such that $\|T_t(x) - T_t(y)\| \leqslant \alpha_t(x)\|x - y\|$ for $x, y \in C$. Such a situation is quite typical in mathematics and applications. For instance, in the theory of dynamical systems, the Banach space X would define the state space and the mapping $(t, x) \to T_t(x)$ would represent the evolution function of a dynamical system. Common fixed points of such a semigroup can be interpreted as points that are fixed during the state space transformation T_t at any given point of time t. In the setting of this paper, the state space may be an infinite dimensional

Banach space. Therefore, it is natural to apply these result not only to deterministic dynamical systems but also to stochastic dynamical systems.

Let us note that the existence of common fixed points for asymptotic pointwise nonexpansive semigroups has been recently proved in [18]. The proof is of analytical nature and does not describe any algorithm for constructing fixed points. While it was demonstrated that under reasonable assumptions the generalized Krasnosel'skii-Mann and Ishikawa iteration processes converge weakly to a common fixed point of an pointwise Lipschitzian semigroup [21], provided X has additionally the Opial property, however many important spaces like L^p for $1 < p \neq 2$ do not possess the Opial property. Throughout the current paper we do not assume the Opial property, we do assume that X is uniformly convex and uniformly smooth. L^p for $p > 1$ are prime examples of such spaces. To deal with this situation, new type of assumptions and constructs need to be applied. Hence the current paper opens a new research direction as well as introduces new techniques specific to semigroups acting in uniformly smooth Banach spaces. It is worthwhile to turn the reader's attention to the fact that the weak compactness of C plays an essential role in proving the weak convergence of these processes to common fixed points.

There exist important characterizations of pointwise Lipschitzian semigroups which satisfy asymptotic pointwise nonexpansive conditions, assuming that all members of the semigroup are continuously Fréchet differentiable on an open convex set A containing C, see [18]. The semigroup is asymptotic pointwise nonexpansive on C if and only if for each $x \in C$

$$\limsup_{t\to\infty} \|(T_t)_x^{'}\| \leqslant 1. \tag{1}$$

The existence of common fixed points for families of contractions and nonexpansive mappings have been investigated since the early 1960s. The asymptotic approach for finding common fixed points of semigroups of Lipschitzian (but not pointwise Lipschitzian) mappings has been also investigated for some time, see e.g. [33]. It is worthwhile mentioning the recent studies on the discrete case, when the parameter set for the semigroup is equal to $\{0, 1, 2, 3, ...\}$ and $T_n = T^n$, the n-th iterate of an asymptotic pointwise nonexpansive mapping, i.e. such a $T : C \to C$ that there exists a sequence of functions $\alpha_n : C \to [0, \infty)$ with $\|T^n(x)-T^n(y)\| \leqslant \alpha_n(x)\|x-y\|$ and $\limsup_{n\to\infty} \alpha_n(x) \leqslant 1$. In [16] the authors proved the existence of fixed points for asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings in Banach spaces, which was then extended to metric spaces [8], and to modular function spaces [13], [14], see also [4]. Recently, the author proved existence of common fixed points for semigroups of nonlinear contractions and nonexpansive mappings in modular functions spaces, [19].

Several authors studied the generalizations of known iterative fixed point construction processes like the Mann process (see e.g. [23, 6]) or the Ishikawa process (see e.g. [9]) to the case of asymptotic and pointwise asymptotic nonexpansive mappings. There exists an extensive literature on the subject of iterative fixed point construction processes for asymptotically nonexpansive mappings in Hilbert, Banach and metric spaces, see e.g. [1, 28, 26, 7, 29, 30, 32, 36, 37, 33, 34, 3, 35, 27, 25, 12, 8, 5, 10, 24, 11, 17, 4, 20, 21] and the works referred there.

The paper is organized as follows:

(a) Section 2 provides necessary preliminary material.

(b) Section 3 defines pointwise Lipschitzian semigroups and associated notions.

(c) Section 4 introduces the important technique of the asymptotic pointwise nonexpansive sequences.

(d) Section 5 introduces the relevant version of the Demiclosedness Principle.

(e) Section 6 provides the proof of the weak convergence theorem for the generalized Mann process.

(f) Section 6 provides the proof of the weak convergence theorem for the generalized Ishikawa process.

2. Preliminaries. The notion of bounded away sequences of real numbers will be used extensively throughout this paper.

Definition 2.1 A sequence $\{t_n\} \subset (0,1)$ is called bounded away from 0 if there exists $0 < a < 1$ such that $t_n > a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0,1)$ is called bounded away from 1 if there exists $0 < b < 1$ such that $t_n < b$ for every $n \in \mathbb{N}$.

The following elementary, easy to prove, lemma will be used in this paper.

Lemma 2.2 *[3] Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{d_{k,n}\}$ is a doubly-index sequence of real numbers which satisfy*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} \leqslant 0, \text{ and } r_{k+n} \leqslant r_k + d_{k,n}$$

for each $k, n \geqslant 1$. Then $\{r_k\}$ converges to an $r \in \mathbb{R}$.

The technique of approximate fixed point sequences will play an important role in proving fixed convergence to common fixed points for semigroups of mappings. Let us recall that given $T : C \to C$, a sequence $\{x_k\} \subset C$ is called an approximate fixed point sequence for T if $\|T(x_k) - x_k\| \to 0$ as $k \to \infty$.

The following property of uniformly convex Banach spaces will play an important role in this paper.

Lemma 2.3 *[30, 38] Let X be a uniformly convex Banach space. Let $\{c_n\} \subset (0,1)$ be bounded away from 0 and 1, and $\{u_n\}, \{v_n\} \subset X$ be such that*

$$\limsup_{n\to\infty} \|u_n\| \leqslant a, \ \limsup_{n\to\infty} \|v_n\| \leqslant a, \ \lim_{n\to\infty} \|c_n u_n + (1-c_n)v_n\| = a.$$

Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

It is known that if X is uniformly smooth then the following inequality is true, see e.g. [31]:

Proposition 2.4 *Let X be a uniformly smooth Banach space. Then*

$$\frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \leqslant \frac{1}{2}\|x+h\|^2 \leqslant \frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + \|h\|^2 \tag{2}$$

for all $x, h \in X$, where J is the normalized duality map from X to X^* defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},$$

$\langle \cdot, \cdot \rangle$ is the duality pairing between X and X^*.

We will also use the following parallelogram inequality.

Proposition 2.5 *[16, 36] Let X be a uniformly convex Banach space. For each $d > 0$ there exists a continuous function $\lambda : [0,\infty) \to [0,\infty)$ such that $\lambda(t) = 0 \Leftrightarrow t = 0$, and*

$$\|\alpha x + (1-\alpha)y\|^2 \leqslant \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\lambda(\|x-y\|), \tag{3}$$

for any $\alpha \in [0,1]$ and all $x, y \in X$ such that $\|x\| \leqslant d$ and $\|y\| \leqslant d$.

3. Pointwise Lipschitzian Semigroups. Throughout this paper X will denote a Banach space, and C a nonempty, bounded, closed and convex subset of X. Throughout this paper we will denote $J = [0,\infty)$. The notation $t \to \infty$ will mean that t tends to infinity over J.

Let us start with more formal definitions of pointwise Lipschitzian mappings and pointwise Lipschitzian semigroups of mappings, and associated notational conventions.

Definition 3.1 We say that $T : C \to C$ is a pointwise Lipschitzian mapping if there exists a function $\alpha : C \to [0,\infty)$ such that

$$\|T(x) - T(y)\| \leqslant \alpha(x)\|x-y\| \ for\ all\ x, y \in C. \tag{4}$$

If the function $\alpha(x) < 1$ for every $x \in C$, then we say that T is a pointwise contraction. Similarly, if $\alpha(x) \leqslant 1$ for every $x \in C$, then T is said to be a pointwise nonexpansive mapping.

