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Tomás Ereú, Nelson Merentes, José L. Sánchez

Some remarks on the algebra of functions of two variables with bounded total Φ-variation in Schramm sense

Abstract. This paper is devoted to discuss some generalizations of the bounded total Φ-variation in the sense of Schramm. This concept was defined by W. Schramm for functions of one real variable. In the paper we generalize the concept in question for the case of functions of of two variables defined on certain rectangle in the plane. The main result obtained in the paper asserts that the set of all functions having bounded total Φ-variation in Schramm sense has the structure of a Banach algebra.

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1. Introduction. The concept of the variation of a function was introduced by C. Jordan in 1881. Subsequently numerous mathematicians presented generalizations of that concept ([1, 2, 5, 7, 9, 10], for example). Moreover, they developed the theory of functions of bounded variation discovering very deep and important results concerning that important class of functions.
Next contributions to this interesting theory were made by Musielak and Orlicz ([4]). They utilized some concepts connected with the theory of Orlicz spaces in order to formulate some results developing the theory in question. Later on, Medvediev ([2]) using some ideas of Riesz, introduced the class of functions with bounded φ-variation in Riesz sense on a bounded and closed real interval.
We will not quote here all contributions to the theory of functions of bounded variation. Reader interesting in this subject are referred to the book ([3]).
Nevertheless, taking into account the principal goal of the paper it is worthwhile mentioning the generalization of the concept of bounded variation which was given by Schramm ([7]). The details of that definition will be given in the next section. However, let us notice that all definitions mentioned above and concerning functions of bounded variation in various sense, were associated with functions of one

variable.
It is the main purpose of this paper to present the concept of the so called bounded Φ-variation in the sense of Schramm for functions of two variables. Moreover, we show that the set of those functions has the structure of Banach algebra with respect to usual algebraic operations and under suitable norm.

2. Notation, definitions and auxiliary facts. In this section we recall some facts which will be needed further on.
Denote by $\mathbb{R}$ the set of all real numbers and put $\mathbb{R}_+ = [0, \infty)$. We will say that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a φ-function if φ is continuous on $\mathbb{R}_+$, $\varphi(0) = 0$, φ is increasing on $\mathbb{R}_+$ and $\varphi(t) \to \infty$ as $t \to \infty$.
Let us recall first the concept of the bounded φ-variation in the sense of Wiener ([9]). Namely, we say that function $u : [a, b] \to \mathbb{R}$ has φ-bounded variation in Wiener sense with respect to a φ-function φ provided the quality $V_\varphi^W(u)$ defined by the formula

$$V_\varphi^W(u) = V_\varphi^W(u; [a, b]) = \sup_\pi \sum_{j=1}^{n} \varphi\left(|u(t_j) - u(t_{j-1})|\right)$$

is finite. Here the supremum is taken over all partitions π of the interval $[a, b]$.

Next, let $\Phi = \{\phi_n\}$ be a sequence of increasing convex functions, defined on the set of nonnegative real numbers and such that $\Phi_n(0) = 0$ and $\Phi_n(t) > 0$, for $t > 0$ and $n = 1, 2, ...$ We shall say that Φ is Φ^*-sequence if $\phi_{n+1}(t) \leqslant \phi_n(t)$ for all n and t, and a Φ-sequence if in addition $\sum \phi_n$ diverges for $t > 0$. If Φ is either a the Φ^*-sequence or a Φ-sequence, we say that a function u is of Φ-bounded variation in the Schramm sense if the Φ-sums $\sum_n \phi_n\left(|u(I_n)|\right) < \infty$ for any appropriate collection $\{I_n\}$ of I ([7]) such that the intersection $I_i \cap I_j$ is empty or contains one point only for all $i, j = 1, 2, ..., i \neq j$. If $I_n = [a_n, b_n]$ is a subinterval of the interval I $(n = 1, 2, ...)$ we write $u(I_n) = u(b_n) - u(a_n)$.
We introduced the $\Phi = \{\phi_{n,m}\}$ bidimensional sequence of increasing convex functions, such that $\phi_{n,m}(0) = 0$ and $\phi_{n,m}(t) > 0$, for $t > 0$ and $n, m = 1, 2, ...$ We shall say that Φ is Φ^*-sequence if $\phi_{n',m'}(t) \leqslant \phi_{n,m}(t)$ for each $n' \leqslant n$, $m' \leqslant m$ and $t \in [0, \infty)$. If $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty}$ diverge for $t > 0$, we will say that Φ is a Φ-sequence.

