Bijendra Singh, Arihant Jain, Bholaram Lodha

On common fixed point theorems for semi-compatible mappings in Menger space

Abstract. In this paper, the concept of semi-compatibility and weak compatibility in Menger space has been applied to prove a common fixed point theorem for six self maps. Our result generalizes and extends the result of Pathak and Verma [6].

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1. Introduction. There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [4]. It is a probabilistic generalization in which we assign to any two points \( x \) and \( y \), a distribution function \( F_{x,y} \). Schweizer and Sklar [8] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [9] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed point theory in Menger space.

Recently, Jungck and Rhoades [3] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [10] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [2] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [5].


In this paper, a fixed point theorem for six self maps has been proved using the concept of semi-compatible maps and weak compatibility which turns out be a material generalization of the result of Pathak and Verma [6].
2. Preliminaries.

Definition 2.1 A mapping \( F : \mathbb{R} \to \mathbb{R}^+ \) is called a distribution if it is nondecreasing left continuous with
\[
\inf \{ F(t) : t \in \mathbb{R} \} = 0 \quad \text{and} \quad \sup \{ F(t) : t \in \mathbb{R} \} = 1.
\]
We shall denote by \( L \) the set of all distribution functions while \( H \) will always denote the specific distribution function defined by
\[
H(t) = \begin{cases} 
0, & t \leq 0, \\
1, & t > 0.
\end{cases}
\]

Definition 2.2 ([5]) A mapping \( t : [0, 1] \times [0, 1] \to [0, 1] \) is called a \( t \)-norm if it satisfies the following conditions:
\begin{align*}
(t-1) & \quad t(a, 1) = a, t(0, 0) = 0; \\
(t-2) & \quad t(a, b) = t(b, a); \\
(t-3) & \quad t(c, d) \geq t(a, b) \quad \text{for} \quad c \geq a, d \geq b, \\
(t-4) & \quad t(t(a, b), c) = t(a, t(b, c)) \quad \text{for all} \quad a, b, c, d \in [0, 1].
\end{align*}

Definition 2.3 ([5]) A probabilistic metric space (PM-space) is an ordered pair \((X, F)\) consisting of a non empty set \( X \) and a function \( F : X \times X \to L \), where \( L \) is the collection of all distribution functions and the value of \( F \) at \((u, v) \in X \times X\) is represented by \( F_{u,v} \). The function \( F_{u,v} \) assumed to satisfy the following conditions:
\begin{align*}
(PM-1) & \quad F_{u,v}(x) = 1, \quad \text{for all} \quad x > 0, \quad \text{if and only if} \quad u = v; \\
(PM-2) & \quad F_{u,v}(0) = 0; \\
(PM-3) & \quad F_{u,u} = F_{v,v}; \\
(PM-4) & \quad \text{If} \quad F_{u,v}(x) = 1 \quad \text{and} \quad F_{v,w}(y) = 1 \quad \text{then} \quad F_{u,w}(x + y) = 1, \quad \text{for all} \quad u, v, w \in X \quad \text{and} \quad x, y > 0.
\end{align*}

Definition 2.4 ([5]) A Menger space is a triplet \((X, F, t)\) where \((X, F)\) is a PM-space and \( t \) is a \( t \)-norm such that the inequality
\begin{align*}
(PM-5) & \quad F_{u,w}(x + y) \geq t\{F_{u,v}(x), F_{v,w}(y)\}, \quad \text{for all} \quad u, v, w \in X, \quad x, y \geq 0.
\end{align*}

Definition 2.5 ([5]) A sequence \( \{x_n\} \) in a Menger space \((X, F, t)\) is said to be convergent and converges to a point \( x \) in \( X \) if and only if for each \( \varepsilon > 0 \) and \( \lambda > 0 \), there is an integer \( M(\varepsilon, \lambda) \) such that \( F_{x,x}(\varepsilon) > 1 - \lambda \) for all \( n \geq M(\varepsilon, \lambda) \).
Further the sequence \( \{x_n\} \) is said to be Cauchy sequence if for \( \varepsilon > 0 \) and \( \lambda > 0 \), there is an integer \( M(\varepsilon, \lambda) \) such that
\[
F_{x_m, x_n}(\varepsilon) > 1 - \lambda \quad \text{for all } m, n \in M(\varepsilon, \lambda).
\]

A Menger PM-space \((X, F, t)\) is said to be complete if every Cauchy sequence in \( X \) converges to a point in \( X \). A complete metric space can be treated as a complete Menger space in the following way:

**Proposition 2.6** ([5]) If \((X, d)\) is a metric space then the metric \(d\) induces mapping \(F: X \times X \to L\), defined by \(F_{p, q}(x) = H(x - d(p, q))\), \(p, q \in X\), where \(H(k) = 0\), for \(k \leq 0\) and \(H(k) = 1\), for \(k > 0\).