Definition 3.2 A one-parameter family $\mathcal{F} = \{T_t; t \in J\}$ of mappings from C into itself is said to be a pointwise Lipschitzian semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;

(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in J$;

(iii) for each $t \in J$, T_t is a pointwise Lipschitzian mapping, i.e. there exists a function $\alpha_t : C \to [0,\infty)$ such that

$$\|T_t(x) - T_t(y)\| \leqslant \alpha_t(x)\|x-y\| \ for\ all\ x, y \in C. \tag{5}$$

(iv) for each $x \in C$, the mapping $t \mapsto T_t(x)$ is strong continuous.

For each $t \in J$ let $F(T_t)$ denote the set of its fixed points. Note that if $x \in F(T_t)$ then x is a periodic point (with period t) for the semigroup $\mathcal{F}$, i.e. $T_{kt}(x) = x$ for every natural k. Define then the set of all common set points for mappings from $\mathcal{F}$ as the following intersection

$$F(\mathcal{F}) = \bigcap_{t \in J} F(T_t).$$

The common fixed points are frequently interpreted as the stationary points of the semigroup $\mathcal{F}$.

Definition 3.3 Let $\mathcal{F}$ be a pointwise Lipschitzian semigroup. $\mathcal{F}$ is said to be asymptotic pointwise nonexpansive if $\limsup_{t\to\infty} \alpha_t(x) \leqslant 1$ for every $x \in C$.

Denoting $a_0 \equiv 1$ and $a_t(x) = \max(\alpha_t(x), 1)$ for $t > 0$, we note that without loss of generality we can assume that $\mathcal{F}$ is asymptotically nonexpansive if

$$\|T_t(x) - T_t(y)\| \leqslant a_t(x)\|x - y\| \; for\ all\ x, y \in C, \; t \in J, \tag{6}$$

$$\lim_{t\to\infty} a_t(x) = 1, \; a_t(x) \geqslant 1 \; for\ all\ x \in C, \; and\ t \in J. \tag{7}$$

Define $b_t(x) = a_t(x) - 1$. In view of (7), we have

$$\lim_{t\to\infty} b_t(x) = 0. \tag{8}$$

The above notation will be consistently used throughout this paper.

Definition 3.4 By $\mathcal{S}(C)$ we will denote the class of all asymptotic pointwise nonexpansive semigroups on C such that

$$M_t = \sup\{a_t(x) : x \in C\} < \infty, \; for\ every\ t \in J, \tag{9}$$

$$\limsup_{t\to\infty} M_t = 1. \tag{10}$$

Note that we do not assume that all functions a_t are bounded by a common constant. Therefore, we do not assume that $\mathcal{F}$ is uniformly Lipschitzian.

Definition 3.5 We will say that a semigroup $\mathcal{F} \in \mathcal{S}(C)$ is equicontinuous if the family of mappings $\{t \mapsto T_t(x) : x \in C\}$ is equicontinuous at $t = 0$.

The next lemma is an important generalization of the nonexpansive mapping result by Bruck [2] to the case of any pointwise Lipschitzian pointwise mapping.

LEMMA 3.6 *[18] Let X be a uniformly convex Banach space, and let $C \subset X$ be nonempty, bounded, closed and convex. There exists a strictly increasing, convex continuous function $\gamma_2 : [0, \infty) \to [0, \infty)$ with $\gamma_2(0) = 0$ such that for every pointwise Lipschitzian mapping $T : C \to C$, every $c \in [0, 1]$ and every $x, y \in C$ there holds*

$$\gamma_2\Big(\frac{\|T(cx + (1-c)y) - cT(x) - (1-c)T(y)\|}{\alpha(cx + (1-c)y)}\Big) \leqslant \|x - y\| - \frac{\|T(x) - T(y)\|}{\alpha(cx + (1-c)y)} \ . \tag{11}$$

The following result will be used in this paper to ensure existence of common fixed points.

THEOREM 3.7 *[18] Assume X is uniformly convex. Let $\mathcal{F}$ be an asymptotically nonexpansive pointwise Lipschitzian semigroup on C. Then $\mathcal{F}$ has a common fixed point and the set $F(\mathcal{F})$ of common fixed points is closed and convex.*

Using Kirk's result [15] (Proposition 2.1), Kozlowski [18] proved the following proposition.

PROPOSITION 3.8 *Let $\mathcal{F}$ be a semigroup of mappings on C. Assume that all mappings $S_t \in \mathcal{F}$ are continuously Fréchet differentiable on an open convex set A containing C then $\mathcal{F}$ is asymptotic pointwise nonexpansive on C if and only if for each $x \in C$*

$$\limsup_{t\to\infty} \|(T_t)_x^{'}\| \leqslant 1. \tag{12}$$

This result, combined with Theorem 3.7, produces the following fixed point theorem.

THEOREM 3.9 *[18] (Theorem 3.5) Assume X is uniformly convex. Let $\mathcal{F}$ be a pointwise Lipschitzian semigroup on C. Assume that all mappings $T_t \in \mathcal{F}$ are continuously Fréchet differentiable on an open convex set A containing C and for each $x \in C$*

$$\limsup_{t\to\infty} \|(T_t)_x^{'}\| \leqslant 1. \tag{13}$$

Then $\mathcal{F}$ has a common fixed point and the set $F(\mathcal{F})$ of common fixed points is closed and convex.

Because of the above, all the results of this paper can be applied to the semigroups of nonlinear mappings satisfying condition (13). This approach may be very useful for applications provided the Fréchet derivatives can be estimated.

We will use extensively the following notion of a generating set.

DEFINITION 3.10 A set $A \subset J$ is called a generating set for the parameter semigroup J if for every $0 < u \in J$ there exist $m \in \mathbb{N}$, $s \in A$, $t \in A$ such that $u = ms + t$.

Remark 3.11 Note that for $J = [0, \infty)$ any interval $A = [0, \alpha]$ for $\alpha > 0$ is a generating set.

The following three technical results related to the role of generating sets will be used in the sequel, see also [21].

Lemma 3.12 *Let C be a nonempty, bounded, closed and convex subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. If $\|T_s(x_n) - x_n\| \to 0$ for an $s \in J$ as $n \to \infty$ then for any $m \in \mathbb{N}$, $\|T_{ms}(x_n) - x_n\| \to 0$ as $n \to \infty$*

Proof It follows from the fact that every a_t is a bounded function that there exists a finite constant $M > 0$ such that

$$\sum_{j=1}^{m-1} sup\{a_{js}(x) : x \in C\} \leqslant M. \tag{14}$$

It follows from

$$\begin{aligned} \|T_{ms}(x_n) - x_n\| &\leqslant \sum_{j=1}^{m-1} \|T_{(j+1)s}(x_n) - T_{js}(x_n)\| + \|T_s(x_n) - x_n\| \\ &\leqslant \|T_s(x_n) - x_n\| \Big(\sum_{j=1}^{m-1} a_{js}(x_n) + 1 \Big) \leqslant (M+1)\|T_s(x_n) - x_n\| \end{aligned} \tag{15}$$

that

$$\lim_{n\to\infty} \|T_{ms}(x_n) - x_n\| = 0, \tag{16}$$

which completes the proof. ∎

Lemma 3.13 *Let C be a nonempty, bounded, closed and convex subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. If $\{x_k\} \subset C$ is a approximate fixed point sequence for T_s for any $s \in A$ where A is a generating set for J then $\{x_k\}$ is a approximate fixed point sequence for T_s for any $s \in J$.*

