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Wadie Aziz, Hugo Leiva, Nelson Merentes, Beata Rzepka

A Representation Theorem for φ-Bounded Variation of Functions in the Sense of Riesz

Abstract. In this paper we extend the well known Riesz lemma to the class of bounded φ-variation functions in the sense of Riesz defined on a rectangle $I_a^b \subset \mathbb{R}^2$. This concept was introduced in [2], where the authors proved that the space $BV_\varphi^R\left(I_a^b;\mathbb{R}\right)$ of such functions is a Banach Algebra. Moreover, they characterized also the Nemytskii operator acting in this space. Thus our result creates a continuation of the paper [2].

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1. Introduction. In 1881 Jordan in [6] introduced the notion of a function of bounded variation. That concept was generalized in various directions depending on the context of the theories in which it was used. In 1910 Riesz in [10] defined the concept of *p-bounded variation function*and with a help of that concept he proved that the dual space of $L_p[a,b]$ coincides with $L_q[a,b]$ $\left(\frac{1}{p}+\frac{1}{q}=1 \text{ with } 1<p<\infty\right)$.

Apart from that it was shown that the well-known Riesz lemma take place expressing that a function belongs to the class of functions of bounded variation if and only if it is absolutely continuous and its derivative belongs to L_p with some p, $1<p<\infty$.

During the next years Young [16] introduces a class denoted by Φ containing of all functions $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ $(\mathbb{R}_+=[0,+\infty))$ being non-decreasing and continuous on $\mathbb{R}_+$ and such that $\varphi(0)=0$, $\varphi(t)\to+\infty$ if $t\to+\infty$. It is worthwhile mentioning that results obtained by Young generalized those obtained by Wiener [15].

On the other hand let us point out that results obtained in [16] and connected with the concept of bounded p–variation were generalized by Medvedev [8] who introduced the following concept:

We say that a function $u:[a,b]\to\mathbb{R}$ has bounded φ*–variation on the interval* $[a,b]$ *in the sense of Riesz* if the quantity

$$V_\varphi^R(u)=V_\varphi^R(u;[a,b]):=\sup_\pi\sum_{i=1}^m\varphi\left(\frac{|u(t_i)-u(t_{i-1})|}{|t_i-t_{i-1}|}\right)|t_i-t_{i-1}| \tag{1}$$

is finite, where φ is a function belonging to the class Φ, π denoted an arbitrary partition $a=t_0<t_1<\cdots<t_m=b$ of the interval $[a,b]$.

The number $V_\varphi^R(u;[a,b])$ defined in (1) is called the φ-variation in the sense of Riesz of a function u on the interval $[a,b]$. In the sequel we denote by $V_\varphi^R[a,b]$ the class containing all such functions.

Moreover, Medvedev proved a result being a generalization of Riesz lemma.

To quote the mentioned result we introduce the following concept. Namely, we will say that a function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ satisfies ∞_1*–condition* if $\varphi(t)/t\to+\infty$ when $t\to+\infty$.

Lemma 1.1 ([8]) *Let φ be a convex φ-function (i.e. $\varphi\in\Phi$) satisfying ∞_1–condition. Then $u\in V_\varphi^R[a,b]$ if and only if $u\in AC[a,b]$ and $\int_a^b\varphi(|u'(t)|)dt<\infty$. Moreover, the following equality holds*

$$V_\varphi^R(u;[a,b])=\int_a^b\varphi(|u'(t)|)dt.$$

Let us note that the symbol $AC[a,b]$ used above denotes the class of all real functions defined and absolutely continuous on $[a,b]$.

In this paper we are going the present some result generalizing the Riesz lemma for functions of two variables defined on a rectangle.

2. Definitions, notations and auxiliary results. The section is devoted to present some auxiliary facts which will be used later on.