Further if, \(t: [0, 1] \times [0, 1] \to [0, 1]\) is defined by \(t(a, b) = \min\{a, b\}\). Then \((X, F, t)\) is a Menger space. It is complete if \((X, d)\) is complete.

The space \((X, F, t)\) so obtained is called the induced Menger space.

**Definition 2.7** ([6]) Self mappings \(A\) and \(S\) of a Menger space \((X, F, t)\) are said to be weak compatible if they commute at their coincidence points i.e. \(Ax = Sx\) for \(x \in X\) implies \(ASx = SAx\).

**Definition 2.8** ([6]) Self mappings \(A\) and \(S\) of a Menger space \((X, F, t)\) are said to be compatible if \(F_{ASx_n, SAx_n}(x) \to 1\) for all \(x > 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that \(Ax_n, Sx_n \to u\) for some \(u\) in \(X\), as \(n \to \infty\).

**Definition 2.9** Self mappings \(A\) and \(S\) of a Menger space \((X, F, t)\) are said to be semi-compatible if \(F_{ASx_n, Su_n}(x) \to 1\) for all \(x > 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that \(Ax_n, Sx_n \to u\), for some \(u\) in \(X\), as \(n \to \infty\).

Now, we give an example of pair of self maps \((S, T)\) which is semicompatible but not compatible. Further we observe here that the pair \((T, S)\) is not semi-compatible though \((S, T)\) is semi-compatible.

**Example 2.10** Let \((X, d)\) be a metric space where \(X = [0, 1]\) and \((X, F, t)\) be the induced Menger space with \(F_{p, q}(\varepsilon) = H(\varepsilon - d(p, q))\), \(\forall p, q \in X\) and \(\forall \varepsilon > 0\). Define self maps \(S\) and \(T\) as follows:
\[
Sx = \begin{cases} 
x, & \text{if } 0 \leq x < \frac{1}{2}, \\
1, & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]
and
\[
Tx = \begin{cases} 
1 - x, & \text{if } 0 \leq x < \frac{1}{2}, \\
1, & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
\]

Take \(x_n = \frac{1}{2} - \frac{1}{n}\). Now,
\[F_{Sx_n, 1/2}(\varepsilon) = H(\varepsilon - (1/n)).\]
Therefore, \(\lim_{n \to \infty} F_{Sx_n, 1/2}(\varepsilon) = H(\varepsilon) = 1\).
Hence, $Sx_n \to 1/2$ as $n \to \infty$.
Similarly, $Tx_n \to 1/2$ as $n \to \infty$.

Also

$$F_{STx_n, TSx_n}(\varepsilon) = H \left( \varepsilon - \left( \frac{1}{2} - \frac{1}{n} \right) \right) \neq 1, \forall \varepsilon > 0.$$ 

Hence, the pair $(S, T)$ is not compatible.

Again, $\lim_{n \to \infty} F_{STx_n, Tx_n}(\varepsilon) = \lim_{n \to \infty} F_{STx_n, 1}(\varepsilon) = H(\varepsilon - |1 - 1|) = 1, \forall \varepsilon > 0$.

Thus, $(T, S)$ is not semi-compatible.

Remark 2.11 In view of above example, it follows that the concept of semi-compatibility is more general than that of compatibility.

Lemma 2.12 ([6]) Let $(X, F, \ast)$ be a Menger space with t-norm $\ast$ such that the family $\{\ast_n(x)\}_{n \in \mathbb{N}}$ is equicontinuous at $x = 1$ and let $E$ denote the family of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi$ is non-decreasing with $\lim_{n \to \infty} \phi^n(t) = +\infty, \forall t > 0$.

If $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in $X$ satisfying the condition

$$F_{y_n, y_{n+1}}(t) \geq F_{y_{n-1}, y_n}(\phi(t)),$$

for all $t > 0$ and $\alpha \in [-1, 0]$, then $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $X$.

3. Main Result.

Theorem 3.1 Let $A, B, S, T, P$ and $Q$ be self maps of a complete Menger space $(X, F, \ast)$ with $\ast = \min$ satisfying:

(3.1.1) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$;

(3.1.2) $AB = BA$, $ST = TS$, $PB = BP$, $QT = TQ$;

(3.1.3) either $P$ or $AB$ is continuous;

(3.1.4) $(P, AB)$ is semi-compatible and $(Q, ST)$ is weak compatible;

(3.1.5) $[1 + \alpha F_{ABz, STy}(t) + F_{Pz, Qy}(t)] * F_{Px, Qy}(t)$

$$\geq \alpha \min\{F_{Px, ABz}(t) * F_{Qy, STy}(t), F_{Px, STy}(2t) * F_{Qy, ABz}(2t)\}$$

$$+ F_{ABz, STy}(\phi(t)) * F_{Pz, ABz}(\phi(t)) * F_{Qy, STy}(\phi(t))$$

$$+ F_{Px, STy}(2\phi(t)) * F_{Qy, ABz}(2\phi(t))$$

for all $x, y \in X$, $t > 0$ and $\phi(t) \in E$.

