COMMENTATIONES MATHEMATICAE Vol. 50, No. 2 (2010), 155-159

Małgorzata Stawiska

Liouville theorem with parameters: asymptotics of certain rational integrals in differential fields

(*Dedicated to the memory of Andrzej Pelczar*)

Abstract. We study asymptotics of integrals of certain rational functions that depend on parameters in a field K of characteristic zero. We use formal power series to represent the integral and prove certain identities about coefficients of this series following from the generalized Vandermonde determinant expansion. Our result can be viewed as a parametric version of a classical theorem of Liouville. We also give some applications.

2010 Mathematics Subject Classification: 13F25, 41A58, 12Y99.

Key words and phrases: integrals of rational functions, Vandermonde determinant, differential fields, Liouville theorem.

1. Integrating rational functions over differential fields. Let $\mathcal{D}$ be a differential field. This means that $\mathcal{D}$ is a field with an additional mapping $' : \mathcal{D} \mapsto \mathcal{D}$ (differentiation) satisfying the following two conditions: $(u+v)' = u' + v'$ and $(uv)' = u'v + uv'$ for all $u, v \in \mathcal{D}$. The set $K = \{u \in \mathcal{D} : u' = 0\}$ is a subfield of $\mathcal{D}$, called the field of constants. We use the following terminology, adapted from [Ri]: Let U be a universal (differential) extension of $\mathcal{D}$. For $u \in U$, an element u and the field extension $\mathcal{D}(u)$ are said to be simple elementary over $\mathcal{D}$ if and only if one of the following conditions holds: (1) u is algebraic over $\mathcal{D}$; (2) There is a v in $\mathcal{D}$, $v \neq 0$ such that $v' = vu'$ (we will write equivalently $u = \log v$); (3) There is a v in $\mathcal{D}$, $v \neq 0$ such that $u' = uv'$. (or equivalently $u = \exp v$).

We say that the field $\mathcal{F}$ (and any element $w \in \mathcal{F}$) is elementary over $\mathcal{D}$ if $\mathcal{F} = \mathcal{D}(u_1, ..., u_n)$ for some n, where each u_i is simple elementary over $\mathcal{D}(u_1, ..., u_{i-1})$, $\quad i = 1, ..., n$.

The following theorem dates back to J. Liouville (cf. [Ri]):

THEOREM 1.1 *Let $\mathcal{D}$ be a differential field, $\mathcal{F}$ elementary over $\mathcal{D}$. Suppose $\mathcal{D}$ and $\mathcal{F}$ have the same constant field K. Let $g \in \mathcal{F}$, $f \in \mathcal{D}$ with $g' = f$. Then $g = v_0 + \sum c_i \log v_i$, where v_0, v_i are elements of $\mathcal{D}$ and c_i are elements of K.*

We will write $\int f = g$ as equivalent to $g' = f$.

Example and notation: Let K be an arbitrary field of characteristic zero and z be transcendental over K. We introduce differentiation in the polynomial ring $K[z]$ by taking $z' = 1$ and $a' = 0$ for all $a \in K$ (the standard differentiation of polynomials in one variable). The field $K(z)$ of rational fractions of $K[z]$ is a differential field when we extend $(z^{-1})' = -z^{-1}z^{-1}$, and K is its field of constants.

With K as in the example, we will consider the ring $K[[1/z]]$ of the following formal series: $\sum_{n=0}^{\infty} a_n z^{-n}$ with $a_n \in K$. The differentiation ' can be extended term-by-term as a map of $K[[1/z]]$ to itself. One can also define a valuation $o : K[[1/z]] \mapsto \mathbb{N} \cup \{\infty\}$ as follows (cf. [VS], discussion before Proposition 2.3.16): $o(f) = \min\{n : a_n \neq 0\}$ and $o(0) = \infty$.

Consider a a square-free polynomial $Q(z) = z(z - a_1)...(z - a_q)$ with $a_n \in K$. Our result can be now formulated as follows:

THEOREM 1.2 *(a) In an elementary field $\mathcal{F}$ over $K(z)$, consider the elements $g = \int f \in \mathcal{F}$ with $f = 1/Q$, where $Q(z) = z(z - a_1)...(z - a_q)$ is a square-free polynomial with $a_1, ..., a_q \in K$, $q \geqslant 1$. The set G of all such elements is in a bijective correspondence with a subset of $K[[1/z]]$.*
(b) For (the image of) a $g = \int 1/Q$ as in part (a) we have $o(g) = q$, where $q = \deg Q - 1$.