Assume that $a=(a_1,a_2)$, $b=(b_1,b_2)$ are arbitrary fixed points on the $\mathbb{R}^2$ plane. Denote by I_a^b the rectangle defined as $I_a^b=[a_1,b_1]\times[a_2,b_2]$. Based on [4, 8] we can introduced the concept of bounded φ–variation in the sense of Riesz for function of two variables defined on I_a^b. To this end suppose that $a_1=t_0<\cdots<t_m=b_1$ and $a_2=s_0<\cdots<s_n=b_2$ are arbitrary partitions of the intervals $[a_1,b_1]$, $[a_2,b_2]$, respectively. Using the notation introduced in [1, 5], let us write: $\Delta t_i=t_i-t_{i-1}$, $\Delta s_j=s_j-s_{j-1}$. Next, for a function $u:I_a^b\to\mathbb{R}$, $u=u(t,s)$ let us put:

$$\Delta_{10}u(t_i,s_j)=u(t_i,s_j)-u(t_{i-1},s_j),$$

$$\Delta_{01}u(t_i,s_j)=u(t_i,s_j)-u(t_i,s_{j-1}),$$

$$\Delta_{11}u(t_i,s_j)=u(t_{i-1},s_{j-1})+u(t_i,s_j)-u(t_{i-1},s_j)-u(t_i,s_{j-1}).$$

Definition 2.1 Let $\varphi \in \Phi$.

(a) If x_2 is a fixed number in $[a_2, b_2]$ then the quantity defined by the formula

$$V^R_{\varphi,[a_1,\,b_1]}(u) := \sup_{\Pi_1} \sum_{i=1}^{m} \varphi\left[\frac{|\Delta_{10} u(t_i, x_2)|}{|\Delta t_i|}\right] |\Delta t_i| \tag{2}$$

is said to be the *φ–variation in the Riesz sense* of the function $u(\cdot, x_2)$ on the interval $[a_1, b_1]$, were Π_1 denotes the set of all partitions of the interval $[a_1, b_1]$.

(b) If x_1 is fixed in $[a_1, b_1]$ then (similarly as above) we define the quantity

$$V^R_{\varphi,[a_2,b_2]}(u) := \sup_{\Pi_2} \sum_{j=1}^{n} \varphi\left[\frac{|\Delta_{01} u(x_1, s_j)|}{|\Delta s_j|}\right] |\Delta s_j| \tag{3}$$

being the called φ–variation in the sense of Riesz of the function $u(x_1, \cdot)$ on the interval $[a_2, b_2]$, where the supremum is taken over the set Π_2 of all partitions of $[a_2, b_2]$.

(c) The quantity defined by the formula

$$V^R_{\varphi}(u) := \sup_{\Pi_1,\Pi_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi\left[\frac{|\Delta_{11} u(t_i, s_j)|}{|\Delta t_i||\Delta s_j|}\right] \cdot |\Delta t_i||\Delta s_j| \tag{4}$$

where the supremum is taken over all pairs (Π_1, Π_2) of partitions belonging to Π_1, Π_2, respectively, will be referred as *φ–bidimensional Riesz variation* of u.

(d) The quantity $TV^R_{\varphi}(u)$ defined by the formula

$$TV^R_{\varphi}(u) = TV^R_{\varphi}\left(u,\, I^b_a\right) := V^R_{\varphi,[a_1,b_1]}(u) + V^R_{\varphi,[a_2,b_2]}(u) + V^R_{\varphi}(u) \tag{5}$$

will be called *φ–total variation in the sense of Riesz* of a function $u : I^b_a \to \mathbb{R}$.

In what follows we will denote by $V^R_{\varphi}\left(I^b_a\right)$ the class of all functions having bounded φ–total variation in the sense of Riesz on the rectangle I^b_a (cf. [2]).

3. Properties of bounded φ–variation of functions in the Riesz sense. At the beginning denote by $\mathrm{Lip}\left(I^b_a\right)$ the class of real functions being Lipschitzian on the rectangle I^b_a.

It is well-known that in the one dimensional case each function being Lipschitzian on a bounded interval I is a member of the class $V^R_{\varphi}(I)$ for $\varphi \in \Phi$. This assertion is no longer true in two dimensional case.