3. Main results. This section is devoted to give the concept of bounded Φ-variation in the sense of Schramm for functions of two variables and show that the set of such functions has a Banach algebra structure.
At the beginning assume that $a = (a_1, c_1)$, $b = (b_1, d_1)$ are two fixed points in the plane $\mathbb{R}^2$. Denote by I_a^b the rectangle generated by the points a and b, i.e. $I_a^b = [a_1, b_1] \times [c_1, d_1]$.
Next, let us assume that $\{I_n\}$ and $\{J_m\}$ are two sequences of closed subintervals of the interval $[a_1, b_1]$ and $[c_1, d_1]$, respectively. Other words $I_n = [a_n, b_n]$, $(n = 1, 2, ...)$, $J_m = [c_m, d_m]$, $(m = 1, 2, ...)$. Finally assume that $u : I \to \mathbb{R}$ is a given function and let $\Phi = \{\phi_{n,m}\}$ be a fixed double Φ sequence.
Fix $x_2 \in J_1 = [c_1, d_1]$ and consider the function $u(\cdot, x_2) : [a_1, b_1] \to \mathbb{R}$. The quantity

V^S_{Φ,I_1} defined by the formula

$$\begin{aligned} V^S_{\Phi,I_1}(u) &= \sup \sum_{n=1}^{\infty} \phi_{n,m}\left(|u(I_n,x_2)|\right) \\ &= \sup \sum_{n=1}^{\infty} \phi_{n,m}\left(|u(b_n,x_2)-u(a_n,x_2)|\right) \end{aligned} \tag{1}$$

is said to be the Φ-*variation in the Schramm sense* of the function $u(\cdot,x_2)$. In the case when $V^S_{\Phi,I_1}(u)<\infty$ we will say the u has a *bounded* Φ-*variation in the sense of Schramm with respect to the first variable* (with fixed the second one).
In the same way we can define the concept of the Φ-variation of the function $u(x_1,\cdot)$ in the Schramm sense. We denote it by V^S_{Φ,J_1}. Obviously, if $V^S_{\Phi,J_1}(u)<\infty$ then we say that u *has bounded* Φ-*variation in the sense the Schramm with respect to second variable* (with fixed the first one).
Let us pay attention to the fact that in formula (1) the supremum is taken with respect to all sequences $\{I_n\}$ of subintervals the interval I_1. Analogously we understand the supremum in the definition of the quantity V^S_{Φ,J_1}.
Further, we provide the definition of the concept of two dimensional (or bidimensional) variation in the sense of Schramm.

Definition 3.1 The quantity $V^S_{\Phi,I^b_a}(u)$ defined by the formula

$$\begin{aligned} V^S_{\Phi,I^b_a}(u) &= \sup \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \phi_{n,m}\left(|u(I_n,J_m)|\right) \\ &= \sup \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \phi_{n,m}\left(|u(b_n,J_m)-u(a_n,J_m)|\right) \\ &= \sup \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \phi_{n,m}\left(|u(a_n,c_m)+u(b_n,d_m)-u(a_n,d_m)-u(b_n,c_m)|\right) \end{aligned}$$

is said to be the bidimensional variation in the sense of Schramm of the function u.
Finally, we introduce the definition of the main concept considered in the paper.

