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A note on the primitive function of a Bohr almost periodic function

Abstract. In this article we prove a sufficient condition for the primitive function of a uniformly almost periodic function to be Bohr almost periodic.

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1. Preliminaries. A set $E \subset \mathbb{R}$ is called *relatively dense* if there exists a positive number l such that every open interval of the real axis of length l contains at least one element of the set E. Given any $f, g \in C^{(n)}(\mathbb{R})$, $n \in \mathbb{N}_0$, we let

$$D^{(n)}(f,g) = \sup_{x \in \mathbb{R}} \left(|f(x) - g(x)| + \sum_{k=1}^{n} \left| (f-g)^{(k)}(x) \right| \right).$$

A number $\tau \in \mathbb{R}$ is called a $(D^{(n)}, \varepsilon)$-*almost period*($(D^{(n)}, \varepsilon)$-a.p.) of a function $f \in C^{(n)}(\mathbb{R})$ whenever $D^{(n)}(f_\tau, f) \leqslant \varepsilon$ where $\varepsilon > 0$ and $f_\tau(x) \equiv f(x+\tau)$. Let us denote by $E^{(n)}\{\varepsilon; f\}$ the set of all $(D^{(n)}, \varepsilon)$-almost periods of f. A function $f \in C^{(n)}(\mathbb{R})$ is said to be $C^{(n)}$-*almost periodic* ($C^{(n)}$-a.p.) if for an arbitrary $\varepsilon > 0$ the set $E^{(n)}\{\varepsilon; f\}$ is relatively dense (see [1]). In particular, when $n = 0$ one speaks about an ε-almost period of a function $f \in C(\mathbb{R})$ which is *uniformly almost periodic* (B-a.p.) if for all $\varepsilon > 0$ the set $E\{\varepsilon; f\}$ of its ε-almost periods is relatively dense (see [2] and [3]).

Let $\mathcal{R}_0$ denote the space of all finite real-valued functions defined on $\mathbb{R}$. Given any $f \in \mathcal{R}_0$ and $x \in \mathbb{R}$, we put

$$V(f;x) = \sup_{\Pi} \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|,$$

where $\Pi = \{x - 1 = x_0 < x_1 < \ldots < x_n = x + 1\}$. Given $f, g \in \mathcal{R}_0$, let

$$V(f,g) = \sup_{x\in\mathbb{R}} \left(|f(x) - g(x)| + V(f - g; x)\right).$$

Further, let $BV_{\rm loc}$ denote the set of all the functions $f \in \mathcal{R}_0$ which are of a locally finite variation, i.e. $V(f;x) < \infty$ for all $x \in \mathbb{R}$. If $V(f, f_\tau) \leqslant \varepsilon$ with $f \in BV_{\rm loc}$, then $\tau \in \mathbb{R}$ is called a (V,ε)-*almost period*((V,ε)-a.p.) of f. Finally, let $E_V\{\varepsilon; f\}$ be the set of all (V,ε)-almost periods of f. A continuous $f \in BV_{\rm loc}$ is called an *almost periodic function in the sense of variation* (V-a.p.) if for all $\varepsilon > 0$ the set $E_V\{\varepsilon; f\}$ is relatively dense (see [4]). The following holds (see [2]):

Theorem 1.1 (P. Bohl, H. Bohr) *In order that the primitive function*

$$F(x) = \int_0^x f(t)dt, \quad x \in \mathbb{R},$$

of a B-a.p. function f be uniformly a.p. it is necessary and sufficient that F is bounded.

It is known that if the primitive function F of a B-a.p. f is bounded, then F is $C^{(1)}$-a.p. (see [1]) as well as F is V-a.p. (see [4]).

2. Main result. Let f be a B-a.p. function and let $A_N^\gamma = \{\langle \tau - N, \tau + N\rangle : \tau \in E\{\gamma; f\}\}$ where $N, \gamma > 0$. We say that the primitive function F of f is N-*bounded with respect to f* ((N,f)-*bounded*) if for all $N, \gamma > 0$ there is an $M > 0$ such that $|F(x)| \leqslant M$ for all $x \in \bigcup A_N^\gamma$.

We say that the primitive function F of a B-a.p. function f is *strongly* (N,f)-*bounded* if F is (N,f)-bounded and for all $N > 0$ the functions

$$g_N(\gamma) = \inf_{x\in\bigcup A_N^\gamma} F(x) \quad , \quad G_N(\gamma) = \sup_{x\in\bigcup A_N^\gamma} F(x)$$

are respectively lower, upper semi-continuous on the right, uniformly with respect to $\gamma > 0$.

Theorem 2.1 *If f is a B-a.p. function and the primitive function*

$$F(x) = \int_0^x f(t)dt, \quad x \in \mathbb{R},$$

of f is strongly (N,f)-bounded, then F is uniformly a.p.

