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Normal pseudo-BCK-algebras

Abstract. A normal pseudo-BCK-algebra $\mathcal{X}$ is an algebra in which every subalgebra of $\mathcal{X}$ is an ideal of $\mathcal{X}$. Characterizations of normal pseudo-BCK-algebras are given.

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1. Introduction. The notion of BCK-algebras has been introduced by Y. Imai and K. Iséki in 1966 (see [7]). BCK-algebras are algebraic formulation of the BCK-system in combinatory logic which has applications in the language of functional programming.

The notion of pseudo-BCK-algebras has been introduced by G. Georgescu and A. Iorgulescu in [2] as an extension of BCK-algebras. These algebras have connections with pseudo-BL-algebras and pseudo-MV-algebras introduced by G. Georgescu and A. Iorgulescu in [3] and [4], respectively. More about those algebras the reader can find in [8]. A. Walendziak in [14] gives some other axiomatization of pseudo-BCK-algebras. The paper [2] contains basic properties of pseudo-BCK-algebras. Y. B. Jun in [11] obtained some characterizations of pseudo-BCK-algebras. A. Iorgulescu ([9], [10]) studied particular classes of pseudo-BCK-algebras. In [1] the authors give some facts about ideals of pseudo-BCK-algebras.

It is well known that an ideal of a pseudo-BCK-algebra $\mathcal{X}$ is a subalgebra of $\mathcal{X}$. But a subalgebra of an arbitrary pseudo-BCK-algebra $\mathcal{X}$ is not necessarily an ideal of $\mathcal{X}$. This raises a great difficulty in the study of these algebras. A pseudo-BCK-algebra in which every subalgebra is an ideal is called a normal pseudo-BCK-algebra. Normal BCK-algebras are studied in [13]. We give characterizations of normal pseudo-BCK-algebras in Sections 4. In Section 3 we prove some facts about atoms of pseudo-BCK-algebras. For the convenience of the reader, in Section 2 we give the necessary material needed in the sequel, thus making our exposition self-contained.

2. Preliminaries. A *pseudo-BCK-algebra* is a structure $\mathcal{X} = (X, \leqslant, *, \circ, 0)$, where $\leqslant$ is a binary relation on a set X, $*$ and $\circ$ are binary operations on X and 0 is an element of X, satisfying the axioms: for all $x, y, z \in X$,

(a1) $(x * y) \circ (x * z) \leqslant z * y, (x \circ y) * (x \circ z) \leqslant z \circ y$,

(a2) $x * (x \circ y) \leqslant y, x \circ (x * y) \leqslant y$,

(a3) $x \leqslant x$,

(a4) $0 \leqslant x$,

(a5) if $x \leqslant y$ and $y \leqslant x$, then $x = y$,

(a6) $x \leqslant y$ iff $x * y = 0$ iff $x \circ y = 0$.

It is obvious that any pseudo-BCK-algebra $\mathcal{X} = (X, \leqslant, *, \circ, 0)$ can be regarded as a universal algebra $\mathcal{X} = (X, *, \circ, 0)$. Note that every pseudo-BCK-algebra $\mathcal{X} = (X, *, \circ, 0)$ satisfying $x * y = x \circ y$ for all $x, y \in X$ is a BCK-algebra.

EXAMPLE 2.1 ([5]) Let $X = \{0, a, b, 1\}$ and define binary operations $*$ and $\circ$ on X by the following tables:

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	b	0

$\circ$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	1	a	0

Then $\mathcal{X} = (X, *, \circ, 0)$ is a pseudo-BCK-algebra, where $0 < a < b < 1$.

EXAMPLE 2.2 ([12]) Let $Y = [0, \infty)$ and let $\leqslant$ be the usual order on Y. Define binary operations $*$ and $\circ$ on Y by

$$x * y = \begin{cases} 0 & \text{if} \quad x \leqslant y, \\ \frac{2x}{\pi} \operatorname{arc\,tg}(\ln(\frac{x}{y})) & \text{if} \quad y < x, \end{cases}$$

$$x \circ y = \begin{cases} 0 & \text{if} \quad x \leqslant y, \\ x e^{-\operatorname{tg}(\frac{\pi y}{2x})} & \text{if} \quad y < x \end{cases}$$

for all $x, y \in Y$. Then $\mathcal{Y} = (Y, \leqslant, *, \circ, 0)$ is a pseudo-BCK-algebra.

