COMMENTATIONES MATHEMATICAE Vol. 52, No. 1 (2012), 3-9

Monika Herzog

Approximation of functions of two variables by modified Szasz-Mirakyan operators

Abstract. In this paper we study approximative properties of modified Szasz-Mirakyan operators for functions of two variables from polynomial weight spaces. We present some direct theorems giving a degree of approximation for these operators.

2000 Mathematics Subject Classification: 41A36.

Key words and phrases: Linear positive operators, Bessel function, Modulus of continuity, Degree of approximation.

1. Introduction. Let us denote by $C(\mathbb{R}_0)$ a set of all real-valued functions continuous on $\mathbb{R}_0 = [0; +\infty)$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In paper [1] we investigated operators of Szasz-Mirakyan type defined as follows

$$A_n^\nu(f;x) = \begin{cases} \frac{1}{I_\nu(nx)} \sum_{k=0}^{\infty} \frac{(\frac{nx}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} f(\frac{2k}{n}), & x > 0; \\ f(0), & x = 0, \end{cases}$$

where Γ is the Euler-gamma function and I_ν the modified Bessel function defined by the formula ([6], p. 77)

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)}.$$

We studied the operators in polynomial weight spaces

$$C_p = \{f \in C(\mathbb{R}_0) : w_p f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

where w_p was the polynomial weight function defined as follows

$$w_p(x) = \begin{cases} 1, & p = 0; \\ \frac{1}{1+x^p}, & p \in \mathbb{N}, \end{cases}$$

for $x \in \mathbb{R}_0$.

In the present paper we will consider the bivariate version of the operator A_n^ν in appropriate weight spaces.

Let $C(\mathbb{R}_0^2)$ be the set of all real-valued functions continuous on $\mathbb{R}_0^2$. Similarly as in [1] we define the polynomial weight space

$$(1) \qquad C_{p,q} = \{f \in C(\mathbb{R}_0^2) : w_{p,q}\, f \text{ is uniformly continuous and bounded on } \mathbb{R}_0^2\},$$

where $w_{p,q}$ is the polynomial weight function defined as follows

$$(2) \qquad w_{p,q}(x,y) = \begin{cases} 1, & p = q = 0; \\ \frac{1}{1+x^p}, & q = 0,\ p \in \mathbb{N}; \\ \frac{1}{1+y^q}, & p = 0,\ q \in \mathbb{N}; \\ \frac{1}{(1+x^p)(1+y^q)}, & p, q \in \mathbb{N}, \end{cases}$$

for $(x,y) \in \mathbb{R}_0^2$. The space $C_{p,q}$ is a normed space with the norm

$$(3) \qquad \|f\|_{p,q} = \sup\{w_{p,q}(x,y)|f(x,y)|; (x,y) \in \mathbb{R}_0^2\}.$$

Moreover, we consider the modulus of continuity

$$(4) \qquad \omega(f, C_{p,q}; t, s) = \sup\{\|\Delta_{h,d} f\|_{p,q}; h \in [0,t], d \in [0,s]\},$$

where

$$\Delta_{h,d} f(x,y) = f(x+h, y+d) - f(x,y)$$

for $(x,y) \in \mathbb{R}_0^2$, $h, d \in \mathbb{R}_0$.

The note was inspired by the results of [2]-[5]. We introduce the modified Szasz-Mirakyan operator for functions $f \in C_{p,q}$ in the following way

$$(5) \qquad A_{n,m}^{\nu,\mu}(f; x, y) =$$

$$\begin{cases} \frac{1}{I_\nu(nx)} \frac{1}{I_\mu(my)} \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{(\frac{nx}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} \frac{(\frac{my}{2})^{2j+\mu}}{\Gamma(j+1)\Gamma(j+\mu+1)} f(\frac{2k}{n}, \frac{2j}{m}), & x > 0,\ y > 0; \\ \frac{1}{I_\nu(nx)} \sum_{k=0}^\infty \frac{(\frac{nx}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} f(\frac{2k}{n}, 0), & x > 0,\ y = 0; \\ \frac{1}{I_\mu(mx)} \sum_{j=0}^\infty \frac{(\frac{mx}{2})^{2j+\nu}}{\Gamma(j+1)\Gamma(j+\mu+1)} f(0, \frac{2j}{m}), & y > 0,\ x = 0; \\ f(0,0), & x = y = 0. \end{cases}$$

where $p, q \in \mathbb{N}_0$, $\nu, \mu \in \mathbb{R}_0$, $n, m \in \mathbb{N}$.

