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On the Existence of Common Fixed Points for Semigroups of Nonlinear Mappings in Modular Function Spaces

Abstract. Let C be a ρ-bounded, ρ-closed, convex subset of a modular function space L_ρ. We investigate the existence of common fixed points for semigroups of nonlinear mappings $T_t : C \to C$, i.e. a family such that $T_0(x) = x$, $T_{s+t} = T_s(T_t(x))$, where each T_t is either ρ-contraction or ρ-nonexpansive. We also briefly discuss existence of such semigroups and touch upon applications to differential equations.

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1. Introduction. The purpose of this paper is to prove the existence of common fixed points for semigroups of nonlinear mappings acting in modular function spaces which are natural generalizations of both function and sequence variants of many important, from applications perspective, spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others, see the book by Kozlowski [21] for an extensive list of examples and special cases. See also the paper by Cerda, Hudzik and Mastylo [6] for the definition and geometrical properties of Calderon-Lozanovskii spaces. Recently, Khamsi and Kozlowski presented a series of fixed point results for pointwise contractions, asymptotic pointwise contractions, pointwise nonexpansive and asymptotic pointwise nonexpansive mappings acting in modular functions spaces, [14], [15] (all these should be considered in the modular sense, not in the sense of the corresponding norms). These results are also new and of a big interest even in a much simpler context of "plain" modular contractions and nonexpansive mappings, i.e. without any pointwise and asymptotic complications.

In many cases, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts.

Furthermore, there are also important results that can be proved only using the apparatus of modular function spaces. Khamsi, Kozlowski and Reich gave in [16] an example of a mapping which is ρ-nonexpansive but it is not norm-nonexpansive. They demonstrated that for a mapping T to be norm nonexpansive in a modular function space L_ρ, a stronger than ρ-nonexpansiveness assumption is needed: $\rho(\lambda(T(x) - T(x)) \leqslant \rho(\lambda(x - y))$ for any $\lambda \geqslant 0$. From this perspective, the fixed point theory in modular function spaces should be considered as complementary to the fixed point theory in normed spaces and in metric spaces.

Let us recall that a family $\{T_t\}_{t\geqslant 0}$ of mappings forms a semigroup if $T_0(x) = x$, $T_{s+t} = T_s(T_t(x))$, see Definitions 2.12 and 2.13 for details. Such a situation is quite typical in mathematics and applications. For instance, in the theory of dynamical systems, the modular function space L_ρ would define the state space and the mapping $(t, x) \to T_t(x)$ would represent the evolution function of a dynamical system. The question about the existence of common fixed points, and about the structure of the set of common fixed points, can be interpreted as a question whether there exist points that are fixed during the state space transformation T_t at any given point of time t, and if yes - what the structure of a set of such points may look like. In the setting of this paper, the state space may be an infinite dimensional. Therefore, it is natural to apply these result to not only to deterministic dynamical systems but also to stochastic dynamical systems.

The existence of common fixed points for families of contractions and nonexpansive mappings in Banach spaces have been investigated since the early 1960s, see e.g. DeMarr [7], Browder [4], Belluce and Kirk [1, 2], Lim [24], Bruck [5]. The asymptotic approach for finding common fixed points of semigroups of Lipschitzian (but not pointwise Lipschitzian) mappings has been also investigated for some time, see e.g. Tan and Xu [32]. It is worthwhile mentioning the recent studies on the special case, when the parameter set for the semigroup is equal to $\{0, 1, 2, 3, ...\}$ and $T_n = T^n$, the n-th iterate of an asymptotic pointwise nonexpansive mapping, i.e. such a $T : C \to C$ that there exists a sequence of functions $\alpha_n : C \to [0, \infty)$ with $\|T^n(x) - T^n(y)\| \leqslant \alpha_n(x)\|x - y\|$. Kirk and Xu [18] proved the existence of fixed points for asymptotic pointwise contractionsa and asymptotic pointwise nonexpansive mappings in Banach spaces, while Hussain and Khamsi extended this result to metric spaces [10], and Khamsi and Kozlowski to modular function spaces [14], [15]. Kozlowski in [22] proved convergence to fixed points of some iterative algorithms applied to asymptotic pointwise nonexpansive mappings in Banach spaces and the existence of common fixed points in [23]. In the context of modular function spaces, Khamsi discussed in [12] the existence of nonlinear semigroups in Musielak-Orlicz spaces and considered some applications to differential equations, see the last section of the current paper.

The paper is organized as follows:

(a) Section 2 provides necessary preliminary material.

(b) Section 3 provides the fixed point theorem for semigroups of modular contractions.