Indeed, we have the following result.

Proposition 3.1 *There exists a function φ, $\varphi \in \Phi$, such that* $\mathrm{Lip}\left(I^b_a\right) \subset V^R_{\varphi}\left(I^b_a\right)$ *does not hold.*

Proof In order to prove this assertion let us take the partition of the intervals $[0,1]$ lying on both axes of coordinate system $t0s$ with help of the points $0=t_0<\frac{1}{n}<\frac{1}{n-1}<\cdots<\frac{1}{2}<1=t_n$, $0=s_0<\frac{1}{n}<\frac{1}{n-1}<\cdots<\frac{1}{2}<1=s_n$. Next, let us consider the square $I_0^1=[0,1]\times[0,1]$. We consider the function $u=u(t,s)$ defined in such a way that its graph is created by the surface of the pyramid with the base being the rectangle $\left[\frac{1}{i},\frac{1}{i-1}\right]\times\left[\frac{1}{j},\frac{1}{j-1}\right]$ located on the $t0s$ plane and with the vertex situated at the point $\left(\frac{2i-1}{2i(i-1)},\frac{2j-1}{2j(j-1)},\frac{1}{i(i-1)}\right)$ for $i,j=2,3,4,\ldots$. Moreover, we put $u(0,0)=0$.

It is an easy exercise to verify that the function u satisfies the Lipschitz condition with the constant $1/2$.

Next, let us take $\varphi(t)=t^2$ for $t\geqslant 0$. Fix a partition both of the interval $[0,1]$ located on the t–axis and of the same interval located on the s–axis which has the form

$$0=t_0<\frac{1}{n}<\frac{2n-1}{2n(n-1)}<\frac{1}{n-1}<\cdots<\frac{1}{2}<\frac{3}{4}<1=t_{2n-1}$$

and

$$0=s_0<\frac{1}{n}<\frac{2n-1}{2n(n-1)}<\frac{1}{n-1}<\cdots<\frac{1}{2}<\frac{3}{4}<1=s_{2n-1},$$

respectively ($n\geqslant 2$).

Further on, for a fixed i and j let us calculate the quantity $\Delta_{11}u(t_i,s_j)$:

$$\Delta_{11}u(t_i,s_j)=u(t_i,s_j)+u(t_{i-1},s_{j-1})-u(t_{i-1},s_j)-u(t_i,s_{j-1})=\frac{1}{i(i-1)}.$$

Consequently, we derive the following chain of equalities:

$$\begin{aligned}
&\sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1}\varphi\left(\frac{|\Delta_{11}u(t_i,s_j)|}{\Delta t_i\Delta s_j}\right)\Delta t_i\Delta s_j\\
&=\sum_{i=2}^{2n-1}\sum_{j=2}^{2n-1}\varphi\left(\frac{\frac{1}{i(i-1)}}{\frac{1}{i(i-1)}\frac{1}{j(j-1)}}\right)\cdot\frac{1}{i(i-1)}\frac{1}{j(j-1)}\\
&=\sum_{i=2}^{2n-1}\sum_{j=2}^{2n-1}j^2(j-1)^2\cdot\frac{1}{i(i-1)}\frac{1}{j(j-1)}=\sum_{i=2}^{2n-1}\sum_{j=2}^{2n-1}\frac{j(j-1)}{i(i-1)}\\
&=\sum_{i=2}^{2n-1}\frac{1}{i(i-1)}\left[(2n-1)(2n-2)+(2n-2)(2n-3)+\cdots+3\cdot 2+2\cdot 1\right].
\end{aligned}$$

Now, using the simple evaluations, we get

$$\sum_{i=1}^{2n-1}\sum_{j=1}^{2n-1}\varphi\left(\frac{|\Delta_{11}u(t_i,s_j)|}{\Delta t_i\Delta s_j}\right)\Delta t_i\Delta s_j$$

$$\geqslant \sum_{i=2}^{2n-1}\frac{1}{i(i-1)}\left[(2n-1)+(2n-2)+\cdots+2+1\right]$$

$$= n(2n-1)\sum_{i=2}^{2n-1}\frac{1}{i(i-1)} = n(2n-1)\left(1-\frac{1}{2n-1}\right) = 2n(n-1).$$

