A Representation Theorem for ϕ-Bounded Variation of Functions in the Sense of Riesz

Abstract. In this paper we extend the well known Riesz lemma to the class of bounded ϕ-variation functions in the sense of Riesz defined on a rectangle $I^b_a \subset \mathbb{R}^2$. This concept was introduced in [2], where the authors proved that the space $BV^R_{\phi}(I^b_a; \mathbb{R})$ of such functions is a Banach Algebra. Moreover, they characterized also the Nemytskii operator acting in this space. Thus our result creates a continuation of the paper [2].

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1. Introduction. In 1881 Jordan in [6] introduced the notion of a function of bounded variation. That concept was generalized in various directions depending on the context of the theories in which it was used. In 1910 Riesz in [10] defined the concept of $p$-bounded variation function and with a help of that concept he proved that the dual space of $L^p[a,b]$ coincides with $L^q[a,b] \left( \frac{1}{p} + \frac{1}{q} = 1 \right.$ with $1 < p < \infty$).

Apart from that it was shown that the well-known Riesz lemma take place expressing that a function belongs to the class of functions of bounded variation if and only if it is absolutely continuous and its derivative belongs to $L^p$ with some $p$, $1 < p < \infty$.

During the next years Young [16] introduces a class denoted by $\Phi$ containing of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+ \ (\mathbb{R}_+ = [0, +\infty))$ being non-decreasing and continuous on $\mathbb{R}_+$ and such that $\varphi(0) = 0$, $\varphi(t) \to +\infty$ if $t \to +\infty$. It is worthwhile mentioning that results obtained by Young generalized those obtained by Wiener [15].

On the other hand let us point out that results obtained in [16] and connected with the concept of bounded $p$-variation were generalized by Medvedev [8] who introduced the following concept:
We say that a function \( u : [a, b] \to \mathbb{R} \) has bounded \( \varphi \)-variation on the interval \([a, b]\) in the sense of Riesz if the quantity

\[
V_{\varphi}^{R}(u) = V_{\varphi}^{R}(u; [a, b]) := \sup_{\pi} \sum_{i=1}^{m} \varphi \left( \frac{|u(t_{i}) - u(t_{i-1})|}{|t_{i} - t_{i-1}|} \right) |t_{i} - t_{i-1}|
\]

is finite, where \( \varphi \) is a function belonging to the class \( \Phi \), \( \pi \) denoted an arbitrary partition \( a = t_{0} < t_{1} < \cdots < t_{m} = b \) of the interval \([a, b]\).

The number \( V_{\varphi}^{R}(u; [a, b]) \) defined in (1) is called the \( \varphi \)-variation in the sense of Riesz of a function \( u \) on the interval \([a, b]\). In the sequel we denote by \( V_{\varphi}^{R}[a, b] \) the class containing all such functions.

Moreover, Medvedev proved a result being a generalization of Riesz lemma.

To quote the mentioned result we introduce the following concept. Namely, we will say that a function \( \varphi : \mathbb{R}_{+} \to \mathbb{R}_{+} \) satisfies \( \infty_{1} \)-condition if \( \varphi(t)/t \to +\infty \) when \( t \to +\infty \).

**Lemma 1.1** ([8]) Let \( \varphi \) be a convex \( \varphi \)-function (i.e. \( \varphi \in \Phi \)) satisfying \( \infty_{1} \)-condition. Then \( u \in V_{\varphi}^{R}[a, b] \) if and only if \( u \in AC[a, b] \) and \( \int_{a}^{b} \varphi(|u'(t)|)dt < \infty \).

Moreover, the following equality holds

\[
V_{\varphi}^{R}(u; [a, b]) = \int_{a}^{b} \varphi(|u'(t)|)dt.
\]

Let us note that the symbol \( AC[a, b] \) used above denotes the class of all real functions defined and absolutely continuous on \([a, b]\).

In this paper we are going the present some result generalizing the Riesz lemma for functions of two variables defined on a rectangle.