EXAMPLE 2.3 ([6]) Let $Z = \{0, a, b, c\}$ and define a binary operation $*$ on Z by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then $\mathcal{Z} = (Z, *, 0)$ is a BCK-algebra, so also a pseudo-BCK-algebra.

Proposition 2.4 ([2]) *Let $\mathcal{X} = (X, *, \circ, 0)$ be a pseudo-BCK-algebra. Then, for all $x, y, z \in X$, the following hold:*

(i) *if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$,*

(ii) $x * y \leqslant x, x \circ y \leqslant x$,

(iii) $(x * y) \circ z = (x \circ z) * y$,

(iv) $x * 0 = x = x \circ 0$,

(v) *if $x \leqslant y$, then $z * y \leqslant z * x$ and $z \circ y \leqslant z \circ x$,*

(vi) *if $x \leqslant y$, then $x * z \leqslant y * z$ and $x \circ z \leqslant y \circ z$.*

Let $\mathcal{X} = (X, *, \circ, 0)$ be a pseudo-BCK-algebra. A nonempty subset S of X is a *subalgebra* of $\mathcal{X}$ if it satisfies $x * y \in S$ and $x \circ y \in S$ for all $x, y \in S$. It is called *proper*, if $S \neq X$.

Example 2.5 Let $\mathcal{X}$ be the pseudo-BCK-algebra from Example 2.1. Then $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, 1\}$, $\{0, a, b\}$ and X are all subalgebras of $\mathcal{X}$.

Example 2.6 Let $\mathcal{Y}$ be the pseudo-BCK-algebra from Example 2.2. Then, for all $a \geqslant 0$, the set $[0, a]$ is a proper subalgebra of $\mathcal{Y}$.

Example 2.7 Let $\mathcal{Z}$ be the pseudo-BCK-algebra from Example 2.3. Then $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, a, b\}$, $\{0, a, c\}$, $\{0, b, c\}$ and Z are all subalgebras of $\mathcal{Z}$.

Let $\mathcal{X} = (X, *, \circ, 0)$ be a pseudo-BCK-algebra. A subset I of X is called an *ideal* of $\mathcal{X}$ if it satisfies for all $x, y \in X$:

(I1) $0 \in I$,

(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

Example 2.8 Let $\mathcal{X}$ be the pseudo-BCK-algebra from Example 2.1. Then $\{0\}$, $\{0, a\}$ and X are the only ideals of $\mathcal{X}$.

Example 2.9 Let $\mathcal{Y}$ be the pseudo-BCK-algebra from Example 2.2. Then $\{0\}$ and Y are the only ideals of $\mathcal{Y}$.

Example 2.10 Let $\mathcal{Z}$ be the pseudo-BCK-algebra from Example 2.3. Then$\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, a, b\}$, $\{0, a, c\}$, $\{0, b, c\}$ and Z are all ideals of $\mathcal{Z}$.

PROPOSITION 2.11 *Let I be an ideal of a pseudo-BCK-algebra $\mathcal{X}$. Then for any $x, y \in X$, if $y \in I$ and $x \leqslant y$, then $x \in I$.*

PROOF Straightforward. ■

PROPOSITION 2.12 *Let $\mathcal{X} = (X, *, \circ, 0)$ be a pseudo-BCK-algebra and let I be a subset of X. Then I is an ideal of $\mathcal{X}$ if and only if it satisfies conditions* (I1) *and*

(I2') *for all $x, y \in X$, if $x \circ y \in I$ and $y \in I$, then $x \in I$.*

PROOF It suffices to prove that if (I2) is satisfied, then (I2') is also satisfied. The proof of the converse of this implication is analogous. Suppose that $x \circ y \in I$ and $y \in I$. From (a2) we know that $x*(x\circ y) \leqslant y$. Then, by Proposition 2.11, $x*(x\circ y) \in I$. Hence, since $x \circ y \in I$, by (I2), we have $x \in I$. ■

By Proposition 2.4(ii) and Proposition 2.11 we have the following.

