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Uniformly continuous composition operators in the space of functions of two variables of bounded φ-variation in the sense of Wiener

Abstract. Assume that the generator of a Nemytskii composition operator is a function of three variables: the first two real and third in a closed convex subset of a normed space, with values in a real Banach space. We prove that if this operator maps a certain subset of the Banach space of functions of two real variables of bounded Wiener φ-variation into another Banach space of a similar type, and is uniformly continuous, then the one-sided regularizations of the generator are affine in the third variable.

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1. Introduction. Let I_a^b denote the rectangle $[a_1, b_1] \times [a_2, b_2]$ with the vertices $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{R}^2$. Let $(X, |\cdot|)$, $(Y, |\cdot|)$ be real normed spaces and C be a set in X. For a function $h : I_a^b \times C \longrightarrow Y$ define the Nemytskii composition operator $H : C^{I_a^b} \longrightarrow Y^{I_a^b}$ by

$$H(f)(t, s) = h(t, s, f(t, s)), \qquad f \in C^{I_a^b},\ (t, s) \in I_a^b,$$

where $C^{I_a^b}$ stands for the family of all functions $f : I_a^b \longrightarrow C$. The function h is called the generator of the composition operator H.

Let $(BV_\varphi(I_a^b, X), \|\cdot\|_\varphi)$ be the Banach space of functions $f \in X^{I_a^b}$ which are of bounded φ-variation in the sense of Wiener [9], where the norm $\|\cdot\|_\varphi$ is defined with the aid of Luxemburg-Nakano-Orlicz seminorm [4, 7, 8].

Assume that H maps the set of functions $f \in BV_\varphi(I_a^b, X)$ such that $f(I_a^b) \subset C$ into $BV_\psi(I_a^b, Y)$. In the present paper we prove that, under some conditions, if H is uniformly continuous, then the left-left, right-right, left-right and right-left regularizations of its generator h with respect to the two first variables are affine functions in the third variable.

This generalizes the result of Chistyakov [2] where it is assumed that H is Lipschitzian.

2. Preliminaries. We begin this section with some notations.

Let $a = (a_1, a_2),\ b = (b_1, b_2) \in \mathbb{R}^2$ be such that $a_i < b_i,\ i = 1, 2$. In the sequel $I_a^b := [a_1, b_1] \times [a_2, b_2]$ denotes the basic rectangle. Let $\xi = (t_i)_{i=0}^m$ and $\eta = (s_j)_{j=0}^n$ be partitions of $[a_1, b_1]$ and $[a_2, b_2]$, respectively (i.e., $m, n \in \mathbb{N}$, $a_1 = t_0 < t_1 < \cdots < t_m = b_1$ and $a_2 = s_0 < s_1 < \cdots < s_n = b_2$). For each function $f \in X^{I_a^b}$, we define

$$\begin{array}{rcl} \Delta_{10} f(t_i, s_j) & = & f(t_i, s_j) - f(t_{i-1}, s_j) \\ \Delta_{01} f(t_i, s_j) & = & f(t_i, s_j) - f(t_i, s_{j-1}) \\ \Delta_{11} f(t_i, s_j) & = & f(t_{i-1}, s_{j-1}) - f(t_{i-1}, s_j) - f(t_i, s_{j-1}) + f(t_i, s_j). \end{array}$$

We denote by $\mathcal{F}$ the set of all non-decreasing continuous functions $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ such that $\varphi(t) = 0$ if and only if $t = 0$, and $\lim_{t \longrightarrow \infty} \varphi(t) = \infty$.

Definition 2.1 (Chistyakov [2])

Let $\varphi \in \mathcal{F}, X$ be a real normed space and $f \in X^{I_a^b}$ be a function.