**2. Definitions, notations and auxiliary results.** The section is devoted to present some auxiliary facts which will be used later on.

Assume that \( a = (a_{1}, a_{2}) \), \( b = (b_{1}, b_{2}) \) are arbitrary fixed points on the \( \mathbb{R}^2 \) plane. Denote by \( I_{a}^{b} \) the rectangle defined as \( I_{a}^{b} = [a_{1}, b_{1}] \times [a_{2}, b_{2}] \). Based on [4, 8] we can introduced the concept of bounded \( \varphi \)-variation in the sense of Riesz for function of two variables defined on \( I_{a}^{b} \). To this end suppose that \( a_{1} = t_{0} < \cdots < t_{m} = b_{1} \) and \( a_{2} = s_{0} < \cdots < s_{n} = b_{2} \) are arbitrary partitions of the intervals \([a_{1}, b_{1}] \), \([a_{2}, b_{2}] \), respectively. Using the notation introduced in [1, 5], let us write: \( \Delta t_{i} = t_{i} - t_{i-1} \), \( \Delta s_{j} = s_{j} - s_{j-1} \). Next, for a function \( u : I_{a}^{b} \to \mathbb{R} \), \( u = u(t, s) \) let us put:

\[
\Delta_{10}u(t_{i}, s_{j}) = u(t_{i}, s_{j}) - u(t_{i-1}, s_{j}),
\]
\[
\Delta_{01}u(t_{i}, s_{j}) = u(t_{i}, s_{j}) - u(t_{i}, s_{j-1}),
\]
\[
\Delta_{11}u(t_{i}, s_{j}) = u(t_{i-1}, s_{j-1}) + u(t_{i}, s_{j}) - u(t_{i-1}, s_{j}) - u(t_{i}, s_{j-1}).
\]
Definition 2.1 Let $\varphi \in \Phi$.

(a) If $x_2$ is a fixed number in $[a_2, b_2]$ then the quantity defined by the formula

$$V^R_{\varphi,[a_1, b_1]}(u) := \sup_{\Pi_1} \sum_{i=1}^{m} \varphi \left( \frac{\left| \Delta_{10}u(t_i, x_2) \right|}{|\Delta t_i|} \right) |\Delta t_i|$$

is said to be the $\varphi$–variation in the Riesz sense of the function $u(\cdot, x_2)$ on the interval $[a_1, b_1]$, where $\Pi_1$ denotes the set of all partitions of the interval $[a_1, b_1]$.

(b) If $x_1$ is fixed in $[a_1, b_1]$ then (similarly as above) we define the quantity

$$V^R_{\varphi,[a_2, b_2]}(u) := \sup_{\Pi_2} \sum_{j=1}^{n} \varphi \left( \frac{\left| \Delta_{01}u(x_1, s_j) \right|}{|\Delta s_j|} \right) |\Delta s_j|$$

being the called $\varphi$–variation in the sense of Riesz of the function $u(x_1, \cdot)$ on the interval $[a_2, b_2]$, where the supremum is taken over the set $\Pi_2$ of all partitions of $[a_2, b_2]$.

(c) The quantity defined by the formula

$$V^R_{\varphi}(u) := \sup_{\Pi_1, \Pi_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi \left( \frac{\left| \Delta_{11}u(t_i, s_j) \right|}{|\Delta t_i| |\Delta s_j|} \right) |\Delta t_i||\Delta s_j|$$

where the supremum is taken over all pairs $(\Pi_1, \Pi_2)$ of partitions belonging to $\Pi_1, \Pi_2$, respectively, will be referred as $\varphi$–bidimensional Riesz variation of $u$.

(d) The quantity $TV^R_{\varphi}(u)$ defined by the formula

$$TV^R_{\varphi}(u) = TV^R_{\varphi}(u, I^b_a) := V^R_{\varphi,[a_1, b_1]}(u) + V^R_{\varphi,[a_2, b_2]}(u) + V^R_{\varphi}(u)$$

will be called $\varphi$–total variation in the sense of Riesz of a function $u : I^b_a \rightarrow \mathbb{R}$.