PROPOSITION 2.13 *Let $\mathcal{X}$ be a pseudo-BCK-algebra. If I is an ideal of $\mathcal{X}$, then it is a subalgebra of $\mathcal{X}$.*

Next example shows that the converse of Proposition 2.13 does not hold.

EXAMPLE 2.14 Let $\mathcal{X}$ be the pseudo-BCK-algebra from Example 2.1. Then $\{0, b\}$ and $\{0, a, b\}$ are subalgebras of $\mathcal{X}$ which are not ideals of $\mathcal{X}$.

3. Atoms of pseudo-BCK-algebras. Let $\mathcal{X}$ be a pseudo-BCK-algebra. A nonzero element a of X is an *atom* of $\mathcal{X}$ if for every $x \neq 0$,

$$x \leqslant a \text{ implies } x = a.$$

EXAMPLE 3.1 Let $\mathcal{X}$ be the pseudo-BCK-algebra from Example 2.1. Then a is the only atom of $\mathcal{X}$.

EXAMPLE 3.2 Let $\mathcal{Y}$ be the pseudo-BCK-algebra from Example 2.2. Then $\mathcal{Y}$ does not have any atoms.

EXAMPLE 3.3 Let $\mathcal{Z}$ be the pseudo-BCK-algebra from Example 2.3. Then a, b and c are all atoms of $\mathcal{Z}$.

PROPOSITION 3.4 *Let $\mathcal{X} = (X, *, \circ, 0)$ be a pseudo-BCK-algebra. If a is an atom of $\mathcal{X}$, then*

$$a * x = a \circ x \text{ for every } x \in X.$$

Proof Let a be an atom of $\mathcal{X}$ and let $x \in X$. First, assume that a and x are comparable. If $a \leqslant x$, then $a * x = 0 = a \circ x$. If $x \leqslant a$, then, since a is an atom, we have $x = a$. Hence, $a * x = a * a = 0 = a \circ a = a \circ x$. Now, assume that a and x are noncomparable. Then, $a * x \neq 0$ and $a \circ x \neq 0$. Hence, by Proposition 2.4(ii), $a * x \leqslant a$ and $a \circ x \leqslant a$. Since a is an atom, it follows that $a * x = a = a \circ x$. This finishes the proof. ■

Corollary 3.5 *Let $\mathcal{X}$ be a pseudo-BCK-algebra. If every nonzero element of X is an atom of $\mathcal{X}$, then $\mathcal{X}$ is a BCK-algebra.*

Let $\mathcal{X} = (X, *, \circ, 0)$ be a pseudo-BCK-algebra. A nonzero element a of X is called a *strong atom* of $\mathcal{X}$ if a is an atom of $\mathcal{X}$ and $a * x = a \circ x = a$ for every $x \in X$ and $x \neq a$.

Remark 3.6 Note that in the pseudo-BCK-algebra $\mathcal{X}$ from Example 2.1 do not exist any strong atoms and in the pseudo-BCK-algebra $\mathcal{Z}$ from Example 2.3 every atom is strong.

Now we give an example of a pseudo-BCK-algebra which has a strong atom as well as an atom that is not strong.

Example 3.7 ([6]) Let $W = \{0, a, b, c\}$ and define a binary operation $*$ on W by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then $\mathcal{W} = (W, *, 0)$ is a BCK-algebra, so also a pseudo-BCK-algebra. Note that a, c are atoms of $\mathcal{W}$ but only c is a strong atom of $\mathcal{W}$.