(c) Section 4 provides the fixed point theorem for semigroups of modular nonexpansive mappings.

(d) Section 5 illustrates the existence of modular nonexpansive semigroups and some applications to differential equations.

2. Preliminaries. Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leqslant |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.1 Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if:

(i) $\rho(0) = 0$;

(ii) ρ is monotone, i.e. $|f(\omega)| \leqslant |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leqslant \rho(g)$, where $f, g \in \mathcal{M}_\infty$;

(iii) ρ is orthogonally subadditive, i.e. $\rho(f1_{A\cup B}) \leqslant \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}$;

(iv) ρ has the Fatou property, i.e. $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;

(v) ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly as in the case of measure spaces, we we say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty \ \rho - a.e\}, \tag{1}$$

where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.2 Let ρ be a regular function pseudomodular.

1. We say that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0 \ \rho - a.e.$;
2. We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0 \ \rho - a.e.$;

The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\Re$.

Let us denote $\rho(f, E) = \rho(f1_E)$ for $f \in \mathcal{M}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudomodular in the sense of Def.2.1.1 in [21] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function spaces as per the framework defined by Kozlowski in [19, 20, 21], see also Musielak [25] for the basics of the general modular theory.

Remark 2.3 We limit ourselves to convex function modulars in this paper. However, omitting convexity in Definition 2.1 or replacing it by s-convexity would lead to the definition of nonconvex or s-convex regular function pseudomodulars, semimodulars and modulars as in [21].

Definition 2.4 [19, 20, 21] Let ρ be a convex function modular.

(a) A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by
$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}.$$

(b) The following formula defines a norm in L_ρ (frequently called Luxemurg norm):
$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \leqslant 1\}.$$

In the following theorem we recall some of the properties of modular spaces that will be used later on in this paper.

Theorem 2.5 *[19, 20, 21] Let $\rho \in \Re$.*

1. *$L_\rho, \|f\|_\rho$ is complete and the norm $\|\cdot\|_\rho$ is monotone w.r.t. the natural order in $\mathcal{M}$.*
2. *$\|f_n\|_\rho \to 0$ if and only if $\rho(\alpha f_n) \to 0$ for every $\alpha > 0$.*
3. *If $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0$ $\rho - a.e.$*
4. *If $\{f_n\}$ converges uniformly to f on a set $E \in \mathcal{P}$ then $\rho(\alpha(f_n - f), E) \to 0$ for every $\alpha > 0$.*
5. *Let $f_n \to f$ $\rho - a.e.$ There exists a nondecreasing sequence of sets $H_k \in \mathcal{P}$ such that $H_k \uparrow \Omega$ and $\{f_n\}$ converges uniformly to f on every H_k (Egoroff Theorem).*
6. *$\rho(f) \leqslant \liminf \rho(f_n)$ whenever $f_n \to f$ $\rho-a.e.$ (Note: this property is equivalent to the Fatou Property).*
7. *Defining $L^0_\rho = \{f \in L_\rho;\ \rho(f, \cdot)$ is order continuous$\}$ and $E_\rho = \{f \in L_\rho;\ \lambda f \in L^0_\rho$ for every $\lambda > 0\}$ we have:*

   (a) $L_\rho \supset L^0_\rho \supset E_\rho$,

(b) E_ρ *has the Lebesgue property, i.e.* $\rho(\alpha f, D_k) \to 0$ *for* $\alpha > 0$, $f \in E_\rho$ *and* $D_k \downarrow \emptyset$.

(c) E_ρ *is the closure of* $\mathcal{E}$ *(in the sense of* $\|\cdot\|_\rho$*).*

The following definition plays an important role in the theory of modular function spaces.

Definition 2.6 Let $\rho \in \Re$. We say that ρ has the Δ_2-property if $\sup_n \rho(2f_n, D_k) \to 0$ whenever $D_k \downarrow \emptyset$ and $\sup_n \rho(f_n, D_k) \to 0$.

Theorem 2.7 *Let* $\rho \in \Re$. *The following conditions are equivalent:*

(a) ρ *has* Δ_2,

(b) L^0_ρ *is a linear subspace of* L_ρ,

(c) $L_\rho = L^0_\rho = E_\rho$,

(d) if $\rho(f_n) \to 0$ *then* $\rho(2f_n) \to 0$,

(e) if $\rho(\alpha f_n) \to 0$ *for an* $\alpha > 0$ *then* $\|f_n\|_\rho \to 0$, *i.e. the modular convergence is equivalent to the norm convergence.*

We will also use another type of convergence which is situated between norm and modular convergence. It is defined, among other important terms, in the following definition.