The above estimate shows that the φ–bidimensional variation in the Riesz sense of u with respect of φ is unbounded, i.e. $TV_{\varphi}^{R}(u)=\infty$. ■

In what follows we prove a result which indicates a few class of functions belonging to $V_{\varphi}^{R}\left(I_a^b\right)$.

Proposition 3.2 *Let $u : I_a^b \to \mathbb{R}$ be a function satisfying one of the below listed conditions:*

(i) $|\Delta_{10}u(t_i,s)| \leqslant L_1|\Delta t_i|$,

(ii) $|\Delta_{01}u(t,s_j)| \leqslant L_2|\Delta s_j|$,

(iii) $|\Delta_{11}u(t_i,s_j)| \leqslant L_3|\Delta t_i||\Delta s_j|$,

where L_1, L_2, L_3 are nonnegative constants and t, s are arbitrarily chosen numbers, $t\in[a_1,b_1]$, $s\in[a_2,b_2]$. Then $u\in V_{\varphi}^{R}\left(I_a^b\right)$.

Proof Let $\Pi_1=\{t_0,t_1,\ldots,t_m\}$ and $\Pi_2=\{s_0,s_1,\ldots,s_n\}$ be partitions of the intervals $[a_1,b_1]$ and $[a_2,b_2]$, respectively. Then we get

$$
\begin{aligned}
TV_\varphi^R(u) = \sup_{\Pi_1}\sum_{i=1}^{m}\varphi\left[\frac{|\Delta_{10}u(t_i,x_2)|}{|\Delta t_i|}\right]|\Delta t_i| + \sup_{\Pi_2}\sum_{j=1}^{n}\varphi\left[\frac{|\Delta_{01}u(x_1,s_j)|}{|\Delta s_j|}\right]|\Delta s_j| \\
+ \sup_{\Pi_1,\Pi_2}\sum_{i=1}^{m}\sum_{j=1}^{n}\varphi\left[\frac{|\Delta_{11}u(t_i,s_j)|}{|\Delta t_i||\Delta s_j|}\right]|\Delta t_i||\Delta s_j| \\
\leqslant \sup_{\Pi_1}\sum_{i=1}^{m}\varphi\left[\frac{L_1|t_i-t_{i-1}|}{|\Delta t_i|}\right]|\Delta t_i| + \sup_{\Pi_2}\sum_{j=1}^{n}\varphi\left[\frac{L_2|s_j-s_{j-1}|}{|\Delta s_j|}\right]|\Delta s_j| \\
+ \sup_{\Pi_1,\Pi_2}\sum_{i=1}^{m}\sum_{j=1}^{n}\varphi\left[\frac{L_3|\Delta t_i||\Delta s_j|}{|\Delta t_i||\Delta s_j|}\right]|\Delta t_i||\Delta s_j| \\
= \varphi(L_1)\sup_{\Pi_1}\sum_{i=1}^{m}|\Delta t_i| + \varphi(L_2)\sup_{\Pi_2}\sum_{j=1}^{n}|\Delta s_j| + \varphi(L_3)\sup_{\Pi_1,\Pi_2}\sum_{i=1}^{m}\sum_{j=1}^{n}|\Delta t_i||\Delta s_j| \\
\leqslant \varphi(L_1)|b_1-a_1| + \varphi(L_2)|b_2-a_2| + \varphi(L_3)|b_1-a_1||b_2-a_2| < \infty.
\end{aligned}
$$