Then $A, B, S, T, P$ and $Q$ have a unique common fixed point in $X$. 

Proof Suppose $x_0 \in X$. From condition (3.1.1) $\exists x_1, x_2 \in X$ such that 

$$Px_0 = STx_1$$

and

$$Qx_1 = ABx_2.$$

Inductively, we can construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

$$y_{2n} = Px_2 = STx_{2n+1} \quad \text{and} \quad y_{2n+1} = Qx_{2n+1} = ABx_{2n+2}$$

for \( n = 0, 1, 2, \ldots \).

**Step 1.**
Let us show that $F_{y_{n+2},y_{n+1}}(t) \geq F_{y_{n+1},y_n}(\phi(t))$.

For, putting $x_{2n+2}$ for $x$ and $x_{2n+1}$ for $y$ in (3.1.5) and then on simplification, we have

$$[1 + \alpha F_{ABx_{2n+2},STx_{2n+1}}(t)] * F_{ABx_{2n+2},STx_{2n+1}}(t)$$

$$\geq a min \{ F_{F_{2n+2},ABx_{2n+2}}(t) * F_{Q_{2n+1},STx_{2n+1}}(t),$$

$$- F_{F_{2n+2},ABx_{2n+2}}(2t) * F_{Q_{2n+2},ABx_{2n+2}}(2t) \}$$

$$+ F_{F_{2n+2},ABx_{2n+2}}(\phi(t)) * F_{Q_{2n+2},STx_{2n+1}}(\phi(t))$$

$$+ F_{P_{2n+2},STx_{2n+1}}(2\phi(t)) * F_{Q_{2n+2},ABx_{2n+2}}(2\phi(t))$$

$$[1 + \alpha F_{y_{2n+1},y_{2n}}(t)] * F_{y_{2n+1},y_{2n}}(t)$$

$$\geq a min \{ F_{y_{2n+2},y_{2n+1}}(t) * F_{y_{2n+2},y_{2n}}(t), F_{y_{2n+1},y_{2n+2}}(2t) * F_{y_{2n+1},y_{2n+2}}(2t) \}$$

$$+ F_{y_{2n+1},y_{2n}}(\phi(t)) * F_{y_{2n+2},y_{2n+1}}(\phi(t)) * F_{y_{2n+1},y_{2n}}(\phi(t))$$

$$+ F_{y_{2n+2},y_{2n}}(2\phi(t)) * F_{y_{2n+1},y_{2n}}(2\phi(t))$$

$$F_{y_{2n+2},y_{2n+1}}(t) + \alpha F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n+2},y_{2n+1}}(t)$$

$$\geq a min \{ F_{y_{2n+2},y_{2n}}(2t), F_{y_{2n+2},y_{2n}}(2t) \}$$

$$+ F_{y_{2n+1},y_{2n+2}}(\phi(t)) * F_{y_{2n+2},y_{2n+1}}(\phi(t)) * F_{y_{2n+1},y_{2n+2}}(\phi(t)) * 1$$

$$F_{y_{2n+2},y_{2n+1}}(t) + \alpha F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n+2},y_{2n+1}}(t)$$

$$\geq a F_{y_{2n+1},y_{2n+2}}(2t) + F_{y_{2n+1},y_{2n+2}}(\phi(t)) * F_{y_{2n+1},y_{2n+2}}(\phi(t)) * F_{y_{2n+1},y_{2n+2}}(\phi(t))$$

$$F_{y_{2n+2},y_{2n+1}}(t) \geq a F_{y_{2n+1},y_{2n+2}}(2t) + F_{y_{2n+1},y_{2n+2}}(\phi(t)) * F_{y_{2n+1},y_{2n+2}}(\phi(t)) * F_{y_{2n+1},y_{2n+2}}(\phi(t))$$

$$F_{y_{2n+2},y_{2n+1}}(t) \geq a F_{y_{2n+1},y_{2n+2}}(\phi(t)) * F_{y_{2n+1},y_{2n+2}}(\phi(t))$$
or 

\[ F_{y_2n+2,y_2n+1}(t) \geq F_{y_2n+1,y_2n+2}(\phi(t)) \ast F_{y_2n,y_2n+1}(\phi(t)) \]

or 

\[ F_{y_2n+2,y_2n+1}(t) \geq \min\{F_{y_2n+1,y_2n+2}(\phi(t)), F_{y_2n,y_2n+1}(\phi(t))\}. \]

If \( F_{y_2n+1,y_2n+2}(\phi(t)) \) is chosen 'min' then we obtain 

\[ F_{y_2n+2,y_2n+1}(t) \geq F_{y_2n+1,y_2n+2}(\phi(t)), \; \forall t > 0, \]

a contradiction as \( \phi(t) \) is non-decreasing function. Thus, 

\[ F_{y_2n+2,y_2n+1}(t) \geq F_{y_2n+1,y_2n+2}(\phi(t)), \; \forall t > 0. \]