Proof Let $s, t \in A$ and $m \in \mathbb{N}$. We need to show that $\|T_{ms+t}(x_n) - x_n\| \to 0$ as $n \to \infty$. Indeed,

$$\begin{aligned} \|T_{ms+t}(x_n) - x_n\| &\leqslant \|T_{ms+t}(x_n) - T_{ms}(x_n)\| + \|T_{ms}(x_n) - x_n\| \\ &\leqslant a_{ms}(x_n)\|T_t(x_n) - x_n\| + \|T_{ms}(x_n) - x_n\|, \end{aligned}$$

which tends to zero by boundedness of the function a_{ms} and by Lemma 3.12. ∎

Lemma 3.14 *Let $\mathcal{F} \in \mathcal{S}(C)$ be equicontinuous and $\overline{B} = A \subset J$. If $\{x_k\} \subset C$ is an approximate fixed point sequence for T_t for every $t \in B$ then $\{x_k\}$ is an approximate fixed point sequence for T_t for every $t \in A$.*

Proof Let $s \in A$, then there exists a sequence $\{s_n\} \subset B$ such that $s_n \to s$. Note that

$$\begin{aligned} \|T_s(x_k) - x_k\| &\leqslant \|T_s(x_k) - T_{s_n}(x_k)\| + \|T_{s_n}(x_k) - x_k\| \\ &\leqslant \sup_{x\in C} a_{min(s,s_n)}(x) \sup_{x\in C} \|T_{|s-s_n|}(x) - x\| + \|T_{s_n}(x_k) - x_k\|. \end{aligned} \tag{17}$$

Fix $\varepsilon > 0$. By equicontinuity of $\mathcal{F}$ and by (9) there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in C} a_{min(s,s_{n_0})}(x) \sup_{x\in C} \|T_{|s-s_{n_0}|}(x) - x\| < \frac{\varepsilon}{2}. \tag{18}$$

Since $\{x_k\}$ is an approximate fixed point for $T_{s_{n_0}}$ we can find $k_0 \in \mathbb{N}$ such that for every natural $k \geqslant k_0$

$$\|T_{s_{n_0}}(x_k) - x_k\| < \frac{\varepsilon}{2}. \tag{19}$$

By substituting (18) and (19) into (17) we get $\|T_s(x_k) - x_k\| < \varepsilon$ for large k. Hence $\{x_k\}$ is an approximate fixed point for T_s as claimed. ■

4. Asymptotic Pointwise Nonexpansive Sequences. We extend the notion of asymptotic pointwise nonexpansiveness to sequences of mappings acting within the set C.

Definition 4.1 Let C be a subset of a Banach space X. We say that a sequence of mappings $T_n : C \to C$ is an asymptotic pointwise nonexpansive sequence if there exists a sequence of mappings $c_n : C \to [1, \infty)$ such that

$$\|T_n(x) - T_n(y)\| \leqslant c_n(x)\|x - y\| \ for\ all\ x, y \in C,\ n \in \mathbb{N}, \tag{20}$$

$$\lim_{n\to\infty} c_n(x) = 1,\ for\ all\ x \in C. \tag{21}$$

Define $d_n(x) = c_n(x) - 1$. In view of (21), we have

$$\lim_{n\to\infty} d_n(x) = 0. \tag{22}$$

By $\mathcal{A}(C)$ we will denote the class of all asymptotic pointwise nonexpansive sequences of mappings $T_n : C \to C$.

Definition 4.2 Let $\{T_n\} \in \mathcal{A}(C)$. We say that a sequence $\{x_n\}$ of elements of C is generated by $\{T_n\}$ if $x_1 \in C$ and $x_{n+1} = T_n(x_n)$ for $n \geqslant 1$.

Let us introduce a narrower class of the asymptotic pointwise nonexpansive sequences satisfying some boundedness properties.

Definition 4.3 Let us define $\mathcal{A}_c(C)$ as a class of all $\{T_n\} \in \mathcal{A}(C)$ such that

(a) For every $x \in C$ there exists a positive constant ε_x and a sequence of finite positive constants $\{N_n(x)\}$ such that

$$\sum_{n=1}^{\infty} N_n(x) < \infty, \tag{23}$$

and for every $y \in C \cap B(x, \varepsilon_x)$ there holds

$$d_n(y) \leqslant N_n(x), \tag{24}$$

(b) Every function c_n is bounded.

The following result generalizes Lemma 4.1 and Lemma 5.1 from [17], see also [20].

Lemma 4.4 *Let C be a subset of a Banach space X. Let $\{T_k\} \in \mathcal{A}_c(C)$ and let $\{x_k\}$be a sequence generated by $\{T_k\}$. Let $w \in \bigcap_{k=1}^{\infty} F(T_k)$. Then there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$.*

Proof Take any $k, n \in \mathbb{N}$ and observe the following

$$\|x_{k+1} - w\| = \|T_k(x_k) - T_k(w)\| \leqslant c_k(w)\|x_k - w\|, \tag{25}$$

hence

$$\|x_{k+n} - w\| \leqslant \prod_{i=k}^{k+n-1} c_i(w)\|x_k - w\|. \tag{26}$$

Passing with $n \to \infty$ we have

$$\limsup_{m\to\infty} \|x_m - w\| \leqslant \prod_{i=k}^{\infty} c_i(w)\|x_k - w\|. \tag{27}$$

By (23) and (24), $\sum_{i=1}^{\infty} d_i(w) < \infty$. Remembering that $c_i = 1 + d_i$ we conclude that $\prod_{i=k}^{\infty} c_i(w) \to 1$ as $k \to \infty$. Hence, passing with $k \to \infty$ we obtain

$$\limsup_{m\to\infty} \|x_m - w\| \leqslant \liminf_{k\to\infty} \|x_k - w\|, \tag{28}$$

as desired. ∎

The following result provides an important technique to be used for proving the weak convergence of the fixed point processes.

LEMMA 4.5 *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X.Let $\{T_k\} \in \mathcal{A}_c(C)$ and let $\{x_k\}$be a sequence generated by $\{T_k\}$. Let $w_1,\ w_2 \in \bigcap_{k=1}^{\infty} F(T_k)$.Then there exists $t_0 \in [0,1]$ such that the following limit*

$$\lim_{k\to\infty} \|tx_k + (1-t)w_1 - w_2\| \tag{29}$$

exists for every $t \in [0, t_0]$.

PROOF Let us denote

$$S_{k,0}(x) = x, \tag{30}$$

$$S_{k,m} = T_{k+m-1}T_{k+m-2}\dots T_k,\ for\ m \geqslant 1 \tag{31}$$

$$f_k(t) = \|tx_k + (1-t)w_1 - w_2\|, \tag{32}$$

and

$$g_{k,m}(t) = \|S_{k,m}(tx_k + (1-t)w_1) - (tx_{k+m} + (1-t)w_1)\|. \tag{33}$$

Since

$$\|T_k(u) - T_k(v)\| \leqslant c_k(u)\|u - v\|, \tag{34}$$

it follows that

$$\|S_{k,m}(u) - S_{k,m}(v)\| \leqslant \prod_{j=k}^{k+m-1} c_j\Big(S_{k,j-k}(u)\Big)\|u - v\|, \tag{35}$$

for every $u, v \in C$, i.e. every $S_{k,m}$ is a pointwise Lipschitzian mapping in the sense of Definition 3.1. Let us denote $u_t^k = tx_k + (1-t)w_1$ and

$$h_{k,m}(u) = \prod_{j=k}^{k+m-1} c_j\Big(S_{k,j-k}(u)\Big). \tag{36}$$

Note that $S_{k,m}(w_1) = w_1$ which implies the following

$$\|S_{k,m}(u) - w_1\| = \|S_{k,m}(u) - S_{k,m}(w_1)\| \leqslant \prod_{j=k}^{\infty} c_j(w_1)\|u - w_1\|, \tag{37}$$

where $\prod_{j=k}^{\infty} c_j(w_1) \to 1$ as $k \to \infty$ in view of (23) and (24). Observe also that

$$\|u_t^k - w_1\| = \|tx_k + (1-t)w_1 - w_1\| = t\|x_k - w_1\| \leqslant t\ diam(C). \tag{38}$$