Proof Let us fix arbitrary $N > 0$ and $\varepsilon > 0$. By assumption, there is a $\delta = \delta(N,\varepsilon) > 0$ such that for all $0 < h < \delta$ the following hold

$$g_N(\gamma + h) - g_N(\gamma) \geqslant -\frac{\varepsilon}{3} \quad \text{and} \quad G_N(\gamma + h) - G_N(\gamma) \leqslant \frac{\varepsilon}{3}$$

uniformly with respect to $\gamma > 0$. With a fixed $\gamma_0 > 0$, the set $\{F(x) : x \in \bigcup A_N^{\gamma_0}\}$ is bounded, so that we have constants

$$g_N(\gamma_0) = \inf_{x\in\bigcup A_N^{\gamma_0}} F(x) \quad \text{and} \quad G_N(\gamma_0) = \sup_{x\in\bigcup A_N^{\gamma_0}} F(x).$$

Thus, there exist $x_1, x_2 \in \bigcup A_N^{\gamma_0}$ such that $d = |x_1 - x_2| \geqslant 0$ as well as

$$F(x_1) < g_N(\gamma_0) + \frac{\varepsilon}{24} \quad \text{and} \quad F(x_2) > G_N(\gamma_0) - \frac{\varepsilon}{24}.$$

Now, let $d_0 > d$ and $l_0 > 0$ with l_0 being characterizing the relative density of the set $E\{\frac{\varepsilon}{12d_0}; f\}$ in such a way that $\varepsilon(\frac{1}{12d_0} + \frac{1}{6(l_0+d_0)}) < \delta$. With $\xi = \min(x_1, x_2)$ and an arbitrary fixed $\alpha \in \mathbb{R}$ there exists an $(\frac{\varepsilon}{12d_0})$-a.p. τ of f such that $\xi + \tau \in (\alpha, \alpha + l_0)$. Then the numbers $y_1 = x_1 + \tau$ and $y_2 = x_2 + \tau$ are in the interval $(\alpha, \alpha + L_0)$ where $L_0 = l_0 + d_0$. We now have

$$\begin{aligned} F(y_2) - F(y_1) &= F(x_2) - F(x_1) + \int_{x_1}^{x_2} (f(t+\tau) - f(t))dt \\ &\geqslant F(x_2) - F(x_1) - \frac{\varepsilon}{12} \\ &> G_N(\gamma_0) - g_N(\gamma_0) - \frac{\varepsilon}{6}. \end{aligned}$$

Since $x_i \in \langle \tau_i - N, \tau_i + N\rangle$ with $\tau_i \in E\{\gamma_0; f\}$ $(i = 1, 2)$, hence $y_i \in \langle \tau_i + \tau - N, \tau_i + \tau + N\rangle$, so that $y_i \in \bigcup A_N^{\gamma_0 + \frac{\varepsilon}{12d_0}}$ $(i = 1, 2)$. We further obtain

$$\begin{aligned} F(y_2) &> F(y_1) + G_N(\gamma_0) - g_N(\gamma_0) - \frac{\varepsilon}{6} \\ &\geqslant G_N(\gamma_0) - g_N(\gamma_0) + g_N\left(\gamma_0 + \frac{\varepsilon}{12d_0}\right) - \frac{\varepsilon}{6} \\ &\geqslant G_N(\gamma_0) - \frac{\varepsilon}{2}, \end{aligned}$$

so that

(1) $$F(y_2) > G_N(\gamma_0) - \frac{\varepsilon}{2}.$$

Analogously, we have

(2) $$F(y_1) < g_N(\gamma_0) + \frac{\varepsilon}{2}.$$

Fix an arbitrary $x \in \mathbb{R}$. Let us select y_1 in $(x, x + L_0)$ in order that (2) holds true. Then for $\tau^* \in E\{\frac{\varepsilon}{6L_0}; f\}$ we have $y_1 + \tau^* \in \bigcup A_N^{\gamma_0 + \frac{\varepsilon}{12d_0} + \frac{\varepsilon}{6L_0}}$ and

$$\begin{aligned} F(x + \tau^*) - F(x) &= F(y_1 + \tau^*) - F(y_1) + \int_x^{y_1} (f(t) - f(t + \tau^*))dt \\ &> g_N\left(\gamma_0 + \frac{\varepsilon}{12d_0} + \frac{\varepsilon}{6L_0}\right) - g_N(\gamma_0) - \frac{2}{3}\varepsilon \\ &\geqslant -\varepsilon, \end{aligned}$$

so that

(3) $$F(x + \tau^*) - F(x) > -\varepsilon.$$

Similarly, using (1), we obtain

$$F(x+\tau^*) - F(x) < \varepsilon. \tag{4}$$

By (3) and (4) it follows that if $\tau^* \in E\{\frac{\varepsilon}{6L_0}; f\}$, then

$$\sup_{x\in\mathbb{R}} |F(x+\tau^*) - F(x)| \leqslant \varepsilon,$$

i.e. $E\{\frac{\varepsilon}{6L_0}; f\} \subset E\{F;\varepsilon\}$. This means that $E\{F;\varepsilon\}$ is relatively dense. So it follows that the continuous F is uniformly a.p. ■

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