Similarly, by putting \( x_{2n+2} \) for \( x \) and \( x_{2n+3} \) for \( y \) in (3.1.5), we have 

\[ F_{y_{2n+3},y_{2n+2}}(t) \geq F_{y_{2n+2},y_{2n+1}}(\phi(t)), \; \forall t > 0. \]

Using these two, we obtain 

\[ F_{y_{n+2},y_{n+1}}(t) \geq F_{y_{n+1},y_n}(\phi(t)), \; \forall n = 0, 1, 2, \ldots, t > 0. \]

Therefore, by lemma 2.12, \( \{y_n\} \) is a Cauchy sequence in \( X \), which is complete. Hence \( \{y_n\} \to z \in X \). Also its subsequences converges as follows : 

\[ \{P_{x_{2n}}\} \to z \text{ and } \{ST_{x_{2n+1}}\} \to z, \]

(3.1.6) 

\[ \{Q_{x_{2n+1}}\} \to z \text{ and } \{AB_{x_{2n+2}}\} \to z. \]

**Case I. Suppose \( P \) is continuous.**

As \( P \) is continuous and \( (P, AB) \) is semi-compatible, we get 

(3.1.7) 

\[ PAB_{x_{2n+2}} \to Pz \text{ and } PAB_{x_{2n+2}} \to ABz. \]

Since the limit in Menger space is unique, we get 

(3.1.8) 

\[ Pz = ABz. \]

**Step 2.**

We prove \( Pz = z \). Put \( x = z, y = x_{2n+1} \) in (3.1.5) and let \( Pz \neq z \). Then 

\[ [1 + \alpha F_{ABz,ST_{x_{2n+1}}}(t)] \ast F_{Pz,Q_{x_{2n+1}}}(t) \]

\[ \geq \alpha \min\{F_{Pz,ABz}(t) \ast F_{Q_{x_{2n+1}},ST_{x_{2n+1}}}(t), F_{Pz,ST_{x_{2n+1}}}(2t) \ast F_{Q_{x_{2n+1}},ABz}(2t)\} \]

\[ + F_{ABz,ST_{x_{2n+1}}}(\phi(t)) \ast F_{Pz,ABz}(\phi(t)) \ast F_{Pz,ST_{x_{2n+1}}}(\phi(t)) \ast F_{Q_{x_{2n+1}},ABz}(2\phi(t)) \]

\[ \ast F_{Pz,ST_{x_{2n+1}}}(2\phi(t)) \ast F_{Q_{x_{2n+1}},ABz}(2\phi(t)) \]

Letting \( n \to \infty \) and using (3.1.6) and (3.1.8), we get 

\[ [1 + \alpha F_{Pz,z}(t)] \ast F_{Pz,z}(t) \]
\[
\begin{align*}
\geq & \ \alpha \min \{F_{Pz,Pz}(t) \ast F_{z,z}(t), F_{Pz,z}(2t) \ast F_{z,Pz}(2t)\} \\
& + F_{Pz,z}(\phi(t)) \ast F_{Pz,Pz}(\phi(t)) \ast F_{z,z}(\phi(t)) \ast F_{Pz,z}(2\phi(t)) \ast F_{z,Pz}(2\phi(t)) \\
& F_{Pz,z}(t) + \alpha \{F_{Pz,z}(t) \ast F_{Pz,z}(t)\} \\
\geq & \ \alpha \min \{1 \ast F_{Pz,z}(2t) \ast F_{Pz,z}(2t)\} \\
& + F_{Pz,z}(\phi(t)) \ast 1 \ast F_{Pz,z}(2\phi(t)) \ast F_{z,Pz}(2\phi(t)) \\
F_{Pz,z}(t) + \alpha F_{Pz,z}(t) \geq & \ \alpha \min \{1, F_{Pz,z}(2t)\} + F_{Pz,z}(\phi(t)) \ast F_{Pz,z}(2\phi(t)) \\
F_{Pz,z}(t) + \alpha F_{Pz,z}(t) \geq & \ \alpha F_{Pz,z}(2t) + F_{Pz,z}(\phi(t)) \ast F_{Pz,z}(2\phi(t)) \\
F_{Pz,z}(t) + \alpha F_{Pz,z}(t) \geq & \ \alpha \{F_{Pz,z}(t) \ast F_{z,z}(t)\} + F_{Pz,z}(\phi(t)) \ast F_{Pz,z}(2t) \\
F_{Pz,z}(t) + \alpha F_{Pz,z}(t) \geq & \ \alpha F_{Pz,z}(t) + F_{Pz,z}(\phi(t)) \\
F_{Pz,z}(t) \geq & \ F_{Pz,z}(\phi(t))
\end{align*}
\]

which is a contradiction and hence, \(P_z = z\) and so \(z = Pz = ABz\).