We shall present direct approximation theorems for these operators. The main results of the paper are theorems giving a degree of approximation of function $f \in C_{p,q}$ by operators $A_{n,m}^{\nu,\mu}$.

2. Auxiliary results. In this section we will show some basic properties of the operator $A_{n,m}^{\nu,\mu}$.

At the beginning we will recall preliminary results from paper [1] for the operator A_n^ν, which we shall apply to the proofs of the main theorems.

LEMMA 2.1 *For each* $n \in \mathbb{N}$, $\nu \in \mathbb{R}_0$ *and* $x \in \mathbb{R}_0$

$$A_n^\nu(1;x) = 1, \qquad A_n^\nu(t;x) = x\frac{I_{\nu+1}(nx)}{I_\nu(nx)},$$

$$A_n^\nu(t^2;x) = x^2\frac{I_{\nu+2}(nx)}{I_\nu(nx)} + x\frac{2}{n}\frac{I_{\nu+1}(nx)}{I_\nu(nx)},$$

$$A_n^\nu(t-x;x) = x\left(\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1\right),$$

$$A_n^\nu((t-x)^2;x) = x^2\left(\frac{I_{\nu+2}(nx)}{I_\nu(nx)} - 2\frac{I_{\nu+1}(nx)}{I_\nu(nx)} + 1\right) + x\frac{2}{n}\frac{I_{\nu+1}(nx)}{I_\nu(nx)}.$$

LEMMA 2.2 *For each* $\nu \in \mathbb{R}_0$ *there exists a positive constant* $M(\nu)$ *such that for all* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}_0$ *we have*

$$\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)}\right| \leqslant M(\nu),$$

$$nx\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1\right| \leqslant M(\nu).$$

Hence we immediately get

LEMMA 2.3 *For each* $\nu \in \mathbb{R}_0$ *there exists a positive constant* $M(\nu)$ *such that for all* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}_0$ *we have*

$$|A_n^\nu(t-x;x)| \leqslant \frac{M(\nu)}{n}, \qquad |A_n^\nu((t-x)^2;x)| \leqslant M(\nu)\frac{x}{n}.$$

LEMMA 2.4 *For all* $p \in \mathbb{N}_0$ *and* $\nu \in \mathbb{R}_0$ *there exists a positive constant* $M(p,\nu)$ *such that for each* $n \in \mathbb{N}$ *we have*

$$\|A_n^\nu(1/w_p;\cdot)\|_p \leqslant M(p,\nu).$$

THEOREM 2.5 *For all* $p \in \mathbb{N}_0$ *and* $\nu \in \mathbb{R}_0$ *there exists a positive constant* $M(p,\nu)$ *such that for each* $n \in \mathbb{N}$ *we have*

$$\|A_n^\nu(f;\cdot)\|_p \leqslant M(p,\nu)\|f\|_p.$$

LEMMA 2.6 *For all* $p \in \mathbb{N}_0$ *and* $\nu \in \mathbb{R}_0$ *there exists a positive constant* $M(p,\nu)$ *such that for all* $x \in \mathbb{R}_0$ *and* $n \in \mathbb{N}$ *we have*

$$w_p(x)A_n^\nu\left(\frac{(t-x)^2}{w_p(t)};x\right) \leqslant M(p,\nu)\frac{x+1}{n}.$$

The definition of the operator $A_{n,m}^{\nu;\mu}$ implies

$$A_{n,m}^{\nu;\mu}(f;x,y) = A_n^{\nu}(f_1;x)A_m^{\mu}(f_2;y) \tag{6}$$

for all functions $f \in C_{p,q}$ of the form $f(x,y) = f_1(x)f_2(y)$.
In particular we get