This shows that $u \in V_\varphi^R\left(I_a^b\right)$ and completes the proof. ■

4. Main Result. In this section we prove some results which generalize the Riesz lemma (cf. Section 1) for functions of two variables defined on a rectangle I_a^b. To this end we will utilize some facts from measure theory [3, 7, 12, 13, 14]. At the beginning let us denote

$$Q(t,s) = [a_1,t]\times[a_2,s] \qquad \text{for } (t,s)\in I_a^b. \tag{6}$$

Next, let us recall that the symbol $L\left(I_a^b\right)$ will denote the space of real functions being Lebesgue integrable on the rectangle I_a^b. Similarly, we denote by $L([\alpha,\beta])$ and $AC([\alpha,\beta])$ the spaces of real functions which are Lebesgue integrable and absolutely continuous on the interval $[\alpha,\beta]$, respectively. Additionally, if E is a measurable set in the space $\mathbb{R}$ or $\mathbb{R}^2$, we use the symbol $\mu(E)$ in order to denote the Lebesgue measure of E.

In the sequel we present the definition of an absolutely continuous function which is defined on the rectangle I_a^b. This definition will use some ideas of Carathéodory [12] given in one dimensional case.

To realize this goal denote by $\mathcal{P}\left(I_a^b\right)$ the set of all rectangles of the form $[t_1,t_2]\times[s_1,s_2]$ which are contained in I_a^b. If $P\in\mathcal{P}\left(I_a^b\right)$, then the area of P will be denoted by $|P|$. We say that the rectangles $P_1,P_2\in\mathcal{P}\left(I_a^b\right)$ do not overlap if they have no common interior points. Moreover, we say that the rectangles $P_1,P_2\in\mathcal{P}\left(I_a^b\right)$ are adjoining if they are not overlapping and $P_1\cup P_2\in\mathcal{P}\left(I_a^b\right)$.

Definition 4.1 ([7, 12]) A function $F:\mathcal{P}\left(I_a^b\right)\to\mathbb{R}$ is said to be additive if for arbitrary adjoining rectangles $P_1,P_2\in\mathcal{P}\left(I_a^b\right)$, the following equality holds

$$F(P_1\cup P_2) = F(P_1)+F(P_2).$$

Definition 4.2 ([7, 12]) A function $F : \mathcal{P}\left(I_a^b\right) \to \mathbb{R}$ is referred to as *absolutely continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if for $P_1, \ldots, P_k \in \mathcal{P}\left(I_a^b\right)$, we have

$$\sum_{j=1}^{k} |P_j| \leqslant \delta,$$

then

$$\sum_{j=1}^{k} |F(P_j)| \leqslant \varepsilon.$$

In what follows for a function $u : I_a^b \to \mathbb{R}$ we introduce the following notation:

$$F_u([t_1, t_2] \times [s_1, s_2]) = \Delta_{11} u(t_2, s_2) \tag{7}$$

for $[t_1, t_2] \times [s_1, s_2] \in \mathcal{P}\left(I_a^b\right)$.

In such a case we will say that the function $F_u : \mathcal{P}\left(I_a^b\right) \to \mathbb{R}$ is a *function of rectangles associated with* u.

Definition 4.3 We say that a function $u : I_a^b \to \mathbb{R}$ is absolutely continuous on I_a^b in the sense Carathéodory if the following conditions are satisfied:

(a) The function of rectangles associated with u is absolutely continuous on I_a^b.

(b) The functions $u(a_1, \cdot) : [a_2, b_2] \to \mathbb{R}$ and $u(\cdot, a_2) : [a_1, b_1] \to \mathbb{R}$ are absolutely continuous (in the classical sense).

Now we recall a few results concerning necessary and sufficient conditions for functions of rectangles to be absolutely continuous.

Theorem 4.4 ([7, 12, 13, 14]) *The function of rectangles $F : \mathcal{P}\left(I_a^b\right) \to \mathbb{R}$ is absolutely continuous if and only if there exists a function $h \in L\left(I_a^b\right)$ such that*

$$F(P) = \iint_P h(t, s) dt ds \quad \text{for} \quad P \in \mathcal{P}\left(I_a^b\right). \tag{8}$$

Let us pay attention to the fact that in [12] Šremr demonstrated the following theorem characterizing functions being absolutely continuous.