Definition 3.2 We say that the quantity $TV^S_{\Phi}(u)$ defined by the formula

$$TV^S_{\Phi}(u) = V^S_{\Phi,1}(u) + V^S_{\Phi,J_1}(u) + V^S_{\Phi,I^b_a}(u)$$

is the total Φ-variation of the function u in the sense of Schramm.
Obviously, a function u is referred to a function with bounded total Φ-variation of the function in the sense provide $TV^S_{\Phi}(u)<\infty$.
In what follows we denote by $BV^S_{\Phi}(I^b_a)$ the set of all functions $u: I^b_a \to \mathbb{R}$ having bounded total Φ-variation in the Schramm sense.
Starting with some considerations concerning the class $BV^S_{\Phi}(I^b_a)$ we first give a few simple remarks.

PROPOSITION 3.3 *Let $u : I_a^b \to \mathbb{R}$ be an arbitrary fixed function. Then the follows hold:*

(i) $TV_\Phi^S(u) \geqslant 0$.

(ii) *The function $TV_\Phi^S(u)$ is even.*

(iii) $TV_\Phi^S(u) = 0$ *if and only if u is constant.*

(iv) *If $V_\Phi^S(I_a^b) < \infty$ then u is bounded on I_a^b.*

PROPOSITION 3.4 *Let $u, v : I_a^b \to \mathbb{R}$ be given functions. Then*

$$TV_\Phi^S(\alpha u + \beta v) \leqslant \alpha TV_\Phi^S + \beta TV_\Phi^S$$

for $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$.

Now, let us denote by P_Φ the functional defined on the set BV_Φ^S in the following way

$$P_\Phi(f) = \inf\{\epsilon > 0 : TV_\Phi^S\left(\frac{f}{\epsilon}\right) \leqslant 1\} \tag{2}$$

REMARK 3.5 In the case we take the Φ-sequence defined as follows

$$\Phi = \{\phi_{n,m} : \ \phi_{n,m}(t) = t^p, \ 1 < p < \infty \ \text{ for all } n, m = 1, 2, ...\}$$

then we can check that $P_\Phi(f) = \left(TV_\Phi^S(f)\right)^{1/p}$.

For further purposes we denote by $\hat{\Phi}$ the class of all φ-functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ begin continuous, convex and such that $\varphi(t) = 0$ if and only if $t = 0$.
Apart from this we denote by $\hat{\Phi}_0$ the subclass of the class $\hat{\Phi}$ containing these functions belonging to $\hat{\Phi}$ for while $\lim\limits_{t \to 0} \varphi(t)/t = 0$.

Our next result is contained in the following Lemma.

LEMMA 3.6 *Let $f \in BV_\Phi^S\left(I_a^b; \mathbb{R}\right)$ and $\Phi \in \hat{\Phi}$. Then f has the following properties:*

(a) *If $(t,s), (t',s') \in I_a^b$ then $|f(t,s) - f(t',s')| \leqslant 4\Phi_{n,m}^{-1}\left(\frac{1}{2}\right) P_\Phi(f)$.*

(b) *If $P_\Phi(f) > 0$ then $TV_\Phi^S(f/P_\Phi(f)) \leqslant 1$.*

(c) *Let $r > 0$. Then $TV_\Phi^S(f/r) \leqslant 1$ if and only if $P_\Phi(f) \leqslant r$.*

(d) *If $r > 0$ and $TV_\Phi^S(f/P_\Phi(f)) = 1$ then $P_\Phi(f) = r$.*

Proof (a) Let $f \in BV_\Phi^S(I_a^b;\mathbb{R})$. Then, for arbitrary (t,s), $(t',s') \in I_a^b$, we obtain:

$$\begin{aligned}
|f(t,s)-f(t',s')| &= |f(t,a_2)-f(t',a_2)+f(a_1,s)-f(a_1,s')+f(t',a_2)+f(t,s)- \\
&\quad -f(t',s)-f(t,a_2)+f(a_1,s')-f(t',s)-f(a_1,s)-f(t',s')| \\
&\leqslant \big|f(t,a_2)-f(t',a_2)\big| + \big|f(a_1,s)-f(a_1,s')\big| + \\
&\quad + \big|f(t',a_2)+f(t,s)-f(t',s)-f(t,a_2)\big| + \\
&\quad + \big|f(a_1,s')+f(t',s)-f(a_1,s)-f(t',s')\big|.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\Big|\frac{f(t,s)-f(t',s')}{4\lambda}\Big| &\leqslant \frac{1}{4}\big|(f/\lambda)(t,a_2)-(f/\lambda)(t',a_2)\big| + \\
&\quad \frac{1}{4}\big|(f/\lambda)(a_1,s)-(f/\lambda)(a_1,s')\big| + \\
&\quad + \frac{1}{4}\big|(f/\lambda)(t',a_2)+(f/\lambda)(t,s)-(f/\lambda)(t',s)-(f/\lambda)(t,a_2)\big| + \\
&\quad + \frac{1}{4}\big|(f/\lambda)(a_1,s')+(f/\lambda)(t',s)-(f/\lambda)(a_1,s)-(f/\lambda)(t',s')\big|.
\end{aligned}$$

Now, taking into account the fact that the functions $\phi_{n,m}$ are convex and increasing for all $n,m=1,2...$ and using the above obtained estimate, we derive that for any $\lambda > P_\Phi(f)$ the following inequality is satisfied:

$$\begin{aligned}
\phi_{n,m}\left(\Big|\frac{f(t,s)-f(t',s')}{4\lambda}\Big|\right) &\leqslant \frac{1}{4}\phi_{n,m}\big(\big|(f/\lambda)(t,a_2)-(f/\lambda)(t',a_2)\big|\big) + \\
&\quad + \frac{1}{4}\phi_{n,m}\big(\big|(f/\lambda)(a_1,s)-(f/\lambda)(a_1,s')\big|\big) + \\
&\quad + \frac{1}{4}\phi_{n,m}\big(\big|(f/\lambda)(t',a_2)+(f/\lambda)(t,s)-(f/\lambda)(t',s)-(f/\lambda)(t,a_2)\big|\big) + \\
&\quad + \frac{1}{4}\phi_{n,m}\big(\big|(f/\lambda)(a_1,s')+(f/\lambda)(t',s)-(f/\lambda)(a_1,s)-(f/\lambda)(t',s')\big|\big) \\
&\leqslant \frac{1}{4}V_{\Phi,I'}^S(f/\lambda)+\frac{1}{4}V_{\Phi,J'}^S(f/\lambda)+\frac{1}{4}V_\Phi^S(f/\lambda)+\frac{1}{4}V_\Phi^S(f/\lambda) \\
&\leqslant \frac{1}{4}V_{\Phi,I'}^S(f/\lambda)+\frac{1}{4}V_{\Phi,J'}^S(f/\lambda)+\frac{1}{2}V_\Phi^S(f/\lambda) \\
&\leqslant \frac{1}{2}V_{\Phi,I'}^S(f/\lambda)+\frac{1}{2}V_{\Phi,J'}^S(f/\lambda)+\frac{1}{2}V_\Phi^S(f/\lambda) \\
&= \frac{1}{2}TV_\Phi^S(f/\lambda) \leqslant \frac{1}{2}.
\end{aligned}$$

Hence, in view of the invertibility of the functions $\phi_{n,m}$ for $n,m=1,2,...$, we get

$$\left(\Big|\frac{f(t,s)-f(t',s')}{4\lambda}\Big|\right) \leqslant \phi_{n,m}^{-1}\left(\frac{1}{2}\right),$$

or, equivalently

$$\big|f(t,s)-f(t',s')\big| \leqslant 4\lambda\phi_{n,m}^{-1}\left(\frac{1}{2}\right).$$

Thus, assuming that $P_\Phi < \lambda$, we derive

$$\left|f(t,s) - f(t',s')\right| \leqslant 4\phi_{n,m}^{-1}\left(\frac{1}{2}\right) P_\Phi(f).$$

This completes the proof of the part (a).

The proof of (b) is an immediate consequence of the definition of the quantity $P_\Phi(f)$.

Next we proceed with the proof of (c). So, in order to prove the necessity observe that on the base of the definition we infer that if $TV_\Phi^S(f/r) \leqslant 1$ then $P_\Phi(f) \leqslant r$.