(i) For $x_2 \in [a_2, b_2]$, the Jordan φ-variation of the function $f(\cdot, x_2)$ in $[x_1, y_1] \subset [a_1, b_1]$, denoted $v_{\varphi,[x_1,y_1]}(f) = v_{\varphi,[x_1,y_1]}(f(\cdot, x_2))$, is defined by

$$v_{\varphi,[x_1,y_1]}(f) := \sup_{\xi} \sum_{i=1}^{m} \varphi\left(|\Delta_{10} f(t_i, x_2)|\right), \tag{1}$$

where the supremum is taken over all the partitions $\xi = (t_i)_{i=0}^m$ of $[x_1, y_1]$.

(ii) For $x_1 \in [a_1, b_1]$, the Jordan φ-variation of the function $f(x_1, \cdot)$ in $[x_2, y_2] \subset [a_2, b_2]$ is defined by

$$v_{\varphi,[x_2,y_2]}(f) := \sup_{\eta} \sum_{j=1}^{n} \varphi\left(|\Delta_{01} f(x_1, s_j)|\right), \tag{2}$$

where the supremum is taken over all the partitions $\eta = (s_j)_{j=0}^n$ of $[x_2, y_2]$.

(iii) The (two-dimensional) Hardy-Vitali-Wiener φ-variation of a function $f \in X^{I_a^b}$, denoted $v_{\varphi, I_a^b}(f)$, is defined by

$$v_{\varphi, I_a^b}(f) := \sup_{(\xi,\eta)} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi\left(|\Delta_{11} f(t_i, s_j)|\right), \tag{3}$$

where the supremum is taken over all pairs $m, n \in \mathbb{N}$ and (ξ, η) with ξ a partition of $[a_1, b_1]$ and η a partition of $[a_2, b_2]$ of the above form.

(iv) The Wiener total φ-variation of $f \in X^{I_a^b}$ is defined by

(4) $TV_\varphi(f) = TV_\varphi(f, I_a^b) := v_{\varphi,[a_1,b_1]}(f(\cdot, a_2)) + v_{\varphi,[a_2,b_2]}(f(a_1, \cdot)) + v_{\varphi,I_a^b}(f).$

(v) A function $f \in X^{I_a^b}$ is of bounded Wiener total φ-variation in I_a^b, if $TV_\varphi(f) < \infty$.

(vi) The class of all functions $f \in X^{I_a^b}$ with finite Wiener total φ-variation is denoted by $V_\varphi(I_a^b, X)$; that is

$$V_\varphi(I_a^b, X) := \left\{ f \in X^{I_a^b} \ : \ TV_\varphi(f) < \infty \right\}.$$

We denote by $BV_\varphi(I_a^b, X)$ the vector space generated by $V_\varphi(I_a^b, X)$, i.e.

$$BV_\varphi(I_a^b, X) = \left\{ f \in X^{I_a^b} : \exists\ \lambda > 0 \ \text{ such that } \ \lambda f \in V_\varphi(I_a^b, X) \right\}.$$

In the space $BV_\varphi(I_a^b, X)$ we define the norm

$$\|f\|_\varphi := |f(a)| + p_\varphi(f),$$

where

$$p_\varphi(f) = \inf \left\{ \epsilon > 0 : TV_\varphi\left(f/\epsilon\right) \leqslant 1 \right\}.$$

Some properties of p_φ are in the following lemma.

Lemma 2.2 (Chistyakov [2]) For $f \in BV_\varphi(I_a^b, X)$, we have

(a) if $(t,s), (t',s') \in I_a^b$, then $|f(t,s) - f(t',s')| \leqslant 4\varphi^{-1}(1) p_\varphi(f)$;

(b) if $p_\varphi(f) > 0$ then $TV_\varphi\left(\frac{f}{p_\varphi(f)}\right) \leqslant 1$;

(c) if $\lambda > 0$ then

(c1) $p_\varphi(f) \leqslant \lambda$ if and only if $TV_\varphi\left(\frac{f}{\lambda}\right) \leqslant 1$;