From (37) and (38) it follows that

$$\|S_{k,m}(u_t^k) - w_1\| \leqslant t \prod_{j=k}^{\infty} c_j(w_1)\ diam(C). \tag{39}$$

Take ε_{w_1} existing in view of Definition 4.3. By (39) there exists k_0 and t_0 such that

$$\|S_{k,m}(u_t^k) - w_1\| \leqslant \varepsilon_{w_1} \tag{40}$$

for every $m \in \mathbb{N}$, every natural $k \geqslant k_0$ and every $t \in (0, t_0)$. By (23) and (24) the following is true

$$\sum_{j=k}^{\infty} d_j\Big(S_{k,j-k}(u_t^k)\Big) \leqslant \sum_{j=k}^{\infty} N_{n_j}(x) \tag{41}$$

for $k \geqslant k_0$ and $t \in (0, t_0)$. Since the right-hand side tends to zero as $k \to \infty$ we obtain the following

$$h_k(u_t^k) := \prod_{j=k}^{\infty} c_j\Big(S_{k,j-k}(u_t^k)\Big) \to 1 \tag{42}$$

as $k \to \infty$. Note that $0 < h_{k,m}(u_t^k) < \infty$ and that $h_{k,m}(u_t^k) \to h_k(u_t^k)$ as $m \to \infty$.

Using (35), the fact that $S_{k,m}(x_k) = x_{k+m}$, $w_1 \in F(S_{k,m})$, and Lemma 3.6 applied to $S_{k,m}$ we can calculate $g_{k,m}(t)$ as follows:

$$\begin{aligned} g_{k,m}(t) &= \|S_{k,m}(tx_k + (1-t)w_1) - (tS_{k,m}(x_k) + (1-t)S_{k,m}(w_1))\| \\ &\leqslant h_{k,m}(u_t^k)\gamma_2^{-1}\Big(\|x_k - w_1\| - \frac{\|x_{k+m} - w_1\|}{h_{k,m}(u_t^k)}\Big), \end{aligned} \tag{43}$$

or

$$g_{k,m}(t) \leqslant h_{k,m}(u_t^k)\gamma_2^{-1}\Big(\|x_k - w_1\| - \|x_{k+m} - w_1\| + \Big(1 - \frac{1}{h_{k,m}(u_t^k)}\Big) diam(C)\Big). \tag{44}$$

Observe that

$$\begin{aligned} f_{k+m}(t) &= \|tx_{k+m} + (1-t)w_1 - w_2\| \\ &\leqslant g_{k,m}(t) + \|S_{k,m}(tx_k + (1-t)w_1) - w_2\| \\ &\leqslant g_{k,m}(t) + \prod_{j=k}^{k+m-1} a_{n_j}(w_2)\|tx_k + (1-t)w_1) - w_2\| \\ &\leqslant g_{k,m}(t) + G_k f_k(t), \end{aligned} \tag{45}$$

where $0 < G_k = \prod_{j=k}^{\infty} a_{n_j}(w_2) < \infty$; note that $G_k \to 1$. Incorporating (44) into (45) we get

$$f_{k+m}(t) \leqslant h_{k,m}(u_t^k)\gamma_2^{-1}\Big(\|x_k - w_1\| - \|x_{k+m} - w_1\| + \Big(1 - \frac{1}{h_{k,m}(u_t^k)}\Big) diam(C)\Big) + G_k f_k(t). \tag{46}$$

Let $r = \lim_{n\to\infty} \|x_n - w_1\|$ existing by Lemma 4.4. Passing with $m \to \infty$ in both sides of (46) and using continuity of γ_2^{-1} we arrive at

$$\limsup_{n\to\infty} f_n(t) \leqslant h_k(u_t^k)\gamma_2^{-1}\Big(\|x_k - w_1\| - r + \Big(1 - \frac{1}{h_k(u_t^k)}\Big) diam(C)\Big) + G_k f_k(t). \tag{47}$$

Passing with $k \to \infty$, and remembering that $\|x_k - w_1\| \to r$, $h_k(u_t^k) \to 1$, $G_k \to 1$, $\gamma_2^{-1}(0) = 0$ and γ_2^{-1} is continuous, we obtain

$$\limsup_{n\to\infty} f_n(t) \leqslant \liminf_{k\to\infty} f_k(t), \tag{48}$$

which completes the proof. ■

The following lemma introduces the key technique for proving the weak convergence of the iterative processes in uniformly smooth Banach spaces.

LEMMA 4.6 *Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X. Let $\{T_k\} \in \mathcal{A}_c(C)$ and let $\{x_k\}$be a sequence generated by $\{T_k\}$. Let $w_1,\ w_2 \in \bigcap_{k=1}^{\infty} F(T_k)$. Then for any two weak cluster points y, z of the sequence $\{x_n\}$ there holds*

$$\langle y - z, J(w_1 - w_2)\rangle = 0. \tag{49}$$

PROOF A straightforward calculation shows that in order to obtain (49) it suffices to show that for any given $w_1, w_2 \in \bigcap_{k=1}^{\infty} F(T_k)$ the limit $\lim_{n\to\infty} \langle x_n, J(w_1 - w_2)\rangle$ exists. Taking $x = w_1 - w_2$ and $h = t(x_n - w_1)$, where $t \in (0, 1)$, into inequality (2) we get

$$\begin{aligned}\frac{1}{2}\|w_1 - w_2\|^2 + \langle t(x_n - w_1), J(w_1 - w_2)\rangle \ &\leqslant \ \frac{1}{2}\|tx_n + (1-t)w_1 - w_2\|^2 \\ \leqslant \frac{1}{2}\|w_1 - w_2\|^2 + \langle t(x_n - w_1), J(w_1 - w_2)\rangle &+ \ \|t(x_n - w_1)\|^2 \\ \leqslant \frac{1}{2}\|w_1 - w_2\|^2 + \langle t(x_n - w_1), J(w_1 - w_2)\rangle &+ \ t^2 diam(C)^2\end{aligned} \tag{50}$$

Passing with $n \to \infty$ and using Lemma 4.5 we obtain the following

$$\begin{aligned}\frac{1}{2}\|w_1 - w_2\|^2 + t \limsup_{n\to\infty} \langle (x_n - w_1), J(w_1 - w_2)\rangle \\ \leqslant \ \lim_{n\to\infty} \frac{1}{2}\|tx_n + (1-t)w_1 - w_2\|^2 \\ \leqslant \frac{1}{2}\|w_1 - w_2\|^2 + t \liminf_{n\to\infty} \langle x_n - w_1, J(w_1 - w_2)\rangle + \ t^2 diam(C)^2.\end{aligned} \tag{51}$$

Hence

$$\limsup_{n\to\infty} \langle (x_n - w_1), J(w_1 - w_2)\rangle \leqslant \liminf_{n\to\infty} \langle x_n - w_1, J(w_1 - w_2)\rangle + \ t\ diam(C)^2, \tag{52}$$

and letting $t \to 0^+$ we conclude that the limit $\lim_{n\to\infty} \langle x_n, J(w_1 - w_2)\rangle$ exists. ■

5. The Demiclosedness Principle. The following version of the Demiclosedness Principle will be used in the proof of our main convergence theorems. There exist several versions of the Demiclosedness Principle for the case of asymptotic nonexpansive mappings, see e.g. [22, 7, 37, 17, 21, 20].