To prove the sufficiency suppose that $P_\Phi(f) = r$. Then, in view of the part (b) we have that $TV_\Phi^S(f/r) \leqslant 1$. Thus, we have only to show the implication

$$P_\Phi(f) < r \Rightarrow TV_\Phi^S(f/r) < 1. \tag{3}$$

In the case $P_\Phi(f) = 0$ we infer that f is constant and using Proposition 3.3(iii) we have that $TV_\Phi^S(f/r) = 0$.

Further, suppose that $P_\Phi(f) > 0$. Then, from $0 < p < \infty$ we get that $0 < \dfrac{P_\Phi(f)}{r} < 1$ for any $r > 0$. In view of the convexity of the functional TV_Φ^S (3.4) and Lemma 3.6 we deduce that

$$\begin{aligned} TV_\Phi^S(f/r) &= TV_\Phi^S\left(\frac{P_\Phi(f)}{r}\cdot\frac{f}{P_\Phi(f)}\right) \\ &\leqslant \frac{P_\Phi(f)}{r} TV_\Phi^S\left(\frac{f}{P_\Phi(f)}\right) \\ &\leqslant \frac{P_\Phi(f)}{r}\cdot 1 \\ &= \frac{P_\Phi(f)}{r} < 1 \end{aligned}$$

(d) For the proof of this part suppose that $TV_\Phi^S(f/r) = 1$. If $P_\Phi(f) > r$, then in view of part **(c)** we have that $TV_\Phi^S(f/r) > 1$, which is in contradiction with our assumption.

This yields that $P_\Phi(f) < r$. But now, in virtue of (2) we infer that $TV_\Phi^S(f/r) < 1$. Hence $P_\Phi(f) = r$.

This completes the proof of our Lemma. ■

Remark 3.7 From part **(a)** of Lemma 3.6 we deduced that each function $f \in BV_\Phi^S(I_a^b;\mathbb{R})$ is bounded. Moreover, the following estimate is satisfied

$$||f||_\infty = \sup\{|f(t,s)| : (t,s) \in I_a^b\} \leqslant |f(a)| + 4\phi_{n,m}^{-1}(\frac{1}{2})P_\Phi(f) \tag{4}$$

for all $n, m = 1, 2, ...$, where the symbol $\|f\|_\infty$ denotes the supremo norm i.e.

$$\|f\|_\infty = \sup\{|f(t,s)| : (t,s) \in I_a^b\}$$

In the sequel we are going to deal with the main result the this paper.

THEOREM 3.8 *If $\Phi \in \hat{\Phi}$ is convex then $\left(BV_{\Phi}^{S}(I_a^b;\mathbb{R}), ||f||_{\Phi}^{S}\right)$ is a Banach algebra, where the quantity $||f||_{\Phi}^{S}$ is defined by the formula*

$$||f||_{\Phi}^{S} = |f(a)| + P_{\Phi}(f)$$

for any $f \in BV_{\Phi}^{S}(I_a^b;\mathbb{R})$

PROOF First of all observe that the quantity $\|f\|_{\Phi}^{S}$ is well defined in view (2). Next, let us mention that in view of result proved in ([6]) we have that the set $BV_{\Phi}^{S}(I_a^b;\mathbb{R})$ forms a Banach space under the norm $\|f\|_{\Phi}^{S}$.
Thus, it is sufficient to show the property of this norm associeted with the operation of multiplication of functions.

We start with the proof of the inequality

$$P_{\Phi}(fg) \leqslant 2\left[P_{\Phi}(f)||g||_{\infty} + P_{\Phi}(g)||f||_{\infty}\right]. \tag{5}$$

Without loss of generality we can assume that $P_{\Phi}(f)$, $P_{\Phi}(g)$, $\|f\|_{\Phi}^{S}$ and $\|g\|_{\Phi}^{S}$ are distinct of zero.
Next, let us put $\lambda_1 = ||f||_{\infty}$, $\lambda_2 = ||g||_{\infty}$ and $\lambda = 2\left[\lambda_2 P_{\Phi}(f) + \lambda_1 P_{\Phi}(g)\right]$.
Moreover, denote

$$\alpha = 2\frac{\lambda_2 P_{\Phi}(f)}{\lambda}, \qquad \beta = 2\frac{\lambda_1 P_{\Phi}(g)}{\lambda}. \tag{6}$$

Obviously $\alpha + \beta = 1$.