(c2) if $TV_\varphi\left(f/\lambda\right) = 1$ then $p_\varphi(f) = \lambda$. ■

Theorem 2.3 (Chistyakov [2]) If $\varphi \in \mathcal{F}$ is convex and X is a Banach space, then $\left(BV_\varphi(I_a^b), \|\cdot\|_\varphi\right)$ is a Banach space. ■

In the sequel we will use the notions of left-left regularization and the left-left continuity of a function of two variables. Let $f : I_a^b \longrightarrow X$. If the function $f^- : I_a^b \longrightarrow X$ given by

$$f^-(x_1, x_2) := \begin{cases} \lim_{\substack{y_1 \to x_1^- \\ y_2 \to x_2^-}} f(y_1, y_2) & \text{for } (x_1, x_2) \in (a_1, b_1] \times (a_2, b_2], \\ \lim_{\substack{y_1 \to x_1^- \\ y_2 \to a_2^+}} f(y_1, y_2) & \text{for } x_1 \in (a_1, b_1] \ \text{ and } \ x_2 = a_2, \\ \lim_{\substack{y_1 \to a_1^+ \\ y_2 \to x_2^-}} f(y_1, y_2) & \text{for } x_1 = a_1 \ \text{ and } \ x_2 \in (a_2, b_2], \\ \lim_{\substack{y_1 \to a_1^+ \\ y_2 \to a_2^+}} f(y_1, y_2) & \text{for } x_1 = a_1 \ \text{ and } \ x_2 = a_2. \end{cases}$$

is well defined, that is, if all the above limits exist, then f^- is called the *left-left regularization* of f.

Remark 2.4

It is to be noted that $(y_1, y_2) \to (x_1^-, x_2^-)$ means that $(y_1, y_2) \in I_a^b$, $y_i < x_i, i = 1, 2$, and $(y_1, y_2) \to (x_1, x_2)$ in $\mathbb{R}^2$, and similarly for the other limits.

Remark 2.5

In a similar way we can define the right-right, left-right, right-left regularizations of a function $f \in BV_\varphi(I_a^b, X)$.

Definition 2.6 A function $f : I_a^b \longrightarrow X$ is said to be left-left continuous if

$$\lim_{(x,y)\to(t^-,s^-)} f(x, y) = f(t, s) \quad \text{for all} \quad (t, s) \in (a_1, b_1] \times (a_2, b_2].$$

Lemma 2.7 (Chistyakov [2]) If $f \in BV_\varphi(I_a^b, X)$ then f^- exists, $f^- \in BV_\varphi^-(I_a^b, X)$ and f^- is left-left continuous. ■

Remark 2.8

The respective counterparts of Lemma 2.7 for the right-right, left-right and right-left regularizations hold also true.

We denote by $BV_\varphi^-(I_a^b, X)$ the subspace of $BV_\varphi(I_a^b, X)$ of those functions which are left-left continuous on $(a_1, b_1] \times (a_2, b_2]$.

3. Main result. Denote by $\mathcal{A}(X, Y)$ the space of all additive mappings $A : X \longrightarrow Y$ and by $\mathcal{L}(X, Y)$ the space of all continuous linear mappings $A : X \longrightarrow Y$.
The main result of this section reads as follows:

Theorem 3.1 Let $I_a^b \subset \mathbb{R}^2$ be rectangle, $(X, |\cdot|)$ be a real normed space, $(Y, |\cdot|)$ be a real Banach space, C be a closed convex subset in X and assume that $\varphi, \psi \in \mathcal{F}$. If the composition operator H generated by $h : I_a^b \times C \longrightarrow Y$ maps $BV_\varphi(I_a^b, C)$ into $BV_\psi(I_a^b, Y)$, and is uniformly continuous, then there exist functions $A : I_a^b \longrightarrow \mathcal{A}(X, Y)$ and $B : I_a^b \longrightarrow Y$ such that

$$h^-(t, s, x) = A(t, s)x + B(t, s), \quad (t, s) \in I_a^b, \quad x \in C,$$

where h^- is the left-left regularization of the function $(t, s) \to h(t, s, x)$. Moreover, if $0 \in C$ and $\mathrm{int}C \neq \emptyset$, then $A : I_a^b \longrightarrow \mathcal{L}(X, Y)$ and $B \in BV_\psi^-(I_a^b, Y)$.