**Step III.** Put \(x = Bz\) and \(y = x_{2n+1}\) in (3.1.5), we get
\[
[1 + \alpha F_{ABBz,STx_{2n+1}}(t)] \ast F_{PBz,STx_{2n+1}}(t)
\]
\[
\geq \ \alpha \min \{F_{PBz,ABBz}(t) \ast F_{Qx_{2n+1},STx_{2n+1}}(t), \ F_{PBz,STx_{2n+1}}(2t) \ast F_{Qx_{2n+1},ABBz}(2t)\} \\
& + F_{ABBz,STx_{2n+1}}(\phi(t)) \ast F_{PBz,ABBz}(\phi(t)) \ast F_{Qx_{2n+1},STx_{2n+1}}(\phi(t)) \\
& + F_{PBz,STx_{2n+1}}(2\phi(t)) \ast F_{Qx_{2n+1},ABBz}(2\phi(t)).
\]

As \(BP = PB, AB = BA\) so we have \(P(Bz) = B(Pz) = Bz\) and \(AB(Bz) = B(AB)z = Bz\). Letting \(n \to \infty\) and using (3.1.6), we get
\[
[1 + \alpha F_{Bz,z}(t)] \ast F_{Bz,z}(t)
\]
\[
\geq \ \alpha \min \{F_{Bz,Bz}(t) \ast F_{z,z}(t), F_{Bz,z}(2t) \ast F_{z,z}(2t)\} \\
& + F_{Bz,z}(\phi(t)) \ast F_{Bz,Bz}(\phi(t)) \ast F_{z,z}(\phi(t)) \ast F_{Bz,z}(2\phi(t)) \ast F_{z,z}(2\phi(t)) \\
& F_{Bz,z}(t) + \alpha \{F_{Bz,z}(t) \ast F_{Bz,z}(t)\} \\
\geq & \ \alpha \min \{1 \ast F_{Bz,z}(2t)\} + F_{Bz,z}(\phi(t)) \ast 1 \ast F_{Bz,z}(2\phi(t)) \\
F_{Bz,z}(t) + \alpha F_{Bz,z}(t) \geq & \ \alpha F_{Bz,z}(2t) + F_{Bz,z}(\phi(t)) \ast F_{Bz,z}(2\phi(t))
\]
\[ F_{Bz,z}(t) + \alpha F_{Bz,z}(t) \geq \alpha \{ F_{Bz,z}(\phi(t)) \} + F_{Bz,z}(\phi(t)) \]

\[ = F_{Bz,z}(t) + \alpha F_{Bz,z}(t) \geq \alpha \{ F_{Bz,z}(\phi(t)) \} + F_{Bz,z}(\phi(t)) \]

which is a contradiction and we get \( Bz = z \) and so

\[ z = ABz = Az. \]

Therefore,

\[ Pz = Az = Bz = z. \]

**Step IV.** Since \( P(X) \subseteq ST(X) \) there exists \( u \in X \) such that \( z = Pz = STu \). Put \( x = x_{2n} \) and \( y = u \) in (3.1.5), we get

\[ [1 + \alpha F_{ABz_{2n},STu}(t)] * F_{Pz_{2n},Qu}(t) \]

\[ \geq amin\{F_{Pz_{2n},STu}(t) * F_{Qu,STu}(t), F_{Pz_{2n},STu}(2t) * F_{Qu,ABz_{2n}}(2t)\} + F_{ABz_{2n},STu}(\phi(t)) * F_{Pz_{2n},ABz_{2n}}(\phi(t)) \]

\[ * F_{Qu,ABz_{2n}}(2\phi(t)) \]

Letting \( n \to \infty \) and using (3.1.6), we get

\[ [1 + \alpha F_{z,z}(t)] * F_{z,Qu}(t) \]

\[ \geq amin\{F_{z,z}(t) * F_{Qu,z}(t), F_{z,z}(2t) * F_{Qu,z}(2t)\} + F_{z,z}(\phi(t)) * F_{z,z}(\phi(t)) \]

\[ * F_{Qu,z}(\phi(t)) * F_{z,z}(2\phi(t)) \]

\[ F_{z,Qu}(t) + \alpha F_{z,Qu}(t) \geq amin\{F_{Qu,z}(t), F_{Qu,z}(2t)\} + F_{Qu,z}(\phi(t)) * F_{Qu,z}(2\phi(t)) \]

\[ F_{Qu,z}(t) + \alpha F_{Qu,z}(t) \geq amin\{F_{Qu,z}(t), F_{Qu,z}(t) * F_{z,z}(t)\} + F_{Qu,z}(\phi(t)) * F_{Qu,z}(\phi(t)) \]

\[ * F_{Qu,z}(2\phi(t)) \]

which is a contradiction by lemma 2.12 and we get

\[ Qu = z \] and so \( Qu = z = STu \).