Now, fix $x_2 \in J_1$. Then, for an arbitrary sequence $\{I_n\}$ of closed subintervals of the interval $I_1 = [a_1, b_1]$, we obtain:

$$\begin{aligned}
|\frac{fg(I_n,x_2)}{\lambda}| &= \left|\frac{1}{\lambda}[fg(b_n,x_2) - fg(a_n,x_2)]\right| \\
&= \left|\frac{1}{\lambda}[f(b_n,x_2)g(b_n,x_2) - f(a_n,x_2)g(a_n,x_2)]\right| \\
&= \left|\frac{1}{\lambda}[f(b_n,x_2)g(b_n,x_2) - f(a_n,x_2)g(b_n,x_2) + f(a_n,x_2)g(b_n,x_2) - \right. \\
&\qquad \left. -f(a_n,x_2)g(a_n,x_2)]\right| \\
&= \left|\frac{1}{\lambda}[(f(b_n,x_2) - f(a_n,x_2))g(b_n,x_2) + f(a_n,x_2)(g(b_n,x_2) - g(a_n,x_2))]\right| \\
&= \left|\frac{1}{\lambda}[\, f(I_n,x_2)g(b_n,x_2) + f(a_n,x_2)g(I_n,x_2)\,]\right| \\
&\leqslant \left|\frac{1}{\lambda}\right| \left[\left|f(I_n,x_2)\right|\left|g(b_n,x_2)\right| + \left|f(a_n,x_2)\right|\left|g(I_n,x_2)\right|\right] \\
&\leqslant \frac{1}{\lambda}\left[\left|f(I_n,x_2)\right| \|g\|_{\infty} + \|f\|_{\infty}\left|g(I_n,x_2)\right|\right] \\
&\leqslant 2\frac{\lambda_2 P_{\Phi}(f)}{\lambda}\left|\frac{f(I_n,x_2)}{P_{\Phi}(f)}\right| + 2\frac{\lambda_1 P_{\Phi}(g)}{\lambda}\left|\frac{g(I_n,x_2)}{P_{\Phi}(g)}\right|,
\end{aligned}$$

Further, keeping in mind that the function $\phi_{n,m}$, is increasing and convex for all $n, m = 1, 2, ...$, we get

$$\phi_{n,m}\left(|\frac{fg(I_n,x_2)}{\lambda}|\right) \leqslant \alpha\phi_{n,m}\left(|\frac{f(I_n,x_2)}{P_\Phi(f)}|\right) + \beta\phi_{n,m}\left(|\frac{g(I_n,x_2)}{P_\Phi(g)}|\right).$$

This yields

$$\begin{aligned} V_\Phi^S\left(\frac{fg}{\lambda}\right) &= \sup\left\{\sum_{n=1}^{\infty}\phi_{n,m}\left(|\frac{fg(I_n,x_2)}{\lambda}|\right) : \{I_n\}\right\} \\ &\leqslant \alpha V_{\Phi,I_1}^S\left(\frac{f}{P_\Phi(f)}\right) + \beta V_{\Phi,I_1}^S\left(\frac{g}{P_\Phi(g)}\right). \end{aligned} \tag{7}$$

Analogously, if we fix arbitrarily $x_1 \in I_1$ and take an arbitrary sequence $\{J_n\}$ of closed subintervals of the interval $J_1 = [c_1, d_1]$, we derive the following inequality

$$\begin{aligned} V_\Phi^S\left(\frac{fg}{\lambda}\right) &= \sup\left\{\sum_{n=1}^{\infty}\phi_{n,m}\left(|\frac{fg(x_1,J_m)}{\lambda}|\right) : \{J_m\}\right\} \\ &\leqslant \alpha V_{\Phi,J_1}^S\left(\frac{f}{P_\Phi(f)}\right) + \beta V_{\Phi,J_1}^S\left(\frac{g}{P_\Phi(g)}\right). \end{aligned} \tag{8}$$