Proof For every $x \in C$, the constant function $I_a^b \ni (t,s) \longrightarrow x$ belongs to $BV_\varphi(I_a^b, C)$. Since H maps $BV_\varphi(I_a^b, C)$ into $BV_\psi(I_a^b, Y)$, the function $I_a^b \ni (t,s) \mapsto h(t,s,x)$ belongs to $BV_\psi(I_a^b, Y)$. Now Lemma 2.7 implies the existence of the left-left regularization h^- of function $(t,s) \to h(t,s,x)$. By assumption, H is uniformly continuous on $BV_\varphi(I_a^b, C)$. Let $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the *modulus continuity* of H, that is

$$\omega(\rho) := \sup\left\{\|H(f_1) - H(f_2)\|_\psi : \|f_1 - f_2\|_\varphi \leqslant \rho;\ f_1,\ f_2 \in BV_\varphi(I_a^b, C)\right\}, \quad \text{for} \quad \rho > 0.$$

Hence we get

$$\|H(f_1) - H(f_2)\|_\psi \leqslant \omega\left(\|f_1 - f_2\|_\varphi\right), \quad \text{for} \quad f_1, f_2 \in BV_\varphi(I_a^b, C). \tag{5}$$

From the definition of the norm $\|\cdot\|_\psi$ we obtain

$$p_\psi\left(H(f_1) - H(f_2)\right) \leqslant \|H(f_1) - H(f_2)\|_\psi, \quad \text{for} \quad f_1, f_2 \in BV_\varphi(I_a^b, C). \tag{6}$$

From Lemma 2.2(c1) and (6), if $\omega\left(\|f_1 - f_2\|_\varphi\right) > 0$, then

$$v_{\psi,[a_1,b_1]}\left(\frac{\left(H(f_1) - H(f_2)\right)(\cdot, a_2)}{\omega\left(\|f_1 - f_2\|_\varphi\right)}\right) \leqslant TV_\psi\left(\frac{H(f_1) - H(f_2)}{\omega\left(\|f_1 - f_2\|_\varphi\right)}\right) \leqslant 1. \tag{7}$$

Therefore, for any $a_1 = \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m = b_1;\ \ a_2 = \overline{\alpha}_1 < \overline{\beta}_1 < \overline{\alpha}_2 < \overline{\beta}_2 < \cdots < \overline{\alpha}_m < \overline{\beta}_m = b_2,\ \ m \in \mathbb{N}$, the definitions of the operator H and the functional $v_{\psi,[a_1,b_1]}(\cdot, a_2)$, imply that

(8)

$$\sum_{i=1}^m \psi\Bigg(\frac{|h(\alpha_i,\overline{\alpha_i},f_1(\alpha_i,\overline{\alpha_i}))-h(\alpha_i,\overline{\alpha_i},f_2(\alpha_i,\overline{\alpha_i}))-h(\alpha_i,\overline{\beta_i},f_1(\alpha_i,\overline{\beta_i}))+h(\alpha_i,\overline{\beta_i},f_2(\alpha_i,\overline{\beta_i}))}{\omega(\|f_1-f_2\|_\varphi)}$$
$$\frac{-h(\beta_i,\overline{\alpha_i},f_1(\beta_i,\overline{\alpha_i}))+h(\beta_i,\overline{\alpha_i},f_2(\beta_i,\overline{\alpha_i}))+h(\beta_i,\overline{\beta_i},f_1(\beta_i,\overline{\beta_i}))-h(\beta_i,\overline{\beta_i},f_2(\beta_i,\overline{\beta_i}))|}{\omega(\|f_1-f_2\|_\varphi)}\Bigg)$$