Since \((Q, ST)\) is weak-compatible, we have

\[ STQu = QSTu \] i.e. \( STz = Qz \).
Step V. Put \( x = x_{2n} \) and \( y = z \) in (3.1.5), we have

\[
[1 + \alpha F_{ABx_{2n},STz}(t)] * F_{P_{x_{2n}},Qz}(t)
\]

\[
\geq \alpha \min \{ F_{P_{x_{2n}},ABz_{2n}}(t) * F_{Qz,STz}(t), F_{P_{x_{2n}},STz}(2t) * F_{Qz,ABz_{2n}}(2t) \}
+ F_{ABx_{2n},STz}(\phi(t)) * F_{P_{x_{2n}},ABz_{2n}}(\phi(t)) * F_{Qz,STz}(\phi(t)) * F_{P_{x_{2n}},STz}(2\phi(t))
+ F_{Qz,ABx_{2n}}(2\phi(t)).
\]

Letting \( n \to \infty \) and using (3.1.6) and step IV, we get

\[
[1 + \alpha F_{z,Qz}(t)] * F_{z,Qz}(t)
\]

\[
\geq \alpha \min \{ F_{z,z}(t) * F_{Qz,t}(2t) \} + F_{z,z}(\phi(t)) * F_{z,z}(2\phi(t))
\]

\[
F_{z,Qz}(t) + \alpha \{ F_{z,Qz}(t) * F_{Qz,z}(t) \}
\]

\[
\geq \alpha \min \{ 1 * F_{z,Qz}(2t) \} + F_{z,Qz}(\phi(t)) * F_{z,Qz}(2\phi(t))
\]

\[
F_{Qz,z}(t) + \alpha F_{Qz,z}(t) \geq \alpha F_{Qz,z}(2t) + F_{Qz,z}(\phi(t)) * F_{Qz,z}(2\phi(t))
\]

\[
F_{Qz,z}(t) + \alpha F_{Qz,z}(t) \geq \alpha \{ F_{Qz,z}(t) * F_{z,z}(t) \} + F_{Qz,z}(\phi(t))
\]

\[
F_{Qz,z}(t) + \alpha F_{Qz,z}(t) \geq \alpha F_{Qz,z}(t) * F_{Qz,z}(\phi(t))
\]

\[
F_{Qz,z}(t) \geq F_{Qz,z}(\phi(t))
\]

which is a contradiction and we get \( Qz = z \).

Step VI. Put \( x = x_{2n} \) and \( y = Tz \) in (3.1.5), we have

\[
[1 + \alpha F_{ABx_{2n},STTz}(t)] * F_{P_{x_{2n}},QTz}(t)
\]

\[
\geq \alpha \min \{ F_{P_{x_{2n}},ABz_{2n}}(t) * F_{QTz,STTz}(t), F_{P_{x_{2n}},STTz}(2t) * F_{QTz,ABz_{2n}}(2t) \}
+ F_{ABx_{2n},STTz}(\phi(t)) * F_{P_{x_{2n}},ABz_{2n}}(\phi(t)) * F_{QTz,STTz}(\phi(t))
+ F_{P_{x_{2n}},STTz}(2\phi(t)) * F_{QTz,ABz_{2n}}(2\phi(t)).
\]

As \( QT = TQ = ST = TS \), we have

\[
QTz = TQz = Tz \quad \text{and} \quad ST(Tz) = T(STz) = Tz.
\]

Letting \( n \to \infty \), we get

\[
[1 + \alpha F_{z,Tz}(t)] * F_{z,Tz}(t)
\]

\[
\geq \alpha \min \{ F_{z,z}(t) * F_{Tz,Tz}(t), F_{z,Tz}(2t) * F_{Tz,z}(2t) \} + F_{z,Tz}(\phi(t)) * F_{z,z}(\phi(t))
+ F_{Tz,Tz}(\phi(t)) * F_{z,Tz}(2\phi(t)) * F_{Tz,z}(2\phi(t))
\]
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Letting \( Tz \),

\[
F_{Tz,z}(t) = \alpha F_{Tz,z}(2t) + F_{Tz,z}(\phi(t)) * F_{Tz,z}(\phi(t))
\]

Now, \( STz = Tz = z \) implies \( Sx = z \). Hence,

\[
Sx = Tz = Qz = z.
\]

Combining (3.1.9) and (3.1.10), we get

\[
A_{x} = B_{x} = P_{x} = Q_{x} = S_{x} = T_{z} = z
\]

i.e. \( z \) is a common fixed point of \( A, B, P, Q, S \) and \( T \).