Let $\mathcal{X}$ be a pseudo-BCK-algebra. Set

$$At_s(X) = \{a \in X : a \text{ is a strong atom}\} \cup \{0\}.$$

Theorem 3.8 *Let $\mathcal{X}$ be a pseudo-BCK-algebra. Then the set $At_s(X)$ is a subalgebra of $\mathcal{X}$ which is also an ideal of $\mathcal{X}$.*

Proof Let $\mathcal{X} = (X, *, \circ, 0)$ be a pseudo-BCK-algebra. Suppose $a, b \in At_s(X)$. Then, $a * b = a$ or $a * b = 0$, so $a * b \in At_s(X)$. Similarly, $a \circ b \in At_s(X)$. Hence, $At_s(X)$ is a subalgebra of $\mathcal{X}$. Now, let $a, b \in X$ be such that $a * b \in At_s(X)$ and $b \in At_s(X)$. If $a * b = 0$, then $a = 0$ or $a = b$, and thus, $a \in At_s(X)$. If $a * b \neq 0$, then, from $a * b \leqslant a$, we conclude $a = a * b$, i.e., $a \in At_s(X)$. Therefore, $At_s(X)$ is an ideal of $\mathcal{X}$. ■

It is easy to prove the following.

PROPOSITION 3.9 *Let $\mathcal{X}$ be a pseudo-BCK-algebra. Then every nonzero element of X is an atom of $\mathcal{X}$ if and only if $At_s(X) = X$.*

From Corollary 3.5 and Proposition 3.9 we conclude the following.

COROLLARY 3.10 *Let $\mathcal{X}$ be a pseudo-BCK-algebra. If $At_s(X) = X$, then $\mathcal{X}$ is a BCK-algebra.*

Let $\mathcal{X} = (X, *, \circ, 0)$ and $\mathcal{X}' = (X', *', \circ', 0')$ be pseudo-BCK-algebras. A map $f : X \to X'$ is called a *homomorphism* if $f(x * y) = f(x) *' f(y)$ and $f(x \circ y) = f(x) \circ' f(y)$ for all $x, y \in X$.

THEOREM 3.11 *Let $f : X \to X'$ be a homomorphism of pseudo-BCK-algebras $\mathcal{X}$ and $\mathcal{X}'$. Then*

(i) *$f(At_s(X)) \subseteq At_s(f(X))$,*

(ii) *if $At_s(X) = X$, then $At_s(f(X)) = f(X)$.*

PROOF (i) Let $a' \in f(At_s(X))$. Then, there exists $a \in At_s(X)$ such that $f(a) = a'$. Let $x' \in f(X)$ be such that $a' \neq x'$. Then, $f(x) = x'$ for some $x \in X$. Note that $a \neq x$. Indeed, if $a = x$, then $a' = f(a) = f(x) = x'$ and we get a contradiction. Hence, $a * x = a \circ x = a$. So $a' *' x' = f(a) *' f(x) = f(a * x) = f(a) = a'$ and similarly, $a' \circ' x' = a'$. Thus, $a' \in At_s(f(X))$ and we have the inclusion.

(ii) Straightforward. ∎

COROLLARY 3.12 *Let $f : X \to X'$ be a surjective homomorphism of pseudo-BCK-algebras $\mathcal{X}$ and $\mathcal{X}'$. If $At_s(X) = X$, then $At_s(X') = X'$.*

4. Normal pseudo-BCK-algebra. A pseudo-BCK-algebra $\mathcal{X}$ is called *normal* if every subalgebra of $\mathcal{X}$ is an ideal of $\mathcal{X}$.

REMARK 4.1 Note that pseudo-BCK-algebras $\mathcal{X}$ from Example 2.1 and $\mathcal{Y}$ from Example 2.2 are not normal, and pseudo-BCK-algebra $\mathcal{Z}$ from Example 2.3 is normal.

The following proposition is obvious.