In what follows we will denote by $V^R_{\varphi}(I^b_a)$ the class of all functions having bounded $\varphi$–total variation in the sense of Riesz on the rectangle $I^b_a$ (cf. [2]).

3. Properties of bounded $\varphi$–variation of functions in the Riesz sense. At the beginning denote by $\text{Lip}(I^b_a)$ the class of real functions being Lipschitzian on the rectangle $I^b_a$.

It is well-known that in the one dimensional case each function being Lipschitzian on a bounded interval $I$ is a member of the class $V^R_{\varphi}(I)$ for $\varphi \in \Phi$. This assertion is no longer true in two dimensional case.

Indeed, we have the following result.

Proposition 3.1 There exists a function $\varphi$, $\varphi \in \Phi$, such that $\text{Lip}(I^b_a) \subset V^R_{\varphi}(I^b_a)$ does not hold.
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**Proof** In order to prove this assertion let us take the partition of the intervals $[0, 1]$ lying on both axes of coordinate system $t0$s with help of the points $0 = t_0 < \frac{1}{n} < \frac{1}{n-1} < \cdots < \frac{1}{2} < 1 = t_n$, $0 = s_0 < \frac{1}{n} < \frac{1}{n-1} < \cdots < \frac{1}{2} < 1 = s_n$. Next, let us consider the square $I^d_0 = [0, 1] \times [0, 1]$. We consider the function $u = u(t, s)$ defined in such a way that its graph is created by the surface of the pyramid with the base being the rectangle $\left[\frac{1}{j}, \frac{1}{j+1}\right] \times \left[\frac{1}{i}, \frac{1}{i+1}\right]$ located on the $t0$s plane and with the vertex situated at the point $(\frac{2^{i-1}}{2(i-1)!}, \frac{2^{j-1}}{2(j-1)!}, \frac{1}{(i-1)!})$ for $i, j = 2, 3, 4, \ldots$. Moreover, we put $u(0, 0) = 0$.

It is an easy exercise to verify that the function $u$ satisfies the Lipschitz condition with the constant $1/2$.

Next, let us take $\varphi(t) = t^2$ for $t \geq 0$. Fix a partition both of the interval $[0, 1]$ located on the $t$–axis and of the same interval located on the $s$–axis which has the form

$$0 = t_0 < \frac{1}{n} < \frac{2n-1}{2n(n-1)} < \frac{1}{n-1} < \cdots < \frac{1}{2} < \frac{3}{4} < 1 = t_{2n-1}$$

and

$$0 = s_0 < \frac{1}{n} < \frac{2n-1}{2n(n-1)} < \frac{1}{n-1} < \cdots < \frac{1}{2} < \frac{3}{4} < 1 = s_{2n-1},$$

respectively ($n \geq 2$).

Further on, for a fixed $i$ and $j$ let us calculate the quantity $\Delta u(t_i, s_j)$:

$$\Delta u(t_i, s_j) = u(t_i, s_j) + u(t_{i-1}, s_{j-1}) - u(t_{i-1}, s_j) - u(t_i, s_{j-1}) = \frac{1}{i(i-1)}.$$ 

Consequently, we derive the following chain of equalities:

$$\sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} \varphi \left( \frac{\Delta u(t_i, s_j)}{\Delta t_i \Delta s_j} \right) \Delta t_i \Delta s_j$$

$$= \sum_{i=2}^{2n-1} \sum_{j=2}^{2n-1} \varphi \left( \frac{1}{i(i-1)j(j-1)} \right) \cdot \frac{1}{i(i-1)} \cdot \frac{1}{j(j-1)}$$

$$= \sum_{i=2}^{2n-1} \sum_{j=2}^{2n-1} j(j-1)^2 \cdot \frac{1}{i(i-1)j(j-1)} = \sum_{i=2}^{2n-1} \sum_{j=2}^{2n-1} j(j-1) \cdot \frac{1}{i(i-1)}$$