THEOREM 5.1 *Let X be a uniformly convex and uniformly smooth Banach space X. Let C be a nonempty, bounded, closed and convex subset of X, and let $\mathcal{F} \in \mathcal{S}(C)$. Assume that there exists $w \in X$ and $\{x_n\} \subset C$ such that $x_n \rightharpoonup w$. Assume that there exists an $s \in J$ such that $\|T_s(x_n) - x_n\| \to 0$ as $n \to \infty$. Then $w \in F(T_{ks})$ for any natural k.*

PROOF Define a type $\varphi(x) = \limsup_{n\to\infty} \|x_n - x\|$ for $x \in C$. Let us fix $m \in \mathbb{N}$, $m > 2$ and observe that

$$\begin{aligned}\|T_{ms}(x_n) - x\| &\leqslant \sum_{i=1}^{m} \|T_{is}(x_n) - T_{(i-1)s}(x_n)\| + \|x_n - x\| \\ &\leqslant \|T_s(x_n) - x_n\|\Big(\sum_{i=2}^{m} a_{(i-1)s}(x_n) + 1\Big) + \|x_n - x\|.\end{aligned}$$

Since all functions a_i are bounded and $\|T_s(x_n) - x_n\| \to 0$, it follows that

$$\limsup_{n\to\infty} \|T_{ms}(x_n) - x\| \leqslant \limsup_{n\to\infty} \|x_n - x\| = \varphi(x).$$

On the other hand, by Lemma 3.12, we have

$$\varphi(x) \leqslant \limsup_{n\to\infty} \|x_n - T_{ms}(x_n)\| + \limsup_{n\to\infty} \|T_{ms}(x_n) - x\| = \limsup_{n\to\infty} \|T_{ms}(x_n) - x\|.$$

Hence,

$$\varphi(x) = \limsup_{n\to\infty} \|T_{ms}(x_n) - x\|. \tag{53}$$

Because $\mathcal{F}$ is asymptotic pointwise nonexpansive, it follows that $\varphi\Big(T_{ms}(x)\Big) \leqslant a_{ms}(x)\varphi(x)$ for every $x \in C$. Applying this to w and passing with $m \to \infty$, we obtain

$$\lim_{m\to\infty} \varphi\Big(T_{ms}(w)\Big) \leqslant \varphi(w). \tag{54}$$

By (2) we get for any $x \in X$

$$\frac{1}{2}\|x_n - w\|^2 + \ \langle w - x, J(x_n - w)\rangle \ \leqslant \frac{1}{2}\|x_n - x\|^2. \tag{55}$$

Passing with n to ∞ and using the fact that $x_n \rightharpoonup w$ we arrive at

$$\varphi(w)^2 \leqslant \varphi(x)^2 \tag{56}$$

which implies that $\varphi(w) = inf\{\varphi(x) : x \in C\}$. This together with (54) gives us

$$\lim_{m\to\infty} \varphi\Big(T^m(w)\Big) = \varphi(w). \tag{57}$$

Applying Proposition 2.5 to $x = x_n - w$, $y = x_n - T_{ms}(w)$ and $\alpha = \frac{1}{2}$ we obtain the following inequality

$$\|x_n - \frac{1}{2}(w + T_{ms}(w))\|^2 \leqslant \frac{1}{2}\|x_n - w\|^2 + \frac{1}{2}\|x_n - T_{ms}(w)\|^2 - \frac{1}{4}\lambda\Big(\|T_{ms}(w) - w\|\Big).$$

Applying to both side $\limsup_{n\to}$ and remembering that $\varphi(w) = inf\{\varphi(x : x \in C\}$ we have

$$\varphi(w)^2 \leqslant \frac{1}{2}\varphi(w)^2 + \frac{1}{2}\varphi\Big(T_{ms}(w)\Big)^2 - \frac{1}{4}\lambda\Big(\|T_{ms}(w) - w\|\Big),$$

which implies

$$\lambda\Big(\|T_{ms}(w) - w\|\Big) \leqslant 2\varphi\Big(T_{ms}(w)\Big)^2 - 2\varphi(w)^2.$$

Letting $m \to \infty$ and applying (57) we conclude that

$$\lim_{m\to\infty} \lambda\Big(\|T_{ms}(w) - w\|\Big) = 0.$$

By the properties of λ, we have $T_{ms}(w) \to w$. Fix any natural number k. Observe that, using the same argument, we conclude that $T_{(m+k)s}(w) \to w$. Note that

$$T_{ks}(T_{ms}(w)) = T_{(m+k)s}(w) \to w$$

By the continuity of T_{ks},

$$T_{ks}(T_{ms}(w)) \to T_{ks}(w)$$

and finally $T_{ks}(w) = w$ as claimed. ∎

6. Weak convergence of generalized Krasnosel'skii-Mann iteration processes. Let us start with the precise definition of the generalized Krasnosel'skii-Mann iteration process for semigroups of nonlinear mappings.

Definition 6.1 Let $\mathcal{F} \in \mathcal{S}(\mathcal{C})$, $\{t_k\} \subset J$ and $\{c_k\} \subset (0,1)$. The generalized Krasnosel'skii-Mann iteration process gKM$(\mathcal{F}, \{c_k\}, \{t_k\})$ generated by the semigroup $\mathcal{F}$, the sequences $\{c_k\}$ and $\{t_k\}$, is defined by the following iterative formula:

$$x_{k+1} = c_k T_{t_k}(x_k) + (1 - c_k)x_k, \ \textit{where } x_1 \in C \textit{ is chosen arbitrarily}, \tag{58}$$

and

(i) $\{c_k\}$ is bounded away from 0 and 1,

(ii) $\lim_{k\to\infty} t_k = \infty$,

(iii) For every $x \in C$ there exists a positive constant ε_x and a sequence of finite positive constants $\{M_n(x)\}$ such that

$$\sum_{n=1}^{\infty} M_n(x) < \infty, \tag{59}$$

and for every $y \in C \cap B(x, \varepsilon_x)$ there holds

$$b_{t_n}(y) \leqslant M_n(x), \tag{60}$$

where $b_{t_n}(y) = a_{t_n}(y) - 1$ and $\{a_{t_n}(y)\}$ is defined as in Definition 3.2.

Definition 6.2 We say that a generalized Krasnosel'skii-Mann iteration process gKM$(\mathcal{F}, \{c_k\}, \{t_k\})$ is well defined if

$$\limsup_{k\to\infty} a_{t_k}(x_k) = 1. \tag{61}$$

Lemma 6.3 *Let C be a bounded, closed and convex subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$, $w \in F(\mathcal{F})$, and let $\{x_k\}$ be a sequence generated by a generalized Krasnosel'skii-Mann process $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$. Then there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$.*

Proof Let $w \in F(\mathcal{F})$. Since

$$\begin{aligned}\|x_{k+1} - w\| &\leqslant c_k\|T_{t_k}(x_k) - w\| + (1 - c_k)\|x_k - w\| \\ &\leqslant c_k\|T_{t_k}(x_k) - T_{t_k}(w)\| + (1 - c_k)\|x_k - w\| \\ &\leqslant c_k(1 + b_{t_k}(w))\|x_k - w\| + (1 - c_k)\|x_k - w\| \\ &\leqslant c_k b_{t_k}(w)\|x_k - w\| + \|x_k - w\| \\ &\leqslant b_{t_k}(w) diam(C) + \|x_k - w\|,\end{aligned}$$

it follows that for every $n \in \mathbb{N}$,

$$\|x_{k+n} - w\| \leqslant \|x_k - w\| + diam(C) \sum_{i=k}^{k+n-1} b_{t_i}(w). \tag{62}$$

Denote $r_p = \|x_p - w\|$ for every $p \in \mathbb{N}$ and $d_{k,n} = diam(C) \sum_{i=k}^{k+n-1} b_{t_i}(w)$. Observe that $\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} = 0$. By Lemma 2.2 then, there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$. ■