**Case II.** Suppose \( AB \) is continuous.

Since \( AB \) is continuous and \( (P, AB) \) is semi-compatible, we get

\[
(AB)^2 x_{2n} \rightarrow ABz, \quad PABx_{2n} \rightarrow ABz.
\]

Now, we prove \( ABz = z \).

**Step VII.** Put \( x = ABx_{2n} \) and \( y = x_{2n+1} \) in (3.1.5) and assuming \( ABz \neq z \), we get

\[
[1 + \alpha F_{ABx_{2n},STx_{2n+1},z}(t)] * F_{PABx_{2n},Qx_{2n+1},z}(t)
\]

\[
\geq \alpha \min \{ F_{PABx_{2n+1},ABx_{2n+1},z}(t) * F_{Qx_{2n+1},STx_{2n+1},z}(t), F_{PABx_{2n},STx_{2n+1},z}(2t) * F_{Qx_{2n+1},ABABx_{2n+1},z}(2t) + F_{PABx_{2n},ABABx_{2n+1},z}(\phi(t)) * F_{Qx_{2n+1},ABABx_{2n+1},z}(\phi(t)) * F_{PABx_{2n},STx_{2n+1},z}(2\phi(t)) * F_{Qx_{2n+1},ABABx_{2n+1},z}(2\phi(t)).
\]

Letting \( n \rightarrow \infty \) and using (3.1.11), we get

\[
[1 + \alpha F_{ABx_{2n},z}(t)] * F_{ABx_{2n},z}(t)
\]

\[
\geq \alpha \min \{ F_{ABx_{2n},ABx_{2n},z}(t) * F_{z,z}(t), F_{ABx_{2n},z}(2t) * F_{z,ABx_{2n},z}(2t) + F_{ABx_{2n},z}(\phi(t)) * F_{ABx_{2n},z}(\phi(t)) * F_{ABx_{2n},z}(\phi(t)) * F_{ABx_{2n},z}(\phi(t)) * F_{z,ABx_{2n},z}(2\phi(t))
\]

\[
F_{ABx_{2n},z}(t) + \alpha \{ F_{ABx_{2n},z}(t) * F_{ABx_{2n},z}(t) \}
\]

\[
\geq \alpha \min \{ 1 * 1, F_{ABx_{2n},z}(2t) + F_{ABx_{2n},z}(\phi(t)) * 1 * 1 * F_{ABx_{2n},z}(2\phi(t))
\]
\[ F_{ABz,z}(t) + \alpha F_{ABz,z}(t) \geq \alpha \min \{1, F_{ABz,z}(2t)\} + F_{ABz,z}(\phi(t)) \]

\[ F_{ABz,z}(t) + \alpha F_{ABz,z}(t) \geq \alpha \{F_{ABz,z}(t) + F_{z,z}(\phi(t))\} + F_{ABz,z}(\phi(t)) \]

\[ F_{ABz,z}(t) + \alpha F_{ABz,z}(t) \geq \alpha F_{ABz,z}(t) + F_{ABz,z}(\phi(t)) \]

which is a contradiction and we get \( ABz = z \).

**Step VIII.** Put \( x = z \) and \( y = x_{2n+1} \) in (3.1.5), we get

\[ [1 + \alpha F_{ABz,STx_{2n+1}}(t)] * F_{Pz,Qx_{2n+1}}(t) \]

\[ \geq \alpha \min \{F_{Pz,ABz}(t), F_{Qx_{2n+1},STx_{2n+1}}(t), F_{Pz,STx_{2n+1}}(2t), F_{Qx_{2n+1},ABz}(2t)\} \]

\[ + F_{ABz,STx_{2n+1}}(\phi(t)) * F_{Pz,ABz}(\phi(t)) * F_{Qx_{2n+1},STx_{2n+1}}(\phi(t)) \]

\[ + F_{Pz,STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1},ABz}(2\phi(t)) \].

Letting \( n \to \infty \) and using (3.1.6), we get

\[ [1 + \alpha F_{z,z}(t)] * F_{Pz,z}(t) \]

\[ \geq \alpha \min \{F_{Pz,z}(t), F_{Pz,z}(2t)\} + F_{z,z}(\phi(t)) * F_{Pz,z}(\phi(t)) \]

\[ + F_{z,z}(\phi(t)) * F_{Pz,z}(2\phi(t)) * F_{z,z}(2\phi(t)) \]

\[ F_{Pz,z}(t) + \alpha F_{Pz,z}(t) \]

\[ \geq \alpha \min \{F_{Pz,z}(t), F_{Pz,z}(2t)\} + F_{Pz,z}(\phi(t)) * F_{Pz,z}(2\phi(t)) \]

\[ F_{Pz,z}(t) + \alpha F_{Pz,z}(t) \geq \alpha \min \{F_{Pz,z}(t), F_{Pz,z}(t)\} + F_{Pz,z}(\phi(t)) * F_{Pz,z}(\phi(t)) \]

\[ F_{Pz,z}(t) + \alpha F_{Pz,z}(t) \geq \alpha F_{Pz,z}(t) + F_{Pz,z}(\phi(t)) \]

\[ F_{Pz,z}(t) \geq F_{Pz,z}(\phi(t)) \]

which is a contradiction and hence, we get \( Pz = z \). Hence, \( Pz = z = ABz \).