Definition 2.8 Let $\rho \in \Re$.

(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to 0$ (ρ) if and only if $\rho(f_n - f) \to 0$.

(b) A sequence $\{f_n\}$ where $f_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

(c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f$ (ρ) implies that f belongs to B.

(d) A set $B \subset L_\rho$ is called ρ-bounded if $\sup\{\rho(f - g); f \in B, g \in B\} < \infty$

(e) A set $C \subset L_\rho$ is called ρ-a.e. closed if for any $\{f_n\}$ in C which ρ-a.e. converges to some f, then we must have $f \in C$;

(f) A set $C \subset L_\rho$ is called ρ-a.e. compact if for any $\{f_n\}$ in C, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e. converges to some $f \in C$.

(g) Let $f \in L_\rho$ and $C \subset L_\rho$. The ρ-distance between f and C is defined as

$$d_\rho(f, C) = inf\{\rho(f - g); g \in C\}.$$

Let us note that ρ-convergence does not necessarily imply ρ-Cauchy condition. Also, $f_n \to f$ does not imply in general $\lambda f_n \to \lambda f,\ \lambda > 1$. Using Theorem 2.5 it is not difficult to prove the following

PROPOSITION 2.9 *Let $\rho \in \Re$.*

(i) L_ρ is ρ-complete,

(ii) ρ-balls $B_\rho(x,r) = \{y \in L_\rho; \rho(x-y) \leqslant r\}$ are ρ-closed and ρ-a.e. closed.

Let us introduce a notion of a ρ-type, a powerful technical tool which will be used in the proofs of our fixed point results.

DEFINITION 2.10 Let $K \subset L_\rho$ be convex and ρ-bounded.

1. A function $\tau : K \to [0,\infty]$ is called a ρ-type (or shortly a type) if there exists a net $\{y_t\}_{t\geqslant 0}$ of elements of K such that for any $z \in K$ there holds

$$\tau(z) = \limsup_{t\to\infty} \rho(y_t - z).$$

2. Let τ be a type. A sequence $\{g_n\}$ is called a minimizing sequence of τ if

$$\lim_{n\to\infty} \tau(g_n) = \inf\{\tau(f); f \in K\}.$$

Note that τ is convex provided ρ is convex.

Let us finish this section with the modular definitions of Lipschitzian, contractive and nonexpansive mappings, and of associated definitions of semigroups of nonlinear mappings.

DEFINITION 2.11 Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called a ρ-Lipschitzian if there exists a constant $0 < L$ such that

$$\rho(T(f) - T(g)) \leqslant L\rho(f-g)\ for\ any\ f,g \in L_\rho.$$

T is called a ρ-contraction if $L < 1$. T is called a ρ-nonexpansive mapping if $L = 1$.

DEFINITION 2.12 A one-parameter family $\mathcal{F} = \{T_t; t \geqslant 0\}$ of mappings from C into itself is said to be a ρ-Lipschitzian (resp. ρ-nonexpansive) semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;

(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t,s \geqslant 0$;

(iii) for each $t \geqslant 0$, T_t is ρ-Lipschitzian (resp. ρ-nonexpansive).

Definition 2.13 A one-parameter family $\mathcal{F} = \{T_t; t \geqslant 0\}$ of mappings from C into itself is said to be a ρ-contractive semigroup on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;

(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \geqslant 0$;

(iii) for each $t \geqslant 0$, T_t is a ρ-contraction with a constant $0 < L_t < 1$ such that $\limsup_{t\to\infty} L_t < 1$.

3. Common Fixed Points for Contractive Semigroups. Let us start by recalling the definition of the Strong Opial property introduced by Khamsi and Kozlowski in [14].

Definition 3.1 We will say that L_ρ satisfies the ρ-a.e. Strong Opial property (or shortly *SO*-Property) if for every $\{f_n\} \in L_\rho$ which is ρ-a.e. convergent to 0 such that there exists a $\beta > 1$ for which

$$\sup\{\rho(\beta f_n)\} < \infty, \tag{2}$$

the following equality holds for any $g \in E_\rho$

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g). \tag{3}$$

Remark 3.2 Note that the ρ-a.e. Strong Opial property implies ρ-a.e. Opial property (see the paper by Khamsi [13] for definition of the Opial Property in modular function spaces).

Remark 3.3 Also, note that, in virtue of Theorem 2.1 in [13], every convex, orthogonally additive function modular ρ has the ρ-a.e. Strong Opial property. Let us recall that ρ is called orthogonally additive if $\rho(f, A\cup B) = \rho(f, A)+\rho(f, B)$ *whenever* $A\cap B = \emptyset$. Therefore, all Orlicz and Musielak-Orlicz spaces must have the Strong Opial Property.