$$= \sum_{i=2}^{2n-1} \frac{1}{i(i-1)} \left[ (2n-1)(2n-2) + (2n-2)(2n-3) + \cdots + 3 \cdot 2 + 2 \cdot 1 \right].$$
Now, using the simple evaluations, we get

\[
\sum_{i=1}^{2n-1} \sum_{j=1}^{2n-1} \varphi \left( \frac{|\Delta_{11} u(t_i, s_j)|}{\Delta t_i \Delta s_j} \right) \Delta t_i \Delta s_j \geq \sum_{i=2}^{2n-1} \frac{1}{i(i-1)} [(2n - 1) + (2n - 2) + \cdots + 2 + 1] \\
= n(2n - 1) \sum_{i=2}^{2n-1} \frac{1}{i(i-1)} = n(2n - 1) \left( 1 - \frac{1}{2n-1} \right) = 2n(n - 1).
\]

The above estimate shows that the \( \varphi \)-bidimensional variation in the Riesz sense of \( u \) with respect of \( \varphi \) is unbounded, i.e. \( TV^R_\varphi(u) = \infty \).

In what follows we prove a result which indicates a few class of functions belonging to \( V^R_\varphi (I^n_a) \).

**Proposition 3.2** Let \( u : I^n_a \to \mathbb{R} \) be a function satisfying one of the below listed conditions:

(i) \( |\Delta_{10} u(t_i, s)\| \leq L_1 |\Delta t_i| \),

(ii) \( |\Delta_{01} u(t, s_j)| \leq L_2 |\Delta s_j| \),

(iii) \( |\Delta_{11} u(t_i, s_j)| \leq L_3 |\Delta t_i| |\Delta s_j| \),

where \( L_1, L_2, L_3 \) are nonnegative constants and \( t, s \) are arbitrarily chosen numbers, \( t \in [a_1, b_1] , s \in [a_2, b_2] \). Then \( u \in V^R_\varphi (I^n_a) \).

**Proof** Let \( \Pi_1 = \{t_0, t_1, \ldots, t_m\} \) and \( \Pi_2 = \{s_0, s_1, \ldots, s_n\} \) be partitions of the intervals \( [a_1, b_1] \) and \( [a_2, b_2] \), respectively. Then we get
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$$TV^R_\varphi (u) = \sup_{n_1} \sum_{i=1}^{m} \varphi \left[ \frac{|\Delta_{10}u(t_i, x_2)|}{|\Delta t_i|} \right] |\Delta t_i| + \sup_{n_2} \sum_{j=1}^{n} \varphi \left[ \frac{|\Delta_{01}u(x_1, s_j)|}{|\Delta s_j|} \right] |\Delta s_j|$$

$$+ \sup_{n_1, n_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi \left[ \frac{|\Delta_{11}u(t_i, s_j)|}{|\Delta t_i||\Delta s_j|} \right] |\Delta t_i||\Delta s_j|$$

$$\leq \sup_{n_1} \sum_{i=1}^{m} \varphi \left[ \frac{L_1|t_i - t_{i-1}|}{|\Delta t_i|} \right] |\Delta t_i| + \sup_{n_2} \sum_{j=1}^{n} \varphi \left[ \frac{L_2|s_j - s_{j-1}|}{|\Delta s_j|} \right] |\Delta s_j|$$

$$+ \sup_{n_1, n_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi \left[ \frac{L_3|\Delta t_i||\Delta s_j|}{|\Delta t_i||\Delta s_j|} \right] |\Delta t_i||\Delta s_j|$$

$$= \varphi(L_1) \sup_{n_1} \sum_{i=1}^{m} |\Delta t_i| + \varphi(L_2) \sup_{n_2} \sum_{j=1}^{n} |\Delta s_j| + \varphi(L_3) \sup_{n_1, n_2} \sum_{i=1}^{m} \sum_{j=1}^{n} |\Delta t_i||\Delta s_j|$$

$$\leq \varphi(L_1)|b_1 - a_1| + \varphi(L_2)|b_2 - a_2| + \varphi(L_3)|b_1 - a_1||b_2 - a_2| < \infty.$$}

This shows that $u \in V^R_\varphi (I^b_a)$ and completes the proof.