PROPOSITION 4.2 *Let $\mathcal{X}$ be a normal pseudo-BCK-algebra. Then every subalgebra of $\mathcal{X}$ is normal.*

THEOREM 4.3 *A pseudo-BCK-algebra $\mathcal{X} = (X, *, \circ, 0)$ is normal if and only if $x * y = x$ for all $x, y \in X$ such that $x \neq y$.*

Proof Assume that $\mathcal{X} = (X, *, \circ, 0)$ is normal nontrivial pseudo-BCK-algebra. Then, since, for all $a \in X$, $\{0, a\}$ is a subalgebra of $\mathcal{X}$, it is also an ideal of $\mathcal{X}$. Let $x, y \in X$ be such that $x \neq y$. We know that $x * y \leqslant x$. So, $x * y \in \{0, x\}$ because $\{0, x\}$ is an ideal. Thus, $x * y = 0$ or $x * y = x$. If $x * y = 0$, then $x \leqslant y$, and hence, $x \in \{0, y\}$, i.e., $x = 0$. Therefore, $x * y = x$.

Conversely, let S be an arbitrary subalgebra of $\mathcal{X}$. Let $x * y, y \in S$. If $x = y$, then, obviously, $x \in S$. Assume that $x \neq y$. Then, by assumption, $x = x * y$, and hence, $x \in S$. Thus, S is an ideal of $\mathcal{X}$, so $\mathcal{X}$ is normal. ∎

Next theorem is analogous to Theorem 4.3.

Theorem 4.4 *A pseudo-BCK-algebra $\mathcal{X} = (X, *, \circ, 0)$ is normal if and only if $x \circ y = x$ for all $x, y \in X$ such that $x \neq y$.*

From Theorems 4.3 and 4.4 we have the following.

Corollary 4.5 *A normal pseudo-BCK-algebra is a normal BCK-algebra.*

Theorem 4.6 *Let $\mathcal{X}$ be a pseudo-BCK-algebra. Then the following are equivalent:*

(i) *$\mathcal{X}$ is a normal pseudo-BCK-algebra,*

(ii) *every nonzero element of X is an atom of $\mathcal{X}$.*

Proof (i) ⇒ (ii): Assume that $\mathcal{X}$ is normal. Let a be a nonzero element of X. Assume that $x \leqslant a$ for $x \neq 0$. Suppose, $x \neq a$. Then, by Theorem 4.3, $x = x * a = 0$ and we get a contradiction. So, a is an atom of $\mathcal{X}$.

(ii) ⇒ (i): Assume that every nonzero element of X is an atom of $\mathcal{X}$. Let $x, y \in X$ be such that $x \neq y$. By Proposition 2.4(ii), $x * y \leqslant x$. If $x = 0$, then for $y \neq 0$, we have $x * y = 0 * y = 0 = x$. If $x \neq 0$, then x is an atom of $\mathcal{X}$. Hence, $x * y \leqslant x$ implies $x * y = x$. Thus, $x * y = x$ for all $x \neq y$. Now, from Theorem 4.3 we conclude that $\mathcal{X}$ is normal. ∎

From Proposition 3.9 and Theorem 4.6 we obtain the following.

Theorem 4.7 *Let $\mathcal{X}$ be a pseudo-BCK-algebra. Then $\mathcal{X}$ is normal if and only if $At_s(X) = X$.*

Results of this paper we can gather in following theorem. Individual equivalences we have from Theorems 4.3, 4.4, 4.6, 4.7, Proposition 3.9 and Corollary 4.5.

Theorem 4.8 *Let $\mathcal{X} = (X, *, \circ, 0)$ be a pseudo-BCK-algebra. Then the following are equivalent:*

(i) *$\mathcal{X}$ is a normal pseudo-BCK-algebra,*

(ii) *$x * y = x$ for all $x, y \in X$ such that $x \neq y$,*

(iii) $x \circ y = x$ *for all* $x, y \in X$ *such that* $x \neq y$,

(iv) $\mathcal{X}$ *is a normal BCK-algebra,*

(v) *every subalgebra of* $\mathcal{X}$ *is an ideal of* $\mathcal{X}$,

(vi) *every nonzero element of* X *is an atom of* $\mathcal{X}$,

(vii) $At_s(X) = X$.

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