Remark 3.4 The ρ-a.e. Strong Opial property can be also defined and proved for nonconvex regular function modulars, e.g. for some s-convex modulars ($s < 1$) like L^s for $0 < s < 1$, [13, 3].

A typical method of proof for the fixed point theorems in Banach and metric spaces is to construct a fixed point by finding an element on which a specific type function attains its minimum. To be able to proceed with this method, one has to know that such an element indeed exists. In Banach and metric space, the types are lower semicontinuous and hence they attain their minimum at some point in C. In modular function spaces the ρ-types are not in general lower semicontinuous and therefore one needs additional assumptions to ensure that ρ-types attain their minima.

LEMMA 3.5 *Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $C \subset E_\rho$ be a nonempty, ρ-a.e. compact convex subset such that $\delta_\rho(\beta\ C) = \sup\{\rho(\beta(x-y)); x, y \in C\} < \infty$, for some $\beta > 1$. Then any ρ-type defined in C attains its minimum in C.*

PROOF Let us fix a ρ-type τ defined by

$$\tau(u) = \limsup_{t\to\infty} \rho(u - u_t) \tag{4}$$

where $u_t \in C$ for every $t \geqslant 0$, and denote $\tau_0 = \inf\{\tau(u); u \in C\}$. Note that $\tau_0 \leqslant \delta_\rho(\beta\ C) < \infty$. Hence, there exists a sequence $\{x_n\} \subset C$ such that

$$\tau_0 = \lim_{n\to\infty} \tau(x_n). \tag{5}$$

Since C is ρ-a.e. compact, by passing to a subsequence if necessary we can assume that there exists an $x_0 \in C$ such that $x_n \to x_0$ ρ-a.e. Let us select a sequence $t_n \to \infty$ so that

$$\tau(x_0) = \limsup_{t\to\infty} \rho(x_0 - u_t) = \lim_{n\to\infty} \rho(x_0 - u_{t_n}), \tag{6}$$

and denote $y_n = u_{t_n}$. By the ρ-a.e. compactness of C again, there exists a subsequence $\{y_{\varphi(n)}\}$ of $\{y_n\}$ which ρ-a.e. converges to some $y_0 \in C$. By the ρ-a.e. Strong Opial property we get

$$\liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_m) = \liminf_{n\to\infty} \rho(y_{\varphi(n)} - y_0) + \rho(y_0 - x_m), \tag{7}$$

for any $m \geqslant 0$. Since

$$\begin{aligned}\liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_m) &\leqslant \limsup_{n\to\infty} \rho(y_n - x_m) = \limsup_{n\to\infty} \rho(x_m - u_{t_n}) \\ &\leqslant \limsup_{t\to\infty} \rho(x_m - u_t) = \tau(x_m),\end{aligned} \tag{8}$$

we conclude via (7) that

$$\liminf_{m\to\infty} \tau(x_m) \geqslant \liminf_{n\to\infty} \rho(y_{\varphi(n)} - y_0) + \liminf_{m\to\infty} \rho(y_0 - x_m). \tag{9}$$

Using the ρ-a.e. Strong Opial property again, to $\{x_m - x_0\}$ this time, we get

$$\liminf_{m\to\infty} \rho(y_0 - x_m) = \liminf_{m\to\infty} \rho(x_m - x_0) + \rho(x_0 - y_0) \tag{10}$$

which implies

$$\liminf_{m\to\infty} \tau(x_m) \geqslant \liminf_{n\to\infty} \rho(y_{\varphi(n)} - y_0) + \liminf_{m\to\infty} \rho(x_m - x_0) + \rho(x_0 - y_0). \tag{11}$$

Hence

$$\liminf_{m\to\infty} \tau(x_m) \geqslant \liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_0) + \liminf_{m\to\infty} \rho(x_m - x_0), \tag{12}$$

which implies

$$\liminf_{m\to\infty} \tau(x_m) \geqslant \liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_0). \tag{13}$$

Using (6) we get

$$\liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_0) = \lim_{n\to\infty} \rho(y_n - x_0) = \tau(x_0). \tag{14}$$

Combining (14) with (13), we get

$$\tau_0 = \lim_{m\to\infty} \tau(x_m) \geqslant \tau(x_0). \tag{15}$$

On the other hand,

$$\tau_0 = \inf\{\tau(u); u \in C\} \leqslant \tau(x_0). \tag{16}$$

Finally $\tau(x_0) = \tau_0 = \inf\{\tau(u); u \in C\}$, which completes the proof of the lemma. ■