In what follows, in order to simplify our consideration, we put:

$$A' = f(a') + f(b') - f(c') - f(d'), \tag{9}$$

$$B' = g(a') + g(b') - g(c') - g(d'). \tag{10}$$

Then, it is easily seen that following equality does hold:

$$\begin{aligned} |\frac{fg(I_n,J_m)}{\lambda}| &= |\frac{1}{\lambda}[fg(a_n,c_m) + uv(b_n,d_m) - fg(a_n,d_m) - uv(b_n,c_m)]| \\ &= |\frac{1}{\lambda}[f(a')g(a') + f(b')g(b') - f(c')g(c') - f(d')g(d')]| \\ &= \Big|\frac{1}{\lambda}\Big[\big(f(a') - f(d')\big)g(a') + f(d')\big(g(a') - g(d')\big) + \\ &\quad +\big(f(b') - f(c')\big)g(b') + f(c')\big(g(b') - g(c')\big)\Big]\Big| \\ &\leqslant |\frac{1}{\lambda}|\Big[\big(|f(a') - f(d')|\big)|g(a')| + |f(d')|\big(|g(a') - g(d')|\big) + \\ &\quad +\big(|f(b') - f(c')|\big)|g(b')| + |f(c')|\big(|g(b') - g(c')|\big)\Big]\Big| \end{aligned}$$

$$\begin{aligned} &\leqslant |\frac{1}{\lambda}|\Big[|A'|\,\|g\|_\infty + \|f\|_\infty|B'| + |A'|\,\|g\|_\infty + \|f\|_\infty|B'|\Big] \\ &= 2\frac{1}{\lambda}\Big[|A'|\,\lambda_2 + \lambda_1|B'|\Big] \\ &= 2\frac{\lambda_2 P_\Phi(f)}{\lambda}\left|\frac{A'}{P_\Phi(f)}\right| + 2\frac{\lambda_1 P_\Phi(g)}{\lambda}\left|\frac{B'}{P_\Phi(g)}\right|, \end{aligned}$$

This implies

$$\left|\frac{fg(I_n,J_m)}{\lambda}\right| \leqslant \alpha\left|\frac{A'}{P_\Phi(f)}\right| + \beta\left|\frac{B'}{P_\Phi(g)}\right| \tag{11}$$

Next, combining the above obtained inequality with the fact that $\phi_{n,m}$ is increasing and convex $(n,m=1,2,...)$, we obtain

$$\phi_{n,m}\left(\left|\frac{fg(I_n,J_m)}{\lambda}\right|\right) \leqslant \alpha\phi_{n,m}\left(\left|\frac{A'}{P_\Phi(f)}\right|\right) + \beta\phi_{n,m}\left(\left|\frac{B'}{P_\Phi(g)}\right|\right).$$

Joining this inequality with (7) and (8), we derive

$$V_\Phi^S\left(\frac{fg}{\lambda}\right) \leqslant \alpha V_\Phi^S\left(\frac{f}{P_\Phi(f)}\right) + \beta V_\Phi^S\left(\frac{g}{P_\Phi(g)}\right).$$