$\leqslant 1.$

For $\alpha, \beta \in \mathbb{R},\ \alpha < \beta$, we define functions $\eta_{\alpha,\beta} : \mathbb{R} \longrightarrow [0,1]$ by

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if} \quad t \leqslant \alpha \\ \frac{t-\alpha}{\beta-\alpha} & \text{if} \quad \alpha \leqslant t \leqslant \beta \\ 1 & \text{if} \quad \beta \leqslant t\,. \end{cases} \tag{9}$$

We first fix $t \in (a_1, b_1],\ s \in (a_2, b_2],\ m \in \mathbb{N}$. For arbitrary finite sequence $a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t,\ \ a_2 < \overline{\alpha}_1 < \overline{\beta}_1 < \overline{\alpha}_2 < \overline{\beta}_2 < \cdots < \overline{\alpha}_m < \overline{\beta}_m < s$ and $x_1, x_2 \in C,\ \ x_1 \neq x_2$, the functions $f_1, f_2 : I_a^b \longrightarrow X$ defined by

(10)

$$f_\ell(\tau,\gamma) := \frac{1}{2}\left[\left(\eta_{\alpha_i,\beta_i}(\tau) + \eta_{\overline{\alpha}_i,\overline{\beta}_i}(\gamma) - 1\right)(x_1 - x_2) + x_\ell + x_2\right], \quad (\tau,\gamma) \in I_a^b,\ \ell = 1,2,$$

belong to $BV_\varphi(I_a^b, C)$. From (10) we have

$$f_1 - f_2 = \frac{x_1 - x_2}{2},$$

therefore

$$\|f_1 - f_2\|_\varphi = \left|\frac{x_1 - x_2}{2}\right|;$$

and, moreover,

$$f_1(\alpha_i, \bar{\alpha}_i) = x_2;\ f_2(\alpha_i, \bar{\alpha}_i) = \frac{-x_1 + 3x_2}{2};\ f_1(\beta_i, \bar{\alpha}_i) = x_2;\ f_2(\beta_i, \bar{\alpha}_i) = \frac{-x_1 + 3x_2}{2};$$

$$f_1(\beta_i, \bar{\beta}_i) = x_1;\ f_2(\beta_i, \bar{\beta}_i) = \frac{x_1 + x_2}{2};\ f_1(\alpha_i, \bar{\beta}_i) = \frac{x_1 + x_2}{2};\ f_2(\alpha_i, \bar{\beta}_i) = x_2.$$

Using (8), we hence get

$$\begin{aligned}(11)\qquad &\sum_{i=1}^{m}\psi\left(\frac{\left|h(\alpha_i, \overline{\alpha_i}, x_2) - h(\alpha_i, \overline{\alpha_i}, \frac{-x_1+3x_2}{2}) - h(\alpha_i, \overline{\beta_i}, \frac{x_1+x_2}{2}) + h(\alpha_i, \overline{\beta_i}, x_2)\right.}{\omega(\|f_1 - f_2\|_\varphi)}\right.\\ &\left.\frac{\left.-h(\beta_i, \overline{\alpha_i}, x_2) + h(\beta_i, \overline{\alpha_i}, \frac{-x_1+3x_2}{2}) + h(\beta_i, \overline{\beta_i}, x_1) - h(\beta_i, \overline{\beta_i}, \frac{x_1+x_2}{2})\right|}{\omega(\|f_1 - f_2\|_\varphi)}\right)\\ &\leqslant 1.\end{aligned}$$