$$A_{n,m}^{\nu;\mu}(1;x,y) = 1,$$

$$A_{n,m}^{\nu;\mu}(1/w_{p,q};x,y) = A_n^{\nu}(1/w_p;x)A_m^{\mu}(1/w_q;y).$$

From the above facts and Lemma 2.4 we derive

LEMMA 2.7 *For all $p,q \in \mathbb{N}_0$ and $\nu,\mu \in \mathbb{R}_0$ there exists a positive constant $M(p,q,\nu,\mu)$ such that for all $n,m \in \mathbb{N}$ we have*

$$\|A_{n,m}^{\nu;\mu}(1/w_{p,q};\cdot)\|_{p,q} \leqslant M(p,q,\nu,\mu).$$

THEOREM 2.8 *For all $p,q \in \mathbb{N}_0$ and $\nu,\mu \in \mathbb{R}_0$ there exists a positive constant $M(p,q,\nu,\mu)$ such that for any $f \in C_{p,q}$ and $n,m \in \mathbb{N}$ we have*

$$\|A_{n,m}^{\nu;\mu}(f;\cdot)\|_{p,q} \leqslant M(p,q,\nu,\mu)\|f\|_{p,q}.$$

PROOF Applying equation (6) and definition (3) we get

$$w_{p,q}(x,y)|A_{n,m}^{\nu;\mu}(f(t,s);x,y)| \leqslant w_{p,q}(x,y)A_{n,m}^{\nu;\mu}(|f(t,s)|;x,y) =$$

$$w_{p,q}(x,y)A_{n,m}^{\nu;\mu}\left(w_{p,q}(t,s)f(t,s)\frac{1}{w_{p,q}(t,s)};x,y\right) \leqslant$$

$$\|f\|_{p,q}w_p(x)w_q(y)A_n^{\nu}\left(\frac{1}{w_p(t)};x\right)A_q^{\mu}\left(\frac{1}{w_q(s)};y\right) \leqslant M(p,\nu)M(q,\mu)\|f\|_{p,q}.$$

Hence the operator $A_{n,m}^{\nu;\mu}$ transforms the space $C_{p,q}$ into $C_{p,q}$. ■

3. Approximation theorems.

THEOREM 3.1 *For all $p,q \in \mathbb{N}_0$, $\nu,\mu \in \mathbb{R}_0$ and for each function $g \in C_{p,q}^1 = \{f \in C_{p,q} : f' \in C_{p,q}\}$ there exists a positive constant $M(p,q,\nu,\mu)$ such that for all $(x,y) \in \mathbb{R}_0^2$ and $n,m \in \mathbb{N}$ we have*

$$w_{p,q}(x,y)|A_{n,m}^{\nu;\mu}(g;x,y) - g(x,y)| \leqslant$$

$$M(p,q,\nu,\mu)\left(\|g_x'\|_{p,q}\left(\frac{x+1}{n}\right)^{\frac{1}{2}} + \|g_y'\|_{p,q}\left(\frac{y+1}{m}\right)^{\frac{1}{2}}\right).$$

Proof Pick $(x,y)\in\mathbb{R}_0^2$. For $(t,s)\in\mathbb{R}_0^2$ and $g\in C^1_{p,q}$ we can write

$$g(t,s)-g(x,y)=\int_x^t g'_k(k,s)\,dk+\int_y^s g'_u(x,u)\,du.$$

By the linearity of the operator $A^{\nu,\mu}_{n,m}$ we obtain

$$|A^{\nu,\mu}_{n,m}(g(t,s);x,y)-g(x,y)|=|A^{\nu,\mu}_{n,m}(g(t,s)-g(x,y);x,y)|=$$

$$\left|A^{\nu,\mu}_{n,m}\left(\int_x^t g'_k(k,s)\,dk+\int_y^s g'_u(x,u)\,du;x,y\right)\right|\leqslant$$

$$A^{\nu,\mu}_{n,m}\left(\left|\int_x^t g'_k(k,s)\,dk\right|;x,y\right)+A^{\nu,\mu}_{n,m}\left(\left|\int_y^s g'_u(x,u)\,du\right|;x,y\right).$$