In the proof of this theorem we will apply the following identities:

LEMMA 1.3

$$\frac{1}{Q'(0)} + \frac{1}{Q'(a_1)} + ... + \frac{1}{Q'(a_q)} = 0,$$

$$\frac{a_1}{Q'(a_1)} + ... + \frac{a_q}{Q'(a_q)} = 0,$$

...

$$\frac{a_1^{q-1}}{Q'(a_1)} + ... + \frac{a_q^{q-1}}{Q'(a_q)} = 0,$$

$$\frac{a_1^{q}}{Q'(a_1)} + ... + \frac{a_q^{q}}{Q'(a_q)} = 1,$$

$$\frac{a_1^{q+l}}{Q'(a_1)} + ... + \frac{a_q^{q+l}}{Q'(a_q)} = S_l(a_1, ..., a_q),$$

where S_l is the complete homogeneous polynomial of degree l, symmetric in its variables, i.e., $S_l(X_1,...,X_n)=\sum_{1\leqslant i_1\leqslant...\leqslant i_l\leqslant n} X_{i_1}...X_{i_l}$ for $l=1,2,....$

PROOF (of Lemma) We use properties of the Vandermonde determinant:

$$V_n(x_1,...,x_n)=\begin{vmatrix}1 & x_1 & x_1^2 & \dots & x_1^{n-1}\\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1}\\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots}\\ 1 & x_n & x_n^2 & \dots & x_n^{n-1}\end{vmatrix}$$

Recall the following formulas: $V_n(x_1,...,x_n)=\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)$ and $V_{n+1}(x_1,...,x_{n+1})=(-1)^n\prod_{i=1}^n(x_i-x_{n+1})V_n(x_1,...,x_n)$. More generally, one can consider

$$V_{n,l}(x_1,...,x_n)=\begin{vmatrix}1 & x_1 & x_1^2 & \dots & x_1^{n-1+l}\\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1+l}\\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots}\\ 1 & x_n & x_n^2 & \dots & x_n^{n-1+l},\end{vmatrix}$$

where $l=1,2,....$ Then $V_{n,l}(x_1,...,x_n)=V_n(x_1,...,x_n)\cdot S_l(x_1,...,x_n)$, where S_l is the complete homogeneous polynomial of degree l in $x_1,...,x_n$ ([Ma], formula I.3.1). Note that for $Q(z)=z(z-a_1)...(z-a_q)$ one has $Q'(0)=(-1)^q\prod_{i=1}^q a_1...a_q$, $Q'(a_i)=\prod_{j\neq i}(a_i-a_j)$. To prove the first stated identity, let us make $V_{q+1}(0,a_1,...,a_q)$ the common denominator of the left hand side. Then $\frac{1}{Q'(0)}=(-1)^q\frac{V_q(a_1,...,a_q)}{V_{q+1}(0,a_1,...,a_q)}$ and $\frac{1}{Q'(a_i)}=(-1)^{n-i-1}\frac{\prod_{k\neq i}a_k}{\prod_{k,j\neq 1,k<j}(a_k-a_j)}$, so the numerator is the cofactor of the element 1 in the i-th row of V_q. Thus in the summation of all terms corresponding to different roots of Q the numerator is V_q minus its Laplace expansion along the column of 1's, which equals 0. The same argument proves the second identity: in the common denominator we now have $V_q(a_1,...,a_q)$ and the numerator is a Vandermonde determinant of size $(q-1)\times(q-1)$ minus its Laplace expansion along the column of 1's . In the sum of $\frac{a_i^k}{Q'(a_i)}$, $\quad k\leqslant q-1$, the numerator is the Laplace expansion of

$$\begin{vmatrix}1 & a_1 & a_1^2...a_1^{q-1} & 1\\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots}\\ 1 & a_n & a_n^2...a_q^{q-1} & 1\end{vmatrix}$$

along the column containing terms of the type a^k, and in the sum of $\frac{a_i^{q+l}}{Q'(a_i)}$ the numerator is the Laplace expansion of $V_{q,l}(a_1,...a_q)$ with respect to the last column, which is a product of $V_q(a_1,...a_q)$ by the complete homogeneous symmetric polynomial $S_l(a_1,...a_q)$ of degree $l>0$.

Remark: The last identity was obtained in a different way as Theorem 3.2 in [Co], where it is also traced back to C.G.J. Jacobi.