Theorem 4.5 ([12, 13, 14]) *The following conditions are equivalent:*

(1) *The function $u : I_a^b \to \mathbb{R}$ is absolutely continuous on I_a^b.*

(2) *The function $u : I_a^b \to \mathbb{R}$ admits a representation in form*

$$u(t, s) = c + \int_{a_1}^{t} f(\tau) d\tau + \int_{a_2}^{s} g(\eta) d\eta + \iint_{Q(t,s)} h(\tau, \eta) d\tau d\eta \tag{9}$$

where c is a real constant and $f \in L([a_1, b_1])$, $g \in L([a_2, b_2])$, $h \in L\left(I_a^b\right)$ are some functions. Moreover, the symbol Q is defined by (6).

(3) *The function* $u : I_a^b \to \mathbb{R}$ *is subject to the following conditions:*

(a) $u(\cdot, s) \in AC([a_1, b_1])$ *for every* $s \in [a_2, b_2]$, *and* $u(a_1, \cdot) \in AC([a_2, b_2])$,

(b) $u_t(t, \cdot) \in AC([a_2, b_2])$ *for almost every* $t \in [a_1, b_1]$,

(c) $u_{ts} \in L\left(I_a^b\right)$.

Let us remark that the symbols u_t and u_{ts} used in the above theorem designate the appropriate partial derivatives.

If the derivative $u_{ts}(t,s) = \frac{\partial^2 u(t,s)}{\partial t \partial s}$ does exist we can express it in terms of $\Delta_{11}u(t,s)/\Delta t \Delta s$ in the following manner

$$\frac{\partial^2 u}{\partial t \partial s}(t,s) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial s}\right)(t,s) = \lim_{\Delta t \to 0} \frac{\frac{\partial u}{\partial s}(t+\Delta t, s) - \frac{\partial u}{\partial s}(t,s)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\lim\limits_{\Delta s \to 0} \dfrac{u(t+\Delta t, s+\Delta s) - u(t+\Delta t, s)}{\Delta s} - \lim\limits_{\Delta s \to 0} \dfrac{u(t, s+\Delta s) - u(t,s)}{\Delta s}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \lim_{\Delta s \to 0} \frac{u(t+\Delta t, s+\Delta s) - u(t+\Delta t, s) - u(t, s+\Delta s) + u(t,s)}{\Delta t \Delta s}.$$

Obviously, these limits exist if u is absolutely continuous. Putting $t = t_i$, $s = s_j$ we can write the following approximate equality

$$\frac{\partial^2 u}{\partial t \partial s}(t_i, s_j) = \frac{\Delta_{11} u(t_i, s_j)}{\Delta t_i \Delta s_j}. \tag{10}$$

Using this fact we can give the following generalization of Riesz lemma.

THEOREM 4.6 *Assume that a function* $\varphi \in \Phi$ *satisfies the condition* ∞_1 *and* $u : I_a^b \to \mathbb{R}$ *is an arbitrary function. Then* $TV_\varphi^R(u) < \infty$ *if and only if* u *is absolutely continuous on* I_a^b *in the sense Carathéodory and*

$$\int_{a_1}^{b_1} \varphi\left(\left|\frac{\partial u}{\partial t}(t,s)\right|\right) dt + \int_{a_2}^{b_2} \varphi\left(\left|\frac{\partial u}{\partial s}(t,s)\right|\right) ds + \int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi\left(\left|\frac{\partial^2 u}{\partial t \partial s}(t,s)\right|\right) dtds < \infty.$$

Moreover, the following inequality is satisfied

$$TV_\varphi^R(u) = \int_{a_1}^{b_1} \varphi\left(\left|\frac{\partial u}{\partial t}(t,s)\right|\right) dt + \int_{a_2}^{b_2} \varphi\left(\left|\frac{\partial u}{\partial s}(t,s)\right|\right) ds + \int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi\left(\left|\frac{\partial^2 u}{\partial t \partial s}(t,s)\right|\right) dtds.$$