Lemma 6.4 *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. Let $\mathcal{F} \in \mathcal{S}(C)$. Let $\{x_k\}$ be a sequence generated by a well defined generalized Krasnosel'skii-Mann process $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$. Then*

$$\lim_{k\to\infty} \|T_{t_k}(x_k) - x_k\| = 0 \tag{63}$$

and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{64}$$

Proof By Theorem 3.7, $F(\mathcal{F}) \neq \emptyset$. Let us fix $w \in F(\mathcal{F})$. By Lemma 6.3 there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$. Because $w \in F(\mathcal{F})$, and the process is well defined, then there holds

$$\begin{aligned}\limsup_{k\to\infty} \|T_{t_k}(x_k) - w\| &= \limsup_{k\to\infty} \|T_{t_k}(x_k) - T_{t_k}(w)\| \\ &\leqslant \limsup_{k\to\infty} a_{t_k}(x_k)\|x_k - w\| = r.\end{aligned}$$

Observe that

$$\lim_{k\to\infty}\|c_k(T_{t_k}(x_k)-w)+(1-c_k)(x_k-w)\|=\lim_{k\to\infty}\|x_{k+1}-w\|=r.$$

By Lemma 2.3 applied to $u_k = x_k - w$, $v_k = T_{t_k}(x_k) - w$,

$$\lim_{k\to\infty}\|T_{t_k}(x_k)-x_k\|=0, \tag{65}$$

which by the construction of the sequence $\{x_k\}$ is equivalent to

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=0. \tag{66}$$

LEMMA 6.5 *Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X. Let $\mathcal{F}\in\mathcal{S}(C)$. Let $\{t_k\}\subset(0,1)$ be bounded away from 0 and 1. Let gKM$(\mathcal{F},\{c_k\},\{t_k\})$ be a well defined generalized Krasnosel'skii-Mann iteration process. Then for every $w_1, w_2 \in F(\mathcal{F})$ and for any two weak cluster points y, z of the sequence $\{x_k\}$ generated by gKM$(\mathcal{F},\{c_k\},\{t_k\})$ there holds*

$$\langle y-z, J(w_1-w_2)\rangle=0. \tag{67}$$

PROOF Let $w_1,\ w_2\in F(\mathcal{F})$. Let us define

$$H_k(u)=c_kT_{t_k}(u)+(1-c_k)u. \tag{68}$$

Using the fact that $a_{t_j}(u)\geqslant 1$, it is easy to observe that

$$\|H_k(u)-H_k(v)\|\leqslant a_{t_k}(u)\|u-v\|. \tag{69}$$

This together with the boundedness conditions imposed in Definition 58 implies that $\{H_k\}\in\mathcal{A}_c(C)$. It is immediate that $H_k(x_k)=x_{k+1}$, hence the sequence $\{x_k\}$ is generated by $\{H_k\}$. Since $w_1,\ w_2\in F(\mathcal{F})\subset\bigcap_{k=1}^{\infty}F(H_k)$, the equality (67) follows immediately from Lemma 4.6. ■

THEOREM 6.6 *Let X be a uniformly convex and uniformly smooth Banach space X. Let C be a bounded, closed and convex subset of X. Let $\mathcal{F}\in\mathcal{S}(C)$ be equicontinuous and $B\subset\overline{B}=A\subset J$ where A is a generating set for J. Let $\{x_k\}$ be generated by a well defined Krasnosel'skii-Mann iteration process $gKM(\mathcal{F},\{c_k\},\{t_k\})$. If to every $s\in B$ there exists a strictly increasing sequence of natural numbers $\{j_k\}$ satisfying the following conditions.*

(a) $t_{j_{k+1}}-t_{j_k}\to s$ *as* $k\to\infty$,

(b) $\|x_k-x_{j_k}\|\to 0$ *as* $k\to\infty$,

then the sequence $\{x_k\}$ converges weakly to a common fixed point $w\in F(\mathcal{F})$.

PROOF Observe that because $\mathcal{F} \in \mathcal{S}(C)$ is equicontinuous the following holds

$$\lim_{k\to\infty} \|T_{d_k}(x_{j_k}) - x_{j_k}\| = 0, \tag{70}$$

where $d_k = |t_{j_{k+1}} - t_{j_k} - s|$.

Let us prove first that $\{x_n\}$ is an approximate fixed points sequence for all mappings $\{T_{ms}\}$ where $s \in A$ and $m \in \mathbb{N}$, that is that

$$\lim_{k\to\infty} \|T_{ms}(x_k) - x_k\| = 0. \tag{71}$$

In view of Lemma 3.12, it is enough to prove (71) for $m = 1$. To this end, let us fix $s \in A$. Note that

$$\|x_{j_k} - x_{j_{k+1}}\| \to 0 \ as\ k \to \infty. \tag{72}$$

Indeed,

$$\|x_{j_k} - x_{j_{k+1}}\| \leqslant \|x_{j_k} - x_k\| + \|x_k - x_{k+1}\| + \|x_{k+1} - x_{j_{k+1}}\| \to 0, \tag{73}$$

in view of the above assumption (b) and of (64) in Lemma 6.4.

Observe that

$$\|x_{j_k} - T_s(x_{j_k})\| \to 0 \ as\ k \to \infty. \tag{74}$$

Indeed,

$$\begin{aligned}\|x_{j_k} - T_s(x_{j_k})\| &\leqslant \|x_{j_k} - x_{j_{k+1}}\| + \|x_{j_{k+1}} - T_{t_{j_{k+1}}}(x_{j_{k+1}})\| + \|T_{t_{j_{k+1}}}(x_{j_{k+1}}) - T_{t_{j_{k+1}}}(x_{j_k})\| \\ &\quad + \|T_{t_{j_{k+1}}}(x_{j_k}) - T_{s+t_{j_k}}(x_{j_k})\| + \|T_{s+t_{j_k}}(x_{j_k}) - T_s(x_{j_k})\| \\ &\leqslant \|x_{j_k} - x_{j_{k+1}}\| + \|x_{j_{k+1}} - T_{t_{j_{k+1}}}(x_{j_{k+1}})\| + a_{t_{j_{k+1}}}(x_{j_{k+1}})\|x_{j_{k+1}} - x_{j_k}\| \\ &\quad + a_{s+t_{j_k}}(x_{j_k})\|T_{d_k}(x_{j_k}) - x_{j_k}\| + \sup_{x\in C} a_s(x)\|T_{t_{j_k}}(x_{j_k}) - x_{j_k}\|\end{aligned}$$

which tends to the zero as $k \to \infty$ because of (72), Lemma 6.4, the fact that the process is well defined, equicontinuity of $\mathcal{F}$, assumption (10), and the boundedness of each function a_s.

On the other hand,

$$\begin{aligned}\|x_k - T_s(x_k)\| &\leqslant \|x_k - x_{j_k}\| + \|x_{j_k} - T_{t_{j_k}}(x_{j_k})\| + \|T_{t_{j_k}}(x_{j_k}) - T_{s+t_{j_k}}(x_{j_k})\| \\ &\quad + \|T_{s+t_{j_k}}(x_{j_k}) - T_s(x_{j_k})\| + \|T_s(x_{j_k}) - T_s(x_k)\| \\ &\leqslant \|x_k - x_{j_k}\| + \|x_{j_k} - T_{t_{j_k}}(x_{j_k})\| + a_{t_{j_k}}(x_{j_k})\|x_{j_k} - T_s(x_{j_k})\| \\ &\quad + a_s(x_{j_k})\|T_{t_{j_k}}(x_{j_k}) - x_{j_k}\| + a_s(x_k)\|x_{j_k} - x_k\| \\ &\leqslant \|x_k - x_{j_k}\| + \|x_{j_k} - T_{t_{j_k}}(x_{j_k})\| + a_{t_{j_k}}(x_{j_k})\|x_{j_k} - T_s(x_{j_k})\| \\ &\quad + M_s\|T_{t_{j_k}}(x_{j_k}) - x_{j_k}\| + M_s\|x_{j_k} - x_k\|\end{aligned}$$

which tends to the zero as $k \to \infty$ because of the assumption (b), (63) in Lemma 6.4, the fact that the process is well defined, and the fact that $M_s < \infty$.