Since

$$\begin{aligned}\left|\int_x^t g'_k(k,s)\,dk\right| &\leqslant \|g'_x\|_{p,q}\left|\int_x^t \frac{1}{w_{p,q}(k,s)}\,dk\right|\\ &\leqslant \|g'_x\|_{p,q}\left(\frac{1}{w_{p,q}(t,s)}+\frac{1}{w_{p,q}(x,s)}\right)|t-x|,\end{aligned}$$

we get

$$w_{p,q}(x,y)A^{\nu,\mu}_{n,m}\left(\left|\int_x^t g'_k(k,s)\,dk\right|;x,y\right)\leqslant$$

$$\|g'_x\|_{p,q}w_{p,q}(x,y)\left(A^{\nu,\mu}_{n,m}\left(\frac{|t-x|}{w_{p,q}(t,s)};x,y\right)+A^{\nu,\mu}_{n,m}\left(\frac{|t-x|}{w_{p,q}(x,s)};x,y\right)\right)=$$

$$\|g'_x\|_{p,q}w_{p,q}(x,y)A^{\mu}_{m}\left(\frac{1}{w_q(s)};y\right)\left(A^{\nu}_{n}\left(\frac{|t-x|}{w_p(t)};x\right)+\frac{1}{w_p(x)}A^{\nu}_{n}(|t-x|;x)\right)=$$

$$\|g'_x\|_{p,q}w_q(y)A^{\mu}_{m}\left(\frac{1}{w_q(s)};y\right)\left(w_p(x)A^{\nu}_{n}\left(\frac{|t-x|}{w_p(t)};x\right)+A^{\nu}_{n}(|t-x|;x)\right).$$

Using the Hölder inequality and Lemmas 2.3 - 2.5 we obtain

$$w_{p,q}(x,y)A^{\nu,\mu}_{n,m}\left(\left|\int_x^t g'_k(k,s)\,dk\right|;x,y\right)\leqslant$$

$$M(p,q,\nu,\mu)\|g'_x\|_{p,q}\left(\frac{x+1}{n}\right)^{\frac{1}{2}}.$$

Analogously we can write the following estimation

$$w_{p,q}(x,y)A^{\nu,\mu}_{n,m}\left(\left|\int_y^s g'_u(x,u)\,du\right|;x,y\right)\leqslant$$

$$M(p,q,\nu,\mu)\|g'_y\|_{p,q}\left(\frac{y+1}{m}\right)^{\frac{1}{2}},$$

which completes the proof. ■

THEOREM 3.2 *For all $p,q \in \mathbb{N}_0$, $\nu,\mu \in \mathbb{R}_0$ and for each $f \in C_{p,q}$ there exists a positive constant $M(p,q,\nu,\mu)$ such that for all $(x,y) \in \mathbb{R}_0^2$ and $n,m \in \mathbb{N}$ we have*

$$w_{p,q}(x,y)|A_{n,m}^{\nu,\mu}(f;x,y)-f(x,y)| \leqslant M(p,q,\nu,\mu)\omega\left(f, C_{p,q}, \left(\frac{x+1}{n}\right)^{\frac{1}{2}}, \left(\frac{y+1}{m}\right)^{\frac{1}{2}}\right).$$

PROOF Pick $(x,y) \in \mathbb{R}_0^2$ and $h,d \in \mathbb{R}_+$. We define the Steklov mean of $f \in C_{p,q}$ as follows

$$f_{h,d}(x,y) = \frac{1}{hd}\int_0^h\int_0^d f(x+u,y+v)\,du\,dv$$

$$f_{h,d}(x,y) = \frac{1}{hd}\int_x^{x+h}\int_y^{y+d} f(u,v)\,du\,dv.$$

Observe that

$$(f_{h,d})'_x(x,y) = \frac{1}{hd}\left(\int_y^{y+d} f(x+h,v)\,dv - \int_y^{y+d} f(x,v)\,dv\right) =$$

$$\frac{1}{hd}\int_y^{y+d}(f(x+h,v)-f(x,v))\,dv = \frac{1}{hd}\int_0^d (f(x+h,y+v)-f(x,y+v))\,dv =$$

$$\frac{1}{hd}\int_0^d (\Delta_{h,v}f(x,y) - \Delta_{0,v}f(x,y))\,dv.$$

Similarly we can show that

$$(f_{h,d}(x,y))'_y = \frac{1}{hd}\int_0^h (\Delta_{u,d}f(x,y) - \Delta_{u,0}f(x,y))\,du.$$