consequently, we have

$$\begin{aligned}
TV_\Phi^S\left(fg/\lambda\right) &= V_{\Phi,I'}^S\left(\frac{fg}{\lambda}\right) + V_{\Phi,J'}^S\left(\frac{fg}{\lambda}\right) + V_\Phi^S\left(\frac{fg}{\lambda}\right)\\
&\leqslant \alpha V_{\Phi,I'}^S\left(\frac{f}{P_\Phi(f)}\right) + \beta V_{\Phi,I'}^S\left(\frac{g}{P_\Phi(g)}\right) +\\
&\quad +\alpha V_{\Phi,J'}^S\left(\frac{f}{P_\Phi(f)}\right) + \beta V_{\Phi,J'}^S\left(\frac{g}{P_\Phi(g)}\right) +\\
&\quad +\alpha V_\Phi^S\left(\frac{f}{P_\Phi(f)}\right) + \beta V_\Phi^S\left(\frac{g}{P_\Phi(g)}\right)\\
&= \alpha\, TV_\Phi^S\left(\frac{f}{P_\Phi(f)}\right) + \beta\, TV_\Phi^S\left(\frac{g}{P_\Phi(g)}\right)\\
&\leqslant \alpha\cdot 1 + \beta\cdot 1 = 1.
\end{aligned}$$

Now, keeping in mind that $\lambda > 0$ and applying Lemma 3.6 (c), we get that $P_\Phi(fg) \leqslant \lambda$. This proves (3).
Finally, taking into account the definition of the norm and applying (3), we obtain

the following chain of calculations:

$$
\begin{aligned}
||fg||_{\Phi}^{S} &= |fg(a)| + P_{\Phi}(fg) \\
&\leqslant |fg(a)| + \lambda \\
&= |fg(a)| + 2\left[\lambda_2 P_{\Phi}(f) + \lambda_1 P_{\Phi}(g)\right] \\
&\leqslant |fg(a)| + 2\left[||g||_{\infty} P_{\Phi}(f) + ||f||_{\infty} P_{\Phi}(g)\right] \\
&\leqslant 2|f(a)||g(a)| + 2\left[|g(a)| + 4\phi_{n,m}^{-1}\left(\frac{1}{2}\right) P_{\Phi}(g)\right] P_{\Phi}(f) + \\
&\quad +2\left[|f(a)| + 4\phi_{n,m}^{-1}\left(\frac{1}{2}\right) P_{\Phi}(f)\right] P_{\Phi}(g) \\
&= 2|f(a)||g(a)| + 2|g(a)|P_{\Phi}(f) + 8wP_{\Phi}(f)P_{\Phi}(g) + 2|f(a)|P_{\Phi}(g) + \\
&\quad +8wP_{\Phi}(f)P_{\Phi}(g) \qquad \text{(where } w := \max\{\phi_{n,m}^{-1}\left(\frac{1}{2}\right), 1\}) \\
&= 2|f(a)||g(a)| + 2|g(a)|P_{\Phi}(f) + 16wP_{\Phi}(f)P_{\Phi}(g) + 2|f(a)|P_{\Phi}(g) \\
&\leqslant 16w|f(a)||g(a)| + 16w|g(a)|P_{\Phi}(f) + 16wP_{\Phi}(f)P_{\Phi}(g) + 16w|f(a)|P_{\Phi}(g) \\
&= 16w\left[\ |f(a)||g(a)| + |g(a)|P_{\Phi}(f) + P_{\Phi}(f)P_{\Phi}(g) + |f(a)|P_{\Phi}(g)\ \right] \\
&= K\left[\ (|f(a)| + P_{\Phi}(f))|g(a)| + (P_{\Phi}(f) + |f(a)|)P_{\Phi}(g)\ \right] \qquad \text{(with } K = 16w) \\
&= K\left[\ ||f||_{\Phi}^{S}|g(a)| + ||f||_{\Phi}^{S}P_{\Phi}(g)\ \right] \\
&= K||f||_{\Phi}^{S}\ ||g||_{\Phi}^{S}. \qquad (12)
\end{aligned}
$$

Thus the proof of our theorem is complete. ■

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Tomás Ereú
Universidad Nacional Abierta
Centro Local Lara (Barquisimeto), Venezuela
E-mail: tomasereu@gmail.com

Nelson Merentes
Universidad Central de Venezuela
Escuela de Matemáticas, Caracas-Venezuela
E-mail: nmerucv@gmail.com

José L. Sánchez
Universidad Central de Venezuela
Escuela de Matemáticas, Caracas-Venezuela
E-mail: jose.sanchez@ciens.ucv.ve

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