Further using step III, we get \( Bz = z \).

Thus \( ABz = z \) gives \( Az = z \) and so \( Az = Bz = Pz = z \).

Also, it follows from steps IV, V and VI that

\[ Sz = Tz = Qz = z. \]

Hence, we get

\[ Az = Bz = Pz = Sz = Tz = Qz = z. \]
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i.e. $z$ is a common fixed point of $A$, $B$, $P$, $Q$, $S$ and $T$ in this case also.

**Uniqueness :**

Let $z_1$ be another common fixed point of $A$, $B$, $P$, $Q$, $S$ and $T$. Then

$A z_1 = B z_1 = P z_1 = S z_1 = T z_1 = Q z_1 = z_1$, assuming $z \neq z_1$

Put $x = z$ and $y = z_1$ in (3.1.5), we get

$$[1 + \alpha F_{ABz,STz_1}(t)] * F_{Pz,Qz_1}(t)$$

$$\geq \alpha \min \{ F_{Pz,ABz}(t) * F_{Qz_1,STz_1}(t), F_{Pz,STz_1}(2t) * F_{Qz_1,ABz}(2t) \}$$

$$+ F_{ABz,STz_1}(\phi(t)) * F_{Pz,ABz}(\phi(t)) * F_{Qz_1,STz_1}(\phi(t))$$

$$* F_{Pz,STz_1}(2\phi(t)) * F_{Qz_1,ABz}(2\phi(t))$$

$$[1 + \alpha F_{z,z_1}(t)] * F_{z,z_1}(t)$$

$$\geq \alpha \min \{ F_{z,z_1}(t) * F_{z,z_1}(2t) + F_{z,z_1}(2t) \} + F_{z,z_1}(\phi(t))$$

$$* F_{z,z_1}(\phi(t)) * F_{z,z_1}(2\phi(t)) * F_{z,z_1}(2\phi(t))$$

$$F_{z,z_1}(t) + \alpha \{ F_{z,z_1}(t) * F_{z,z_1}(t) \}$$

$$\geq \alpha \min \{ 1, F_{z,z_1}(2t) \} + F_{z,z_1}(\phi(t)) * F_{z,z_1}(2\phi(t))$$

$$F_{z,z_1}(t) + \alpha F_{z,z_1}(t) \geq \alpha F_{z,z_1}(2t) + F_{z,z_1}(\phi(t)) * F_{z,z_1}(\phi(t))$$

which is a contradiction.

Hence $z = z_1$ and so $z$ is the unique common fixed point of $A$, $B$, $S$, $T$, $P$ and $Q$.

This completes the proof. ■

**Remark 3.2** If we take $B = T = I$, the identity map on $X$ in theorem 3.1, then condition (3.1.2) is satisfied trivially and we get

**Corollary 3.3** Let $A$, $S$, $P$ and $Q$ be self maps of a complete Menger space $(X, \mathcal{F}, \ast)$ with $\ast = \min$ satisfying :

(a) $P(X) \subseteq S(X)$, $Q(X) \subseteq A(X)$;

(b) either $P$ or $A$ is continuous;

(c) $(P, A)$ is semi-compatible and $(Q, S)$ is weak compatible;
(d) \[ 1 + \alpha F_{Ax,Sy}(t) \ast F_{Px,Qy}(t) \geq \alpha \min \{ F_{Px,Ax}(t) \ast F_{Qy,Sy}(t), F_{Px,Sy}(2t) \ast F_{Qy,Ax}(2t) \} \]
\[ + F_{Ax,Sy}(\phi(t)) \ast F_{Px,Ax}(\phi(t)) \ast F_{Qy,Sy}(\phi(t)) \ast F_{Px,Sy}(2\phi(t)) \ast F_{Qy,Ax}(2\phi(t)) \]

for all \( x, y \in X, t > 0 \) and \( \phi \in E \).

Then \( A, S, P \) and \( Q \) have a unique common fixed point in \( X \).

**Remark 3.4** In view of Remark 3.2, corollary 3.3 is a generalization of the result of Pathak and Verma [6] in the sense that condition of compatibility of the first pair of self maps has been restricted to semi-compatibility.

**References**


**Blindra Singh**
School of Studies in Mathematics, Vikram University
Ujjain (M.P.) 456010, India

**Arihant Jain**
Department of Applied Mathematics, S.G.S.I.T.S.
Ujjain (M.P.) 456650, India
E-mail: arihant2412@gmail.com

**Bholaram Lodha**
School of Studies in Mathematics, Vikram University
Ujjain (M.P.) 456010, India

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