Therefore $(f_{h,d})'_x, (f_{h,d})'_y \in C_{p,q}$. Moreover,

$$f_{h,d}(x,y) - f(x,y) = \frac{1}{hd}\int_0^h\int_0^d \Delta_{u,v}f(x,y)\,du\,dv,$$

hence and definitons (3), (4) we get

$$\|f_{h,d} - f\|_{p,q} \leqslant \omega(f, C_{p,q}, h, d),$$

$$\|(f_{h,d})'_x\|_{p,q} \leqslant \frac{1}{h}(\|\Delta_{h,v}f\|_{p,q} + \|\Delta_{0,v}f\|_{p,q}) \leqslant \frac{2}{h}\omega(f, C_{p,q}, h, d),$$

$$\|(f_{h,d})'_y\|_{p,q} \leqslant \frac{2}{d}\omega(f, C_{p,q}, h, d).$$

By the linearity of $A_{n,m}^{\nu,\mu}$, Theorems 2.2 and 3.1 we obtain

$$w_{p,q}(x,y)|A_{n,m}^{\nu,\mu}(f;x,y) - f(x,y)| \leqslant$$

$$w_{p,q}(x,y)(A_{n,m}^{\nu,\mu}(|f-f_{h,d}|;x,y) + |A_{n,m}^{\nu,\mu}(f_{h,d};x,y) - f_{h,d}(x,y)| + |f_{h,d}(x,y) - f(x,y)|) \leqslant$$

$$M(p,q,\nu,\mu)(\|f-f_{h,d}\|_{p,q} + \|(f_{h,d})'_x\|_{p,q}\left(\frac{x+1}{n}\right)^{\frac{1}{2}} + \|(f_{h,d})'_y\|_{p,q}\left(\frac{y+1}{m}\right)^{\frac{1}{2}} + \|f_{h,d}-f\|_{p,q}) \leqslant$$

$$M(p,q,\nu,\mu)\omega(f,C_{p,q},h,d)\left(2+\frac{2}{h}\left(\frac{x+1}{n}\right)^{\frac{1}{2}}+\frac{2}{d}\left(\frac{y+1}{m}\right)^{\frac{1}{2}}\right),$$

for $(x,y) \in \mathbb{R}_0^2$, $n,m \in \mathbb{N}$ and $h,d \in \mathbb{R}_+$. Setting, for $(x,y) \in \mathbb{R}_0^2$ and $n,m \in \mathbb{N}$, $h = \left(\frac{x+1}{n}\right)^{\frac{1}{2}}$ and $d = \left(\frac{y+1}{m}\right)^{\frac{1}{2}}$ we get the desired estimation. ∎

Theorem 3.2 implies the following corollaries.

Corollary 3.3 *If $\nu,\mu \in \mathbb{R}_0$ and $f \in C_{p,q}$ with some $p,q \in \mathbb{N}_0$, then for all $(x,y) \in \mathbb{R}_0^2$*

$$\lim_{n,m\to\infty} A_{n,m}^{\nu;\mu}(f;x,y) = f(x,y).$$

Moreover, the above convergence is uniform on every set $[x_1,x_2] \times [y_1,y_2]$ with $0 \leqslant x_1 < x_2$, $0 \leqslant y_1 < y_2$.

Corollary 3.4 *For all $\alpha,\beta \in (0;1]$, $p,q \in \mathbb{N}_0$ and for each $f \in \mathrm{Lip}(C_{p,q},\alpha,\beta)) = \{f \in C_{p,q} : \omega(f,C_{p,q};t,s) = O(t^\alpha + s^\beta)\}$ there exists a positive constant $M(p,q,\alpha,\beta)$ such that for all $(x,y) \in \mathbb{R}_0^2$ and $n,m \in \mathbb{N}$ we have*

$$w_{p,q}(x,y)|A_{n,m}^{\nu;\mu}(f;x,y) - f(x,y)| \leqslant M(p,q,\alpha,\beta)\left(\left(\frac{x+1}{n}\right)^{\frac{\alpha}{2}} + \left(\frac{y+1}{m}\right)^{\frac{\beta}{2}}\right).$$

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Monika Herzog

Inistitute of Mathematics, Cracow University of Technology

Warszawska 24, 31-155 Cracow, Poland

E-mail: mherzog@pk.edu.pl

(Received: 16.09.2011)