Since, for any $x \in C$, the constant function $I_a^b \ni (t,s) \longrightarrow x$ belongs to $BV_\varphi(I_a^b, C)$ and H maps $BV_\varphi(I_a^b, C)$ into $BV_\psi(I_a^b, Y)$, the function $I_a^b \ni (t,s) \mapsto h(t,s,x)$ is in $BV_\psi(I_a^b, Y)$ for any fixed $x \in C$. From continuity of ψ and the left-left continuity of h^- (Lemma 2.7), letting $(\alpha_1, \overline{\alpha}_1)$ tend to (t,s) from the left in (11), we obtain

$$\sum_{i=1}^{m}\psi\left(\frac{\left|h^-(t,s,x_1) - 2h^-\left(t,s,\frac{x_1+x_2}{2}\right) + h^-(t,s,x_2)\right|}{\omega\left(\frac{|x_1-x_2|}{2}\right)}\right) \leqslant 1,$$

that is

$$\psi\left(\frac{\left|h^-(t,s,x_1) - 2h^-\left(t,s,\frac{x_1+x_2}{2}\right) + h^-(t,s,x_2)\right|}{\omega\left(\frac{|x_1-x_2|}{2}\right)}\right) \leqslant \frac{1}{m}.$$

Hence, since $m \in \mathbb{N}$ is arbitrary,

$$\psi\left(\frac{\left|h^-(t,s,x_1) - 2h^-\left(t,s,\frac{x_1+x_2}{2}\right) + h^-(t,s,x_2)\right|}{\omega\left(\frac{|x_1-x_2|}{2}\right)}\right) = 0.$$

As $\psi \in \mathcal{F}$, we obtain

$$h^-(t,s,x_1) - 2h^-\left(t,s,\frac{x_1+x_2}{2}\right) + h^-(t,s,x_2) = 0.$$

Therefore

$$(12)\qquad h^-\left(t,s,\frac{x_1+x_2}{2}\right) = \frac{h^-(t,s,x_1) + h^-(t,s,x_2)}{2}$$

for all $(t, s) \in (a_1, b_1] \times (a_2, b_2]$, and for all $x_1,\ x_2 \in C$.

For $t \in (a_1, b_1]$ and $s = a_2$, let us fix $a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t$ and $a_2 < \overline{\alpha}_1 < \overline{\beta}_1 < \overline{\alpha}_2 < \overline{\beta}_2 < \cdots < \overline{\alpha}_m < \overline{\beta}_m < b_2$. Proceeding as above we get (11). Taking the limit as $(\alpha_1, \overline{\beta}_m) \to (t^-, a_2^+)$ and by (11), we get, again (12). The cases when $t = a_1$ and $s \in (a_2, b_2]$ or $t = a_1$ and $s = a_2$ can be treated similarly. Consequently,

$$h^-\left(t, s, \frac{x_1 + x_2}{2}\right) = \frac{h^-(t, s, x_1) + h^-(t, s, x_2)}{2}$$

is valid for all $(t, s) \in I_a^b$ and all $x_1,\ x_2 \in C$.

Therefore, the function $h^-(t, s, \cdot)$ satisfies the Jensen functional equation in C, for each $(t, s) \in I_a^b$. Adapting the standard argument (cf. Kuczma [3]), we conclude that, for each $(t, s) \in I_a^b$ there exist an additive function $A(t, s)$ and $B(t, s) \in Y$ such that

$$h^-(t, s, x) = A(t, s)x + B(t, s), \quad x \in C,\ (t, s) \in I_a^b. \tag{13}$$

The "moreover part", follow from the uniform continuity of the operator $H : BV_\varphi(I_a^b, C) \longrightarrow BV_\psi(I_a^b, Y)$ and $\mathrm{int} C \neq \emptyset$ imply the continuity of the function $A(t, s)$, consequently $A(t, s) \in \mathcal{L}(X, Y)$.

Since $0 \in C$, putting $x = 0$ in (13), we get

$$h^-(t, s, 0) = B(t, s), \quad (t, s) \in I_a^b,$$

which shows that $B \in BV_\psi(I_a^b, Y)$. ■

Remark 3.2 The counterparts of Theorem 3.1 for the right-right, right-left and left-right regularizations are also valid.

Remark 3.3 The uniformly continuous composition operators for functions of bounded variation in a single variable were considered in [5].

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