4. Main Result. In this section we prove some results which generalize the Riesz lemma (cf. Section 1) for functions of two variables defined on a rectangle $I^b_a$. To this end we will utilize some facts from measure theory [3, 7, 12, 13, 14]. At the beginning let us denote

$$Q(t, s) = [a_1, t] \times [a_2, s] \quad \text{for } (t, s) \in I^b_a.$$}

Next, let us recall that the symbol $L(I^b_a)$ will denote the space of real functions being Lebesgue integrable on the rectangle $I^b_a$. Similarly, we denote by $L([\alpha, \beta])$ and $AC([\alpha, \beta])$ the spaces of real functions which are Lebesgue integrable and absolutely continuous on the interval $[\alpha, \beta]$, respectively. Additionally, if $E$ is a measurable set in the space $\mathbb{R}$ or $\mathbb{R}^2$, we use the symbol $\mu(E)$ in order to denote the Lebesgue measure of $E$.

In the sequel we present the definition of an absolutely continuous function which is defined on the rectangle $I^b_a$. This definition will use some ideas of Carathéodory [12] given in one dimensional case.

To realize this goal denote by $P(\{I^b_a\})$ the set of all rectangles of the form $[t_1, t_2] \times [s_1, s_2]$ which are contained in $I^b_a$. If $P \in P(\{I^b_a\})$, then the area of $P$ will be denoted by $|P|$. We say that the rectangles $P_1, P_2 \in P(\{I^b_a\})$ do not overlap if they have no common interior points. Moreover, we say that the rectangles $P_1, P_2 \in P(\{I^b_a\})$ are adjoining if they are not overlapping and $P_1 \cup P_2 \in P(\{I^b_a\})$.

**Definition 4.1 ([7, 12])** A function $F : P(\{I^b_a\}) \to \mathbb{R}$ is said to be additive if for arbitrary adjoining rectangles $P_1, P_2 \in P(\{I^b_a\})$, the following equality holds

$$F(P_1 \cup P_2) = F(P_1) + F(P_2).$$
Definition 4.2 ([7, 12]) A function \( F : \mathcal{P}(I^b_a) \to \mathbb{R} \) is referred to as absolutely continuous if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if for \( P_1, \ldots, P_k \in \mathcal{P}(I^b_a) \), we have
\[
\sum_{j=1}^k |P_j| \leq \delta,
\]
then
\[
\sum_{j=1}^k |F(P_j)| \leq \varepsilon.
\]

In what follows for a function \( u : I^b_a \to \mathbb{R} \) we introduce the following notation:
\[
F_u([t_1, t_2] \times [s_1, s_2]) = \Delta_{11}u(t_2, s_2)
\]
for \([t_1, t_2] \times [s_1, s_2] \in \mathcal{P}(I^b_a)\).
In such a case we will say that the function \( F_u : \mathcal{P}(I^b_a) \to \mathbb{R} \) is a function of rectangles associated with \( u \).

Definition 4.3 We say that a function \( u : I^b_a \to \mathbb{R} \) is absolutely continuous on \( I^b_a \) in the sense Carathéodory if the following conditions are satisfied:

(a) The function of rectangles associated with \( u \) is absolutely continuous on \( I^b_a \).

(b) The functions \( u(a_1, \cdot) : [a_2, b_2] \to \mathbb{R} \) and \( u(\cdot, a_2) : [a_1, b_1] \to \mathbb{R} \) are absolutely continuous (in the classical sense).

Now we recall a few results concerning necessary and sufficient conditions for functions of rectangles to be absolutely continuous.