THEOREM 3.6 *Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $C \subset E_\rho$ be a nonempty, ρ-a.e. compact convex subset such that $\delta_\rho(\beta\ C) = \sup\{\rho(\beta(x-y)); x, y \in C\} < \infty$, for some $\beta > 1$. Let $\mathcal{F}$ be a ρ-contractive semigroup on C. Then $\mathcal{F}$ has a unique common fixed point $z \in C$ and for each $u \in C$, $\rho(T_t(u) - z) \to 0$ as $t \to \infty$.*

PROOF First, let us prove the uniqueness. Assume that $z, w \in \mathcal{F}$. Then we have

$$\rho(z - w) = \rho(T_t(z) - T_t(w)) \leqslant L_t \rho(z - w) \tag{17}$$

for any $t \geqslant 0$. If we let $t \to \infty$, we will get

$$\rho(z - w) \leqslant L\rho(z - w), \tag{18}$$

where

$$L = \limsup_{t\to\infty} L_t. \tag{19}$$

Since $L < 1$, we conclude that $\rho(z - w) = 0$ or equivalently that $z = w$, hence there must exist at most one common fixed point for $\mathcal{F}$.

To prove the existence of the common fixed point, let us fix any $x \in C$ and define the ρ-type τ by

$$\tau(u) = \limsup_{t\to\infty} \rho(T_t(x) - u) \tag{20}$$

for $u \in C$. By Lemma 3.5, there exists an $z \in C$ such that

$$\tau(z) = \inf\{\tau(y); y \in C\}. \tag{21}$$

Let us prove that $\tau(z) = 0$. To this end, take $s, t \geqslant 0$ and observe that by the nonexpansiveness of T_t we have

$$\rho(T_{s+t}(x) - T_t(z)) \leqslant L_t \rho(T_s(x) - z), \tag{22}$$

and then letting $s \to \infty$

$$\tau(T_t(z)) \leqslant L_t \tau(z) \tag{23}$$

which implies the following

$$\tau(z) = \inf\{\tau(y); y \in C\} \leqslant \tau(T_t(z)) \leqslant L_t\ \tau(z). \tag{24}$$

Passing with $t \to \infty$ we get

$$\tau(z) \leqslant L\tau(z). \tag{25}$$

Since $0 < L < 1$ it follows that $\tau(z) = 0$. Hence, by (20),

$$\lim_{t\to\infty} \rho(T_t(x) - z) = 0. \tag{26}$$

Since T_s is a ρ-contraction for any given $s \geqslant 0$, it follows that

$$\lim_{t\to\infty} \rho(T_t(x) - T_s(z)) = \lim_{t\to\infty} \rho(T_{s+t}(x) - T_s(z)) \leqslant \lim_{t\to\infty} \rho(T_t(x) - z) = 0. \tag{27}$$

By the uniqueness of the ρ-limit, we conclude from (26) and (27) that $T_s(z) = z$ for any $s \geqslant 0$, i.e $z \in \mathcal{F}$.

To prove the convergence of orbits, let us fix any $u \in C$. We shall prove that $\{T_t(u)\}$ converges to z. Indeed we have

$$\rho(T_{t+s}(u) - z) = \rho(T_{t+s}(u) - T_t(z)) \leqslant L_t \rho(T_s(u) - z), \tag{28}$$

for any $t, s \geqslant 0$. Hence

$$\limsup_{s\to\infty} \rho(T_{t+s}(u) - z) \leqslant \limsup_{s\to\infty} L_t\ \rho(T_s(u) - z). \tag{29}$$

Since $\limsup_{s\to\infty} \rho(T_{t+s}(u) - z) = \limsup_{t\to\infty} \rho(T_s(u) - z)$, we get

$$\limsup_{s\to\infty} \rho(T_s(u) - z) \leqslant L_t \limsup_{s\to\infty} \rho(T_s(u) - z), \tag{30}$$

for any $t \geqslant 0$. If we let $t \to \infty$, we obtain

$$\limsup_{s\to\infty} \rho(T_s(u) - z) \leqslant L \limsup_{s\to\infty} \rho(T_s(u) - z). \tag{31}$$

Since $L < 1$, we get

$$\limsup_{s\to\infty} \rho(T_s(u) - z) = 0. \tag{32}$$

Clearly we can derive the same equality where lim sup is replaced by lim inf which implies the desired conclusion:

$$\lim_{s\to\infty} \rho(T_s(u) - z) = 0. \tag{33}$$

4. Common Fixed Points for Nonexpansive Semigroups. We recall the following concepts related to the modular uniform convexity introduced in [14]:

DEFINITION 4.1 Let $\rho \in \Re$. We define the following uniform convexity type properties of the function modular ρ:

(i) Let $r > 0, \varepsilon > 0$. Define

$$D_1(r,\varepsilon) = \{(f,g); f,g \in L_\rho, \rho(f) \leqslant r, \rho(g) \leqslant r, \rho(f-g) \geqslant \varepsilon r\}.$$

Let

$$\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\Big(\frac{f+g}{2}\Big); (f,g) \in D_1(r,\varepsilon)\right\}, if D_1(r,\varepsilon) \neq \emptyset,$$

and $\delta_1(r,\varepsilon) = 1 \ if \ D_1(r,\varepsilon) = \emptyset$. We say that ρ satsifes $(UC1)$ if for every $r > 0, \varepsilon > 0$, $\delta_1(r,\varepsilon) > 0$. Note, that for every $r > 0$, $D_1(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(ii) We say that ρ satsifes $(UUC1)$ if for every $s \geqslant 0, \varepsilon > 0$ there exists

$$\eta_1(s,\varepsilon) > 0$$

depending on s and ε such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0 \ for \ r > s.$$

(iii) Let $r > 0, \varepsilon > 0$. Define

$$D_2(r,\varepsilon) = \{(f,g); f,g \in L_\rho, \rho(f) \leqslant r, \rho(g) \leqslant r, \rho\Big(\frac{f-g}{2}\Big) \geqslant \varepsilon r\}.$$

Let

$$\delta_2(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\Big(\frac{f+g}{2}\Big); (f,g) \in D_2(r,\varepsilon)\right\}, if D_2(r,\varepsilon) \neq \emptyset,$$

and $\delta_2(r,\varepsilon) = 1 \ if \ D_2(r,\varepsilon) = \emptyset$. We say that ρ satsifes $(UC2)$ if for every $r > 0, \varepsilon > 0$, $\delta_2(r,\varepsilon) > 0$. Note, that for every $r > 0$, $D_2(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(iv) We say that ρ satsifes $(UUC2)$ if for every $s \geqslant 0, \varepsilon > 0$ there exists

$$\eta_2(s,\varepsilon) > 0$$

depending on s and ε such that

$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0 \ for \ r > s.$$

PROPOSITION 4.2 *[14] It is easy to prove the following conditions characterizing relationship between the above defined notions:*

1. *For* $i = 1, 2,\ \delta_i(r, 0) = 0\ ,\ \ \delta_i(r, 0) \leqslant 1,$ *and* $\delta_i(r,\varepsilon)$ *is an increasing function of* ε *for every fixed* r.
2. $(UUCi) \Rightarrow (UCi)\ for\ i = 1, 2.$
3. $\delta_1(r,\varepsilon) \leqslant \delta_2(r,\varepsilon)$
4. $(UC1) \Rightarrow (UC2)$
5. $(UUC1) \Rightarrow (UUC2)$

LEMMA 4.3 *Let* ρ *satisfies* $(UC2)$. *Let* $f, g \in L_\rho$ *and* $r > 0$ *be such that* $f \neq g$, $\rho(f) \leqslant r$ *and* $\rho(g) \leqslant r$. *Then*

$$\rho\Big(\frac{f+g}{2}\Big) < r. \tag{34}$$

PROOF Set $h = f - g$. Since $f \neq g$ then $\rho(\frac{h}{2}) > 0$. By the definition of δ_2 it follows from

$$\rho\Big(\frac{f-g}{2}\Big) = \rho\Big(\frac{h}{2}\Big) = \Big(\frac{1}{r}\rho\Big(\frac{h}{2}\Big)\Big)r \tag{35}$$

that

$$\rho\Big(\frac{f+g}{2}\Big) \leqslant r\Big[1 - \delta_2\Big(r, \frac{1}{r}\rho\Big(\frac{h}{2}\Big)\Big)\Big]. \tag{36}$$

Notice that

$$\delta_2\Big(r, \frac{1}{r}\rho\Big(\frac{h}{2}\Big)\Big) > 0 \tag{37}$$

because ρ satisfies $(UC2)$ and $\frac{1}{r}\rho\Big(\frac{h}{2}\Big) > 0$. Hence

$$\rho\Big(\frac{f+g}{2}\Big) < r, \tag{38}$$

as claimed. ∎

Lemma 4.4 *Let ρ satisfies $(UC2)$. Then for every $r > 0$, $\delta_2(r, 1) = 1$.*