Since A is a generating set for J then by Lemma 3.13, $\{x_k\}$ is an approximate fixed point sequence for any T_s, that is

$$\lim_{k\to\infty} \|T_t(x_k) - x_k\| = 0 \tag{75}$$

for any $t \in J$, as claimed.

Consider $y, z \in C$, two weak cluster points of the sequence $\{x_k\}$. Then there exist two subsequences $\{y_k\}$ and $\{z_k\}$ of $\{x_k\}$ such that $y_k \rightharpoonup y$ and $z_k \rightharpoonup z$. Fix any $s \in A$. Since $\{x_k\}$ is an approximate fixed point sequence for s it follows that

$$\lim_{k\to\infty} \|T_s(x_k) - x_k\| = 0. \tag{76}$$

It follows from the Demiclosedness Principle (Theorem 5.1) that $T_s(y) = y$ and $T_s(z) = z$. By Lemma 6.5

$$\|y - z\|^2 = \langle y - z, J(y - z)\rangle = 0, \tag{77}$$

which implies that $y = z$. Hence the sequence $\{x_k\}$ has at most one weak cluster point. Since C is weakly sequentially compact, we deduce that the sequence $\{x_k\}$ has exactly one weak cluster point $w \in C$, which means that $x_k \rightharpoonup w$. Applying the Demiclosedness Principle again, we get $T_s(w) = w$. Since $s \in A$ was chosen arbitrarily and the construction of w did not depend on the selection of s, and A is a generating set for J, we conclude that $T_t(w) = w$ for any $t \in J$, as claimed. ■

REMARK 6.7 Observe that the set B in Theorem 6.6 can be made countable. Hence, it is easy to see that we can always construct a sequence $\{t_k\}$ with the properties specified in the assumptions of Theorem 6.6. When constructing concrete implementations of this algorithm, the difficulty will be to ensure that the constructed sequence $\{t_k\}$ is not "too sparse"in the sense that the Krasnosel'skii-Mann process $gKM(\mathcal{F}, \{c_k\}, \{t_k\})$ remains well defined (see Definition 6.2).

7. Weak convergence of generalized Ishikawa iteration processes. The two-step Ishikawa iteration process is a generalization of the one-step Krasnosel'skii-Mann process. The Ishikawa iteration process provides more flexibility in defining the algorithm parameters which is important from the numerical implementation perspective.

DEFINITION 7.1 Let $\mathcal{F} \in \mathcal{S}(\mathcal{C})$, $\{t_k\} \subset J$. Let $\{c_k\} \subset (0,1)$, and $\{d_k\} \subset (0,1)$. The generalized Ishikawa iteration process $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ generated by the semigroup $\mathcal{F}$, the sequences $\{c_k\}$, $\{d_k\}$ and $\{t_k\}$, is defined by the following iterative formula:

$$x_{k+1} = c_k T_{t_k}(d_k T_{t_k}(x_k) + (1 - d_k)x_k) + (1 - c_k)x_k, \tag{78}$$

where $x_1 \in C$ is chosen arbitrarily, and

(i) $\{c_k\}$ is bounded away from 0 and 1, and $\{d_k\}$ is bounded away from 1,

(ii) $\lim_{k\to\infty} t_k = \infty$,

(iii) For every $x \in C$ there exists a positive constant ε_x and a sequence of finite positive constants $\{M_n(x)\}$ such that

$$\sum_{n=1}^{\infty} M_n(x) < \infty, \tag{79}$$

and for every $y \in C \cap B(x, \varepsilon_x)$ there holds

$$b_{t_n}(y) \leqslant M_n(x), \tag{80}$$

where $b_{t_n}(y) = a_{t_n}(y) - 1$ and $\{a_{t_n}(y)\}$ is defined as in Definition 3.2.

DEFINITION 7.2 We say that a generalized Ishikawa iteration process $\mathrm{gI}(\mathcal{F}, \{c_k\}, \{t_k\})$ is well defined if

$$\limsup_{k\to\infty} a_{t_k}(x_k) = 1. \tag{81}$$

LEMMA 7.3 *Let C be a bounded, closed and convex subset of a Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$, $w \in F(\mathcal{F})$, and let $\{x_k\}$ be the sequence generated by a generalized Ishikawa process $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$. Then there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$.*

PROOF Define $G_k : C \to C$ by

$$G_k(x) = c_k T_{t_k}\Big(d_k T_{t_k}(x) + (1-d_k)x\Big) + (1-c_k)x, \ x \in C. \tag{82}$$

It is easy to see that $x_{k+1} = G_k(x_k)$ and that $F(\mathcal{F}) \subset F(G_k)$ for every $k \geqslant 1$. Moreover, a straight calculation shows that each G_k satisfies

$$\|G_k(x) - G_k(y)\| \leqslant A_k(x)\|x - y\|, \tag{83}$$

where

$$A_k(x) = 1 + c_k a_{t_k}\Big(d_k T_{t_k}(x) + (1-d_k)x\Big)(1 + d_k a_{t_k}(x) - d_k) - c_k. \tag{84}$$

Note that $A_k(x) \geqslant 1$ which follows directly from the fact that $a_{t_k}(z) \geqslant 1$ for any $z \in C$. Using (84) and remembering that $w \in F(\mathcal{F})$ we have

$$B_k(w) = A_k(w) - 1 = c_k(1 + d_k a_{t_k}(w))(a_{t_k}(w) - 1) \leqslant (1 + a_{t_k}(w))b_{t_k}(w). \tag{85}$$

Fix any $M > 1$. Since $\lim_{k\to\infty} a_{t_k}(w) = 1$, it follows that there exists a $k_0 \geqslant 1$ such that for $k > k_0$, $a_{t_k}(w) \leqslant M$. Therefore, using the same argument as in the proof of Lemma 6.3, we deduce that for $k > k_0$ and $n > 1$

$$\begin{aligned} \|x_{k+n} - w\| &\leqslant \|x_k - w\| + diam(C) \sum_{i=k}^{k+n-1} B_{t_i}(w) \\ &\leqslant \|x_k - w\| + diam(C)(1+M) \sum_{i=k}^{k+n-1} b_{t_i}(w). \end{aligned} \tag{86}$$

Arguing like in the proof of Lemma 6.3, we conclude that there exists an $r \in \mathbb{R}$ such that $\lim_{k\to\infty} \|x_k - w\| = r$. ∎

Lemma 7.4 *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. Let $\{x_k\}$ be the sequence generated by a well defined generalized Ishikawa process $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$. Then*

$$\lim_{k\to\infty} \|T_{t_k}(x_k) - x_k\| = 0 \tag{87}$$

and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{88}$$

Proof By Theorem 3.7, $F(\mathcal{F}) \neq \emptyset$. Let us fix $w \in F(\mathcal{F})$. By Lemma 7.3, $\lim_{k\to\infty} \|x_k - w\|$ exists. Let us denote it by r. Let us define

$$y_k = d_k T_{t_k}(x_k) + (1 - d_k)x_k. \tag{89}$$

Since $w \in F(\mathcal{F})$, $\mathcal{F} \in \mathcal{S}(C)$, and $\lim_{k\to\infty} \|x_k - w\| = r$, we have the following

$$\begin{aligned} &\limsup_{k\to\infty} \|T_{t_k}(y_k) - w\| = \limsup_{k\to\infty} \|T_{t_k}(y_k) - T_{t_k}(w)\| \\ &\leqslant \limsup_{k\to\infty} a_{t_k}(w)\|y_k - w\| = \limsup_{k\to\infty} a_{t_k}(w)\|d_k T_{t_k}(x_k) + (1-d_k)x_k - w\| \\ &\leqslant \limsup_{k\to\infty} \Big(d_k a_{t_k}(w)\|T_{t_k}(x_k) - w\| + (1-d_k)a_{t_k}(w)\|x_k - w\|\Big) \\ &\leqslant \lim_{k\to\infty} \Big(d_k a_{t_k}^2(w)\|x_k - w\| + (1-d_k)a_{t_k}(w)\|x_k - w\|\Big) \leqslant r. \end{aligned} \tag{90}$$