Theorem 4.4 ([7, 12, 13, 14]) The function of rectangles \( F : \mathcal{P}(I^b_a) \to \mathbb{R} \) is absolutely continuous if and only if there exists a function \( h \in \mathcal{L}(I^b_a) \) such that
\[
F(P) = \int_P h(t,s)dtds \quad \text{for } P \in \mathcal{P}(I^b_a).
\]

Let us pay attention to the fact that in [12] Šremr demonstrated the following theorem characterizing functions being absolutely continuous.

Theorem 4.5 ([12, 13, 14]) The following conditions are equivalent:

(1) The function \( u : I^b_a \to \mathbb{R} \) is absolutely continuous on \( I^b_a \).

(2) The function \( u : I^b_a \to \mathbb{R} \) admits a representation in form
\[
u(t,s) = c + \int_{a_1}^t f(\tau)d\tau + \int_{a_2}^s g(\eta)d\eta + \int_{Q(t,s)} h(\tau, \eta)d\tau d\eta
\]
where \( c \) is a real constant and \( f \in \mathcal{L}([a_1, b_1]) \), \( g \in \mathcal{L}([a_2, b_2]) \), \( h \in \mathcal{L}(I^b_a) \) are some functions. Moreover, the symbol \( Q \) is defined by (6).
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(3) The function \( u: I^b_a \rightarrow \mathbb{R} \) is subject to the following conditions:

(a) \( u(\cdot, s) \in AC([a_1, b_1]) \) for every \( s \in [a_2, b_2] \), and \( u(a_1, \cdot) \in AC([a_2, b_2]) \),

(b) \( u(t, \cdot) \in AC([a_2, b_2]) \) for almost every \( t \in [a_1, b_1] \),

(c) \( u_{ts} \in L \left( I^b_a \right) \).

Let us remark that the symbols \( u_t \) and \( u_{ts} \) used in the above theorem designate the appropriate partial derivatives.

If the derivative \( u_{ts}(t, s) = \frac{\partial^2 u(t, s)}{\partial t \partial s} \) does exist we can express it in terms of \( \Delta_{11} u(t, s)/\Delta t \Delta s \) in the following manner

\[
\frac{\partial^2 u}{\partial t \partial s}(t, s) = \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial s} \right)(t, s) = \lim_{\Delta t \to 0} \frac{u(t + \Delta t, s + \Delta s) - u(t, s + \Delta s) - u(t + \Delta t, s) + u(t, s)}{\Delta t \Delta s}
\]

\[
= \lim_{\Delta t \to 0} \lim_{\Delta s \to 0} \frac{u(t + \Delta t, s + \Delta s) - u(t, s + \Delta s) + u(t + \Delta t, s) - u(t, s)}{\Delta t \Delta s}
\]

Obviously, these limits exist if \( u \) is absolutely continuous. Putting \( t = t_i, s = s_j \) we can write the following approximate equality

\[
(10) \quad \frac{\partial^2 u}{\partial t \partial s}(t_i, s_j) = \frac{\Delta_{11} u(t_i, s_j)}{\Delta t \Delta s}.
\]

Using this fact we can give the following generalization of Riesz lemma.

**THEOREM 4.6** Assume that a function \( \varphi \in \Phi \) satisfies the condition \( \infty_1 \) and \( u : I^b_a \rightarrow \mathbb{R} \) is an arbitrary function. Then \( TV^R \varphi(u) < \infty \) if and only if \( u \) is absolutely continuous on \( I^b_a \) in the sense Carathéodory and

\[
\int_{a_1}^{b_1} \varphi \left( \frac{\partial u}{\partial t}(t, s) \right) dt + \int_{a_2}^{b_2} \varphi \left( \frac{\partial u}{\partial s}(t, s) \right) ds + \int_{a_1}^{b_1} \int_{a_2}^{b_2} \varphi \left( \left| \frac{\partial^2 u}{\partial t \partial s}(t, s) \right| \right) dt ds < \infty.
\]