PROOF Suppose $TV_\varphi^R(u) < \infty$. In order to show that $F_u \in AC\left(I_a^b\right)$ we have to prove that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $P_1, \ldots, P_k$ are rectangles in I_a^b with $\operatorname{int}(P_i) \cap \operatorname{int}(P_j) = \emptyset$ for $i \neq j$ and if

$$\sum_{i=1}^{k} |P_i| \leqslant \delta \quad \text{implies} \quad \sum_{i=1}^{k} |F(P_i)| = \sum_{i=1}^{m}\sum_{j=1}^{n} [|\Delta_{11} u(t_i, s_j)|] < \varepsilon,$$

where the symbol $\operatorname{int}(P)$ indicates the interior of P.

Thus, let us fix arbitrary $\varepsilon > 0$. Next, take rectangles $P_1, \ldots, P_k$ such that $\operatorname{int}(P_i) \cap \operatorname{int}(P_j) = \emptyset$ if $i \neq j$ and $P_1, \ldots, P_k \subset I_a^b$.

Keeping in mind that $\varphi \in \Phi$ is convex function and satisfies the condition ∞_1, we can choose $r > 0$ such that $\frac{\varepsilon r}{2} > V_\varphi^R(u)$. Moreover, there exists t_0 with the property $\varphi(t) \geqslant rt$ for $t \geqslant t_0$.

Now, let us define the following sets:

$$\begin{aligned} C_{t_0} &= \left\{(i,j) : \frac{|\Delta_{11} u(t_i, s_j)|}{|\Delta t_i||\Delta s_j|} \geqslant t_0\right\}, \\ C'_{t_0} &= \left\{i : \frac{|\Delta_{10} u(t_i, x_2)|}{|\Delta t_i|} \geqslant t_0\right\}, \\ C''_{t_0} &= \left\{j : \frac{|\Delta_{01} u(x_1, s_j)|}{|\Delta s_j|} \geqslant t_0\right\}. \end{aligned}$$

Then we infer the following estimates:

$$\begin{aligned} &\sum_{i=1}^{k} |F(P_i)| = \sum_{i=1}^{m}\sum_{j=1}^{n} |\Delta_{11} u(t_i, s_j)| \\ &= \sum\sum_{(i,j)\in C_{t_0}} |\Delta_{11} u(t_i, s_j)| + \sum\sum_{(i,j)\notin C_{t_0}} |\Delta_{11} u(t_i, s_j)| \\ &< \sum\sum_{(i,j)\in C_{t_0}} \frac{|\Delta_{11} u(t_i, s_j)|}{|\Delta t_i||\Delta s_j|} \cdot |\Delta t_i||\Delta s_j| + \sum\sum_{(i,j)\notin C_{t_0}} t_0 \cdot |\Delta t_i||\Delta s_j| \\ &\leqslant \tfrac{1}{r} \sum\sum_{(i,j)\in C_{t_0}} \varphi\left[\frac{|\Delta_{11} u(t_i, s_j)|}{|\Delta t_i||\Delta s_j|}\right] \cdot |\Delta t_i||\Delta s_j| + t_0 \sum\sum_{(i,j)\notin C_{t_0}} |\Delta t_i||\Delta s_j| \\ &\leqslant \tfrac{1}{r} \cdot V_\varphi^R(u) + t_0 \sum_{i=1}^{k} |P_i| < \frac{\varepsilon}{2} + t_0 \sum_{i=1}^{k} |P_i|. \end{aligned}$$

Thus, for δ such that $0 < \delta < \frac{\varepsilon}{2t_0}$ we obtain that the following implication holds:

$$\sum_{i=1}^{k} |P_i| \leqslant \delta \quad \text{implies} \quad \sum_{i=1}^{k} |F(P_i)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently we have that $F_u \in AC\left(I_a^b\right)$.