PROOF (of Theorem 1.2)

First note that for $a\in K$ we can identify $(z-a)^{-1}$ with $\sum_{n=1}^\infty a^n z^{-(n+1)}$. More generally, if $Q(z)=z(z-a_1)...(z-a_q)$ is a square-free polynomial with $a_n\in K$

and $P \in K[z]$, then partial fraction decomposition gives $P/Q = c_0/z + c_1/(z - a_1) + ... + c_q/(z - a_q)$ with $c_n = P(a_n)/Q'(a_n), \quad n = 1, ..., q$ (cf. [Tr]) and P/Q can also be identified with an element of $K[[1/z]]$. It follows that $\log(1 - a/z)$ can be identified with $\sum_{n=1}^{\infty}((-1)^n a^n/n)z^{-n}$. Let now $g = \int(1/Q)$ with Q as above. Then $g = b_0 + \frac{1}{Q'(0)} \log z + \frac{1}{Q'(a_1)} \log(z - a_1) + ... + \frac{1}{Q'(a_q)} \log(z - a_q)$ in $\mathcal{F}$ with b_0 in K. Identifying each $\log(z - a_j), \quad j = 1, ..., q$ with an appropriate formal series in $K[[1/z]]$ as above and adding the results, we get $g = \sum_{n=0}^{\infty} b_n z^{-n}$. By the lemma, $b_1 = ... = b_{q-1} = 0$, $b_q = 1/q$ and $b_{q+l} = S_l(a_1, ..., a_q)/(q + 1)$, where S_l is the complete homogeneous symmetric polynomial of degree l, $l = 1, 2, ...$ To ensure this is the only possible series in $K[[1/z]]$ that can be identified with $g = \int(1/Q)$, note that any such series should be symmetric with respect to $a_1, ..., a_q$. Theorem 9.3 in [LS] says the following: If a formal series F in the variables $X_1, ..., X_q, Y$ is symmetric with respect to $(X_1, ..., X_q)$, then $F = \Phi(\sigma_1, ..., \sigma_q, Y)$, where Φ is a formal series in the variables $X_1, ..., X_q, Y$ and $\sigma_1, ..., \sigma_q$ are elementary symmetric polynomials in $X_1, ..., X_q$. Moreover, the series Φ is unique. Uniqueness of our $\sum_{n=0}^{\infty} b_n z^{-n}$ follows, because $S_l(X_1, ..., X_q)$ can be expressed as polynomials in $\sigma_1, ..., \sigma_q$. Hence also $o(g) = q$. ■

2. Applications. A particular case of our Theorem 1.2 is Proposition 1 in [GS], which was proved for $K = \mathbb{C}$ and $a_j = \zeta^j a_1, j = 1, ..., q$, where ζ is a primitive root of unity of order q. The convergence of $\int 1/(z(z - a_1)...(z - a_q)) \to -1/(qz^q)$ as $a_1, ..., a_q \to 0$ (which is in fact uniform for $|z| > R$) is important in constructing approximate Fatou coordinates for analytic maps f in a neighborhood of an $f_0(z) = z + z^{q+1} + ...$ with $q > 1$. These are coordinates in which f looks like a translation. The first step in constructing Fatou coordinate for f_0 consists in lifting f_0 to a neighborhood of infinity by the coordinate change $z \mapsto -1/(qz^q)$. We considered f belonging to an one-parameter family of polynomials $P_\lambda(z) = \lambda z + z^2$ with $\lambda_0 = e^{2\pi i p/q}$ and $\lambda = e^{2\pi i(p/q+u)}$, with p, q coprime integers and u in a sufficiently small neighborhood of 0 in $\mathbb{C}$. We started the construction of near-Fatou coordinate by applying the transformation $w(z) = \int_{z_0}^{z}(1/Q(u, \zeta))d\zeta$, where $Q(u, z)$ is the Weierstrass polynomial for $P_\lambda^{\circ q}(z) - z$. As u is small, the non-zero solutions of $Q(u, z) = 0$ are also small. Because of convergence of integrals, the coordinates obtained for P_λ depend continuously on u. For more details and references see [GS].

Another application is a generalization of the well-known formula for electrostatic potential of a dipole located at $z = 0$ (cf. [Ne]): consider two charges $1/a$ and $-1/a$ placed respectively at $z = 0$ and $z = a$. Then their combined electrostatic potential is $(1/a) \log z - (1/a) \log(z - a)$, which tends to $-1/z$ as $a \to 0$ (this follows easily from the definition of derivative of the logarithmic function). In the setting of this paper, there are charges $1/Q'(0), 1/Q'(a_1), ..., 1/Q'(a_q)$ placed at $0, a_1, ..., a_q$ (with $Q(z) = z(z - a_1)...(z - a_q)$), and it follows from Theorem 1.2 that their combined potential tends to $-1/(qz^q)$ as $a_1, ..., a_q \to 0$.

Acknowledgement. The main idea of this paper occurred while I was working with Estela Gavosto on [GS]. I thank her for many useful conversations.

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Małgorzata Stawiska
Department of Mathematics, University of Kansas
1460 Jayhawk Blvd., Lawrence, KS 66045, USA
E-mail: stawiska@ku.edu
URL: www.math.ku.edu/~mfriedland

(Received: 17.06.2010)