Proof Fix $r > 0$. Let $(x, y) \in D_2(r, 1)$. i.e. $\rho(x) \leqslant r$, $\rho(y) \leqslant r$, and

$$\rho\Big(\frac{x-y}{2}\Big) \geqslant r. \tag{39}$$

Observe that

$$r \leqslant \rho\Big(\frac{x-y}{2}\Big) \leqslant \frac{\rho(x)+\rho(y)}{2} \leqslant r. \tag{40}$$

Hence

$$\rho\Big(\frac{x-y}{2}\Big) = r. \tag{41}$$

Setting $z = -y$ we get

$$r = \rho\Big(\frac{x-z}{2}\Big) = \rho\Big(\frac{x+y}{2}\Big) \leqslant r. \tag{42}$$

By Lemma 4.3 applied to $f = x$ and $g = z$ we obtain that $x = z$, i.e. $x = -y$, which implies that

$$\rho\Big(\frac{x+y}{2}\Big) = 0. \tag{43}$$

Hence, by the definition of δ_2 we conclude that $\delta_2(r, 1) = 1$ as claimed. ■

The following property plays in the theory of modular function spaces a role similar to the reflexivity in Banach spaces (see e.g. [17]).

Definition 4.5 We say that L_ρ has property (R) if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.

Similarly as in the Banach space case, the modular uniform convexity implies the property (R):

Theorem 4.6 *[17] Let $\rho \in \Re$ be $(UUC1)$ then L_ρ has property (R).*

Let us start with the net version of the minimizing sequence property proved for types defined by sequences in Lemma 4.3 in [14]. The proof for the net version is a straightforward generalization of the proof from [14].

Lemma 4.7 *Assume that $\rho \in \Re$ is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset. Let τ be a type defined by a net $\{x_t\}_{t\geqslant 0} \subset C$. Then any minimizing sequence of τ is ρ-convergent. Its limit is independent of the minimizing sequence.*

Using Lemma 4.7, we are ready to prove the following fixed point result for nonexpansive semigroups.

THEOREM 4.8 *Assume $\rho \in \Re$ is (UUC1). Let C be a ρ-closed ρ-bounded convex nonempty subset. Let $\mathcal{F}$ be a nonexpansive semigroup on C. Then the set $F(\mathcal{F})$ of common fixed points is nonempty, ρ-closed and convex.*

PROOF Let us fix an $x \in C$ and define the ρ-type

$$\tau(y) = \limsup_{t\to\infty} \rho(T_t(x) - y), \tag{44}$$

where $y \in C$. Let $\tau_0 = \inf\{\tau(y); y \in C\}$ and let $\{z_n\}$ be a minimizing sequence for τ, i.e.

$$\lim_{n\to\infty} \tau(z_n) = \tau_0. \tag{45}$$

By Lemma 4.7 there exists a $z \in C$ such that

$$\lim_{n\to\infty} \rho(z_n - z) = 0. \tag{46}$$

We are going to prove that $z \in F(\mathcal{F}$.

Noting that

$$\rho(T_{s+t}(x) - T_t(y)) \leqslant \rho(T_s(x) - t) \tag{47}$$

and passing with $s \to \infty$ we get

$$\tau(T_t(y)) \leqslant \tau(y). \tag{48}$$

In particular, for any $n \geqslant 1$ we have

$$\tau(T_t(z_n)) \leqslant \tau(z_n). \tag{49}$$

Let us fix any sequence $t_k \to \infty$ and note that for every $s > 0$ the sequence $\{T_{t_k+s}(z_k)\}$ is a minimizing sequence for τ. Indeed, using (49) we obtain

$$\tau_0 \leqslant \tau(T_{t_k+s}(z_k)) \leqslant \tau(z_k) \to \tau_0. \tag{50}$$

Hence, Lemma 4.7 implies that

$$\lim_{k\to\infty} \rho(T_{t_k+s}(z_k) - z) = 0, \tag{51}$$

and in particular for $s = 0$,

$$\lim_{k\to\infty} \rho(T_{t_k}(z_k) - z) = 0. \tag{52}$$

Using (52) we get

$$\rho(T_{t_k+s}(z_k) - T_s(z)) \leqslant \rho(T_{t_k}(z_k) - z) \to 0. \tag{53}$$

Since the ρ-limit is unique, (51) and (53) give $T_s(z) = z$, i.e. $z \in F(\mathcal{F})$ as claimed. Let us prove that $F(\mathcal{F})$ is ρ-closed. Let $x_n \in F(\mathcal{F}$ and $\rho(x_n - x) \to 0$. Observe that for every $t \geqslant 0$,

$$\begin{aligned} \rho\Big(\frac{1}{3}(T_t(x) - x)\Big) &\leqslant \rho(T_t(x) - T_t(x_n)) + \rho(T_t(x_n) - x_n) + \rho(x_n - x) \\ &\leqslant \rho(x_n - x) + \rho(x_n - x) \to 0. \end{aligned} \tag{54}$$

Hence, $x \in F(\mathcal{F})$ proving that $x \in F(\mathcal{F})$, and consequently, that $F(\mathcal{F})$ is ρ-closed.