Notice that in order to prove the first part of our theorem we have to show that $u(\cdot, s) \in AC([a_1, b_1])$ and $u(t, \cdot) \in AC([a_2, b_2])$. Indeed, to obtain these assertions it is enough to apply Riesz lemma for one dimensional case [8, 9, 10]. Thus we infer that u is absolutely continuous in the Carathéodory sense. Apart from this, in view of Theorem 4.5 we get

$$\int_{a_1}^{b_1} \varphi\left(\left|\frac{\partial u}{\partial t}(t,s)\right|\right) dt < \infty,$$

$$\int_{a_2}^{b_2} \varphi\left(\left|\frac{\partial u}{\partial s}(t,s)\right|\right) ds < \infty,$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi\left(\left|\frac{\partial^2 u}{\partial t\partial s}(t,s)\right|\right) dtds < \infty.$$

Conversely, suppose that u is absolutely continuous. Then we have

$$V^R_{\varphi,[a_1,b_1]}(u) = \int_{a_1}^{b_1} \varphi\left(\left|\frac{\partial u}{\partial t}(t,s)\right|\right) dt < \infty$$

and

$$V^R_{\varphi,[a_2,b_2]}(u) = \int_{a_2}^{b_2} \varphi\left(\left|\frac{\partial u}{\partial s}(t,s)\right|\right) ds < \infty.$$

Let us pay attention to the fact that the first of the above equalities is a consequence of Riesz lemma (Lemma 1.1) in the horizontal setting while the second one utilizes the same lemma in the vertical setting.

Now we show that

$$V^R_{\varphi}(u) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi\left(\left|\frac{\partial^2 u}{\partial t\partial s}(t,s)\right|\right) dtds < \infty.$$

To this end recall (cf. Theorem 4.5) that in the case when the function u is absolutely continuous in Carathéodory sense then the partial derivatives $\partial^2 u(t_i,s_j)/\partial t\partial s$ and $\partial^2 u(t_i,s_j)/\partial s\partial t$ do exist and coincide. In such a situation we can use equality (10). This yields:

$$\varphi\left(\frac{|\Delta_{11}u(t_i,s_j)|}{|\Delta t_i||\Delta s_j|}\right) = \varphi\left(\left|\frac{\partial^2 u}{\partial t\partial s}(t_i,s_j)\right|\right).$$

Hence

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \varphi\left(\frac{|\Delta_{11}u(t_i,s_j)|}{|\Delta t_i||\Delta s_j|}\right)|\Delta t_i||\Delta s_j|$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi\left(\left|\frac{\partial^2 u}{\partial t\partial s}(t_i,s_j)\right|\right)|\Delta t_i||\Delta s_j| \quad \text{see [11]}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} \varphi\left(\left|\frac{\partial^2 u}{\partial t\partial s}(t_i,s_j)\right|\right) dtds$$

$$= \int_{t_0}^{t_m}\int_{s_0}^{s_n} \varphi\left(\left|\frac{\partial^2 u}{\partial t\partial s}(t_i,s_j)\right|\right) dtds.$$

Taking supremum with respect to all partitions we deduce the following equality

$$V_{\varphi}^{R}(u) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \varphi \left(\left| \frac{\partial^2 u}{\partial t \partial s}(t,s) \right| \right) dtds.$$

Which implies that, $TV_{\varphi}^{R}(u) < \infty$. This completes the proof. ∎

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Wadie Aziz
Escuela de Matemáticas, Universidad Central de Venezuela
Caracas - Venezuela
E-mail: wadie@ula.ve

Hugo Leiva
Escuela de Matemáticas, Universidad Central de Venezuela
Caracas - Venezuela
E-mail: hleiva@ula.ve

Nelson Merentes
Escuela de Matemáticas, Universidad Central de Venezuela
Caracas - Venezuela
E-mail: nmerucv@gmail.com

Beata Rzepka
Department of Mathematics, Rzeszow University of Technology
W. Pola 2, 35-959 Rzeszów, Poland
E-mail: brzepka@prz.edu.pl

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