Moreover, the following inequality is satisfied

\[
TV^R \varphi(u) = \int_{a_1}^{b_1} \varphi \left( \frac{\partial u}{\partial t}(t, s) \right) dt + \int_{a_2}^{b_2} \varphi \left( \frac{\partial u}{\partial s}(t, s) \right) ds
\]

\[
+ \int_{a_1}^{b_1} \int_{a_2}^{b_2} \varphi \left( \left| \frac{\partial^2 u}{\partial t \partial s}(t, s) \right| \right) dt ds.
\]

**PROOF** Suppose \( TV^R \varphi(u) < \infty \). In order to show that \( F_u \in AC \left( I^b_a \right) \) we have to prove that for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( P_1, \ldots, P_k \) are rectangles in \( I^b_a \) with \( \text{int}(P_i) \cap \text{int}(P_j) = \emptyset \) for \( i \neq j \) and if

\[
\sum_{i=1}^k |P_i| < \delta \quad \text{implies} \quad \sum_{i=1}^k |F(P_i)| = \sum_{i=1}^m \sum_{j=1}^n ||\Delta_{11} u(t_i, s_j)|| < \varepsilon,
\]

\[\sum_{i=1}^k |P_i| \leq \delta \] implies \[\sum_{i=1}^k |F(P_i)| = \sum_{i=1}^m \sum_{j=1}^n ||\Delta_{11} u(t_i, s_j)|| < \varepsilon,\]
where the symbol int(P) indicates the interior of P.

Thus, let us fix arbitrary $\varepsilon > 0$. Next, take rectangles $P_1, \ldots, P_k$ such that
\[ \text{int}(P_i) \cap \text{int}(P_j) = \emptyset \text{ if } i \neq j \text{ and } P_1, \ldots, P_k \subset I_a^b. \]

Keeping in mind that $\varphi \in \Phi$ is convex function and satisfies the condition $\varphi(w_0) > 0$, we can choose $r > 0$ such that $\varphi(r) > V^R_w(u)$. Moreover, there exists $t_0$ with the property $\varphi(t) \geq rt$ for $t \geq t_0$.

Now, let us define the following sets:

\[
C_{t_0} = \left\{ (i, j) : \frac{\Delta_{11}u(t_i, s_j)}{\Delta t_i \Delta s_j} \geq t_0 \right\},
\]

\[
C_{t_0}' = \left\{ i : \frac{\Delta_{10}u(t_i, x_2)}{\Delta t_i} \geq t_0 \right\},
\]

\[
C_{t_0}'' = \left\{ j : \frac{\Delta_{01}u(x_1, s_j)}{\Delta s_j} \geq t_0 \right\}.
\]

Then we infer the following estimates:

\[
\sum_{i=1}^k |F(P_i)| = \sum_{i=1}^k \sum_{j=1}^n |\Delta_{11}u(t_i, s_j)|
\]

\[
= \sum_{(i, j) \in C_{t_0}} |\Delta_{11}u(t_i, s_j)| + \sum_{(i, j) \notin C_{t_0}} |\Delta_{11}u(t_i, s_j)|
\]

\[
< \sum_{(i, j) \in C_{t_0}} \frac{|\Delta_{11}u(t_i, s_j)|}{|\Delta t_i| |\Delta s_j|} \cdot |\Delta t_i||\Delta s_j| + \sum_{(i, j) \notin C_{t_0}} \sum_{(i, j) \notin C_{t_0}} t_0 \cdot |\Delta t_i||\Delta s_j|
\]

\[
\leq \frac{1}{2} \sum_{(i, j) \in C_{t_0}} \varphi \left[ \frac{|\Delta_{11}u(t_i, s_j)|}{|\Delta t_i| |\Delta s_j|} \right] \cdot |\Delta t_i||\Delta s_j| + t_0 \sum_{(i, j) \notin C_{t_0}} \sum_{(i, j) \notin C_{t_0}} |\Delta t_i||\Delta s_j|
\]

\[
\leq \frac{1}{2} \cdot V^R_w(u) + t_0 \sum_{i=1}^k |P_i| < \frac{\varepsilon}{2} + t_0 \sum_{i=1}^k |P_i|.
\]

Thus, for $\delta$ such that $0 < \delta < \frac{\varepsilon}{2t_0}$ we obtain that the following implication holds:

\[
\sum_{i=1}^k |P_i| < \delta \quad \text{implies} \quad \sum_{i=1}^k |F(P_i)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Consequently we have that $F_u \in AC(I_a^b)$.