To prove convexity of $F(\mathcal{F})$, we need to demonstrate that

$$w = \frac{u+v}{2} \in F(\mathcal{F}) \tag{55}$$

provided $u, v \in F(\mathcal{F})$. Indeed, let $t > 0$. Define $x = T_t(w) - u$, $y = T_t(w) - v$. Note that

$$\frac{x+y}{2} = T_t(w) - w. \tag{56}$$

Define

$$r = \rho\Big(\frac{v-u}{2}\Big) \tag{57}$$

and observe that

$$\rho(x) = \rho(T_t(w) - u) = \rho(T_t(w) - T_t(u)) \leqslant \rho(w-u) = \rho\Big(\frac{v-u}{2}\Big) = r. \tag{58}$$

Similarly, $\rho(y) \leqslant r$. Hence, $x, y \in D_2(r, 1)$ and therefore

$$\delta_2(r,1) \leqslant 1 - \frac{1}{r}\rho\Big(\frac{x+y}{2}\Big) = 1 - \frac{1}{r}\rho(T_t(w) - w). \tag{59}$$

Using the assumed $(UUC1)$ and Proposition 4.2, we conclude that ρ satisfies $(UC2)$. Hence, by Lemma 4.4, $\delta_2(r,1) = 1$, which yields

$$\frac{1}{r}\rho(T_t(w) - w) \leqslant 1 - \delta_2(r,1) = 0. \tag{60}$$

Hence, $T_t(w) = w$, i.e. $w \in F(\mathcal{F})$, completing the proof. ∎

5. Application - Existence of Nonlinear Semigroups. One can ask a ligitimate question about existence of natural examples of semigroups of nonlinear mappings in modular function spaces and their applications. In this section we present examples addressing these issues.

In [12] Khamsi considered the following initial value problem.

THEOREM 5.1 *[12] Let ρ be a convex Musielak-Orlicz function modular, and $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex. Let $T : C \to C$ be ρ-nonexpansive and norm-continuous, and let $f \in C$ and $A > 0$ be fixed. Consider the following initial value problem:*

$$\begin{cases} u(0) = f \\ u'(t) + (I - T)u(t) = 0, \end{cases} \tag{61}$$

where the unknown function $u : [0, A] \to L_\rho$. *Assume, in addition, that* ρ *satisfies the* Δ_2 *condition. Then there exists a solution* u_f *to* (61), $u_f(t) \in C$ *for every* $t \in [0, A]$ *and the solution* $u_f(t)$ *can be obtain as the* ρ*-limit of* $\{u_n(t)\}$ *where* u_n *are defined by the following recurrent sequence:*

$$\begin{cases} u_0(t) = f \\ u_{n+1} = e^{-t}f + \int_0^t e^{s-t}T(u_n(s))ds. \end{cases} \tag{62}$$

Let us define

$$S_t(f) = u_f. \tag{63}$$

It can be proved that $\{S_t\}$ forms a ρ-nonexpansive semigroup of nonlinear mappings in the sense of Definition 2.12. Hence, if in addition ρ is $(UCC1)$, it follows from Theorem 4.8 that the set of common fixed points for $\{S_t\}$ is nonempty. To interpret this fact, observe that if f_0 is such a common fixed point and we place the initial value of our system (61) at f_0 then this point becomes a stationary point of the system, i.e. the constant function $u_{f_0}(t) = f_0$ for every t is the solution of (61).

These results can be extended to systems where T is a ρ-Lipschitz operator [31], and applied to the perturbed integral equations in modular function spaces [9].

There exists an extensive literature on the question of representation of some types of semigroups of nonlinear mapings acting in Banach spaces, see e.g. [27, 11, 28, 8, 29]. It would be interesting to consider similar representation questions in modular function spaces.

Similarly, it would be interesting to discuss the modular ergodic theory for nonlinear semigroups defined in modular function spaces. For the Banach space results of this type, see e.g. [30, 32, 26].

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