Note that

$$\begin{aligned} &\lim_{k\to\infty} \|d_k(T_{t_k}(y_k) - w) + (1-d_k)(x_k - w)\| \\ &= \lim_{k\to\infty} \|d_k T_{t_k}(y_k) + (1-d_k)x_k - w\| = \lim_{k\to\infty} \|x_{k+1} - w\| = r. \end{aligned} \tag{91}$$

Applying Lemma 2.3 with $u_k = T_{t_k}(y_k) - w$ and $v_k = x_k - w$, we obtain the equality $\lim_{k\to\infty} \|T_{t_k}(y_k) - x_k\| = 0$. This fact, combined with the construction formulas for x_{k+1} and y_k, proves (88).
Since

$$\begin{aligned} \|T_{t_k}(x_k) - x_k\| &\leqslant \|T_{t_k}(x_k) - T_{t_k}(y_k)\| + \|T_{t_k}(y_k) - x_k\| \\ &\leqslant a_{t_k}(x_k)\|x_k - y_k\| + \|T_{t_k}(y_k) - x_k\| \\ &= d_k a_{t_k}(x_k)\|T_{t_k}(x_k) - x_k\| + \|T_{t_k}(y_k) - x_k\|, \end{aligned} \tag{92}$$

it follows that

$$\|T_{t_k}(x_k) - x_k\| \leqslant (1 - d_k a_{t_k}(x_k))^{-1}\|T_{t_k}(y_k) - x_k\|. \tag{93}$$

The right-hand side of this inequality tends to zero because $\|T_{t_k}(y_k) - x_k\| \to 0$, $\limsup_{k\to\infty} a_{t_k}(x_k) = 1$ by the fact that the Ishikawa process is well defined, and $\{d_k\} \subset (0,1)$ is bounded away from 1. ∎

LEMMA 7.5 *Let C be a bounded, closed and convex subset of a uniformly convex and uniformly smooth Banach space X. Let $\mathcal{F} \in \mathcal{S}(C)$. Let $\{t_k\} \subset (0,1)$ be bounded away from 0 and 1, and $\{d_k\} \subset (0,1)$ be such that $d_k \to 0$. Let $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ be a well defined generalized Ishikawa iteration process. Then for every $w_1, w_2 \in F(\mathcal{F})$ and for any two weak cluster points y, z of the sequence $\{x_k\}$ generated by $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ there holds*

$$\langle y - z, J(w_1 - w_2)\rangle = 0. \tag{94}$$

PROOF Let $w_1,\ w_2 \in F(\mathcal{F})$. Let us define

$$H_k(u) = c_k T_{t_k}\Big(d_k T_{t_k}(u) + (1-d_k)u\Big) + (1-c_k)u. \tag{95}$$

We need to show that the sequence $\{H_k\} \in \mathcal{A}_c(C)$. Let us denote $u_k = c_k T_{t_k}(u) + (1-c_k)u$ and $v_k = d_k T_{t_k}(v) + (1-d_k)v$, and calculate

$$\begin{aligned} &\|H_k(u) - H_k(v)\| \\ \leqslant c_k\|T_{t_k}(u_k) - T_{t_k}(v_k)\| + (1-c_k)\|u-v\| &\leqslant c_k a_{t_k}(u_k)\|u_k - v_k\| + (1-c_k)\|u-v\| \\ &\leqslant c_k a_{t_k}(u_k)\Big(d_k\|T_{t_k}(u) - T_{t_k}(v)\| + (1-d_k)\|u-v\|\Big) + (1-c_k)\|u-v\| \\ &\leqslant c_k a_{t_k}(u_k)\Big(d_k a_{t_k}(u)\|u-v\| + (1-d_k)\|u-v\|\Big) + (1-c_k)\|u-v\| \\ &\leqslant c_k a_{t_k}(u_k)a_{t_k}(u)\|u-v\|\Big) + (1-c_k)\|u-v\| \\ &\leqslant a_{t_k}(u_k)a_{t_k}(u)\|u-v\| = A_k(u)\|u-v\|, \end{aligned} \tag{96}$$

where $A_k(u) = a_{n_k}(u_k)a_{n_k}(u)$. By the boundedness condition (iii) from Definition 7.1 there exists $\varepsilon_u > 0$ and a sequence of positive constants $\{M_k(u)\}$ such that $\sum_{k=1}^{\infty} M_k(u) < \infty$ and for every $y \in C \cap B(u, \varepsilon_u)$ there holds $b_{t_k}(y) \leqslant M_k(u)$. Note that

$$\|u_k - u\| = d_k\|T_{t_k}(u) - u\| \leqslant d_k\ diam(C) \to 0 \tag{97}$$

as $k \to \infty$ since $d_k \to 0$. Hence, there exists $k_0 \in \mathbb{N}$ such that $\|u_k - u\| < \varepsilon_u$ for $k > k_0$, and therefore $b_{t_k}(u) \leqslant M_k(u)$ for $k > k_0$. Let us observe that for $k > k_0$,

$$B_k(u) = A_k(u) - 1 = \Big(1 + b_{t_k}(u_k)\Big)a_{t_k}(u) - 1 \leqslant \Big(1 + M_k(u)\Big)^2 - 1. \tag{98}$$

Set

$$N_k(u) = (1 + M_k(u))^2 - 1 \tag{99}$$

for $k > k_0$ and

$$N_k(u) = (1 + m_k)^2 - 1 \tag{100}$$

for $k \leqslant k_0$ where $m_k = sup\{b_{t_k}(x) : x \in C\}$ which is finite in view of the boundedness of each function a_t. Clearly $\sum_{k=1}^{\infty} N_k(u) < \infty$, $B_k(y) \leqslant N_k(u)$ for every $y \in C \cap B(u, \varepsilon_u)$, and every A_k is a bounded function since $A_k(u) \leqslant (1 + m_k)^2$ for every $u \in C$. Hence, $\{H_k\} \in \mathcal{A}_c(C)$. It is immediate that $H_k(x_k) = x_{k+1}$, hence the sequence $\{x_k\}$ is generated by $\{H_k\}$. Since $w_1,\ w_2 \in F(\mathcal{F}) \subset \bigcap_{k=1}^{\infty} F(H_k)$, the equality (94) follows immediately from Lemma 4.6. ■

THEOREM 7.6 *Let X be a uniformly convex and uniformly smooth Banach space X. Let C be a bounded, closed and convex subset of X. Let $\mathcal{F} \in \mathcal{S}(C)$ be equicontinuous and $B \subset \overline{B} = A \subset J$ where A is a generating set for J. Let $\{x_k\}$ be generated by a well defined Ishikawa iteration process $gI(\mathcal{F}, \{c_k\}, \{d_k\}, \{t_k\})$ such that $d_k \to 0$. If to every $s \in B$ there exists a strictly increasing sequence of natural numbers $\{j_k\}$ satisfying the following conditions:*

(a) $t_{j_{k+1}} - t_{j_k} \to s$ *as* $k \to \infty$,

(b) $\|x_k - x_{j_k}\| \to 0$ *as* $k \to \infty$,

then the sequence $\{x_k\}$ converges weakly to a common fixed point $w \in F(\mathcal{F})$.

PROOF The proof is analogous to that of Theorem 6.6 with Lemma 6.3 replaced by Lemma 7.3, Lemma 6.4 replaced by Lemma 7.4, and Lemma 6.5 replaced by Lemma 7.5. ■

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