Notice that in order to prove the first part of our theorem we have to show that $u(\cdot, s) \in AC([a_1, b_1])$ and $u(t, \cdot) \in AC([a_2, b_2])$. Indeed, to obtain these assertions it is enough to apply Riesz lemma for one dimensional case [8, 9, 10]. Thus we infer that $u$ is absolutely continuous in the Carathéodory sense. Apart from this, in view of Theorem 4.5 we get
A Representation Theorem for Functions in \(BV_R^\phi([a,b]; \mathbb{R})\)

\[
\int_{a_1}^{b_1} \varphi \left( \left| \frac{\partial u}{\partial t}(t,s) \right| \right) dt < \infty,
\]

\[
\int_{a_2}^{b_2} \varphi \left( \left| \frac{\partial u}{\partial s}(t,s) \right| \right) ds < \infty,
\]

\[
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \varphi \left( \left| \frac{\partial^2 u}{\partial t \partial s}(t,s) \right| \right) dtds < \infty.
\]

Conversely, suppose that \(u\) is absolutely continuous. Then we have

\[
\mathcal{V}_{\varphi, [a_1,b_1]}(u) = \int_{a_1}^{b_1} \varphi \left( \left| \frac{\partial u}{\partial t}(t,s) \right| \right) dt < \infty
\]

and

\[
\mathcal{V}_{\varphi, [a_2,b_2]}(u) = \int_{a_2}^{b_2} \varphi \left( \left| \frac{\partial u}{\partial s}(t,s) \right| \right) ds < \infty.
\]

Let us pay attention to the fact that the first of the above equalities is a consequence of Riesz lemma (Lemma 1.1) in the horizontal setting while the second one utilizes the same lemma in the vertical setting.

Now we show that

\[
\mathcal{V}^R_{\varphi}(u) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \varphi \left( \left| \frac{\partial^2 u}{\partial t \partial s}(t,s) \right| \right) dtds < \infty.
\]

To this end recall (cf. Theorem 4.5) that in the case when the function \(u\) is absolutely continuous in Carathéodory sense then the partial derivatives \(\frac{\partial^2 u(t_i, s_j)}{\partial t \partial s}\) do exist and coincide. In such a situation we can use equality (10).

This yields:

\[
\varphi \left( \frac{|\Delta_{11} u(t_i, s_j)|}{|\Delta t_i||\Delta s_j|} \right) = \varphi \left( \left| \frac{\partial^2 u}{\partial t \partial s}(t_i, s_j) \right| \right).
\]

Hence

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \varphi \left( \frac{|\Delta_{11} u(t_i, s_j)|}{|\Delta t_i||\Delta s_j|} \right) |\Delta t_i||\Delta s_j|\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi \left( \left| \frac{\partial^2 u}{\partial t \partial s}(t_i, s_j) \right| \right) |\Delta t_i||\Delta s_j| \quad \text{see [11]}
\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{t_{i-1}}^{t_i} \int_{s_{j-1}}^{s_j} \varphi \left( \left| \frac{\partial^2 u}{\partial t \partial s}(t_i, s_j) \right| \right) dtds
\]

\[
= \int_{t_0}^{t_m} \int_{s_0}^{s_n} \varphi \left( \left| \frac{\partial^2 u}{\partial t \partial s}(t_i, s_j) \right| \right) dtds.
\]
Taking supremum with respect to all partitions we deduce the following equality

\[ V^R_{\varphi}(u) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \varphi \left( \frac{\partial^2 u}{\partial t \partial s}(t, s) \right) \, dt \, ds. \]

Which implies that, \( TV^R_{\varphi}(u) < \infty \). This completes the proof. \( \blacksquare \)

REFERENCES


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