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On some forms of quasi-uniform convergence of transfinite sequence of multifunctions

Abstract. In this paper we introduce various forms of convergence of transfinite sequences of multifunctions with values in a quasi-uniform space. We also study some weak types of continuity for such multifunctions. Any such sequence of multifunctions generates the sequence of the sets of weak types of continuity points and the sequence of various types of cluster sets of members of such sequence. We study the connection between convergence of a transfinite sequences of multifunctions and convergence of the corresponding sequences of the sets of the weak continuity points and the sequences of cluster sets. Some of the presented results concern of general nets of multifunctions.

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1. Introduction and preliminaries.

For a subset A of a topological space (X, π) we denote by $Cl(A)$ and $Int(A)$ the closure and the interior of A, respectively. By a multifunction $F : X \to Y$ we mean a correspondence which assigns to each element x of X a nonempty subset F(x) of Y. The upper and lower inverse images of a set $B \subset Y$ under F are defined by $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$, respectively. For any $A \subset X$ it's image under F is the set $F(A) = \bigcup\{F(x) \subset Y : x \in A\}$. We first recall the following well known types of continuity for multifunctions.

Definition 1.1 A multifunction $F : (X, \pi) \to (Y, \tau)$ is said to be

(a) *u.s.c.* (resp. *l.s.c.*) at a point $x \in X$ if for each subset $V \in \tau$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Int(F^+(V))$ (resp. $x \in Int(F^-(V))$). The set of all such points will be denoted by $C_u(F)$ (resp. $C_l(F)$), [15], [33];

(b) *u.α.c.* (resp. *l.α.c.*) at a point $x \in X$ if for each subset $V \in \tau$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Int(Cl(Int(F^+(V))))$ (resp. $x \in Int(Cl(Int(F^-(V)))))$. The set of all such points will be denoted by $\alpha C_u(F)$ (resp. $\alpha C_l(F)$), [30];

(c) *u.p.c.* (resp. *l.p.c.*) at a point $x \in X$ if for each subset $V \in \tau$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Int(Cl(F^+(V)))$ (resp. $x \in Int(Cl(F^-(V))))$. The set of all such points will be denoted by $pC_u(F)$ (resp. $pC_l(F)$), [34];

(d) *u.q.c.* (resp. *l.q.c.*) at a point $x \in X$ if for each subset $V \in \tau$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Cl(Int(F^+(V)))$ (resp. $x \in Cl(Int(F^-(V))))$. The set of all such points will be denoted by $qC_u(F)$ (resp. $qC_l(F)$), [35];

(e) *u.β.c.* (resp. *l.β.c.*) at a point $x \in X$ if for each subset $V \in \tau$ such that $x \in F^+(V)$ (resp. $x \in F^-(V)$), $x \in Cl(Int(Cl(F^+(V))))$ (resp. $x \in Cl(Int(Cl(F^-(V)))))$. The set of all such points will be denoted by $\beta C_u(F)$ (resp. $\beta C_l(F)$), [36].

Of course, if a single-valued function $f : (X, \pi) \to (Y, \tau)$ is treated as a multifunction F given by $F(x) = \{f(x)\}$ then, in the above definition, the conditions in (a) (resp. (b), (c), (d), (e)) are equivalent to the continuity (resp. $\alpha-$continuity [23], pre-continuity [24], quasi continuity [20], [13], $\beta-$continuity [26]) of f at x; we will write this condition as $x \in C(f)$ (resp. $x \in \alpha C(f), x \in pC(f), x \in qC(f)$, $x \in \beta C(f)$).

It is well known that a multifunction $F : (X, \pi) \to (Y, \tau)$ is *u.s.c.* (resp. *l.s.c.*) if and only if the function $F : (X, \pi) \to (P(Y), \tau^V)$ (resp. $F : (X, \pi) \to (P(Y), \tau_V)$) is continuous , where τ^V (resp. τ_V) is the upper (resp. lower) Vietoris topology induced by τ on the family of all nonempty subsets of Y (see [14], [25]).

Analogously, a multifunction $F : (X, \pi) \to (Y, \tau)$ is *u.α.c.* (resp. *u.p.c*, *u.q.c.*, *u.β.c.*) if and only if the finction $F : (X, \pi) \to (P(Y), \tau^V)$ is $\alpha-$continuous (resp. pre-continuous, quasi-continuous, $\beta-$continuous) and, $F : (X, \pi) \to (Y, \tau)$ is *l.α.c.* (resp. *l.p.c*, *l.q.c.*, *l.β.c.*) if and only if the function $F : (X, \pi) \to (P(Y), \tau_V)$ is $\alpha-$continuous (resp. pre-continuous, quasi-continuous, $\beta-$continuous).

In this paper, we consider multifunctions with values in a quasi-uniform space. Our basic references for quasi-uniform spaces are [29], [32], [9], [16]. A quasi-uniformity on Y is a filter $\mathcal{V}$ on $Y \times Y$ which satisfies:

(a) $\Delta \subset V$ for all $V \in \mathcal{V}$ and

(b) given $V \in \mathcal{V}$ there exists $W \in \mathcal{V}$ such that $W \circ W \subset V$,

where $\Delta = \{(x, y) \in Y \times Y : y \in Y\}$ and

$W \circ W = \{(x, y) \in Y \times Y : (x, z), (z, y) \in W \ for \ some \ z \in Y\}$.

The pair $(Y, \mathcal{V})$ is called a quasi-uniform space.

A family $\mathcal{B} \subset \mathcal{V}$ is base for $\mathcal{V}$ if each member of $\mathcal{V}$ contains a member of $\mathcal{B}$. If $\mathcal{V}$ is a quasi-uniformity, then so its conjugate $\mathcal{V}^{-1} = \{V^{-1} : V \in \mathcal{V}\}$, where $V^{-1} = \{(x, y) \in Y \times Y : (y, x) \in V\}$.

The collection of sets of the form $V^s = V \cap V^{-1}, V \in \mathcal{V}$, is a subbase for a uniformity $\mathcal{V}^s$.

Every quasi-uniformity $\mathcal{V}$ on Y generates the topology $t(\mathcal{V})$ given by the neighborhood base $\{V(y) : V \in \mathcal{V}\}$ for each point $y \in Y$, where $V(y) = \{z \in Y : (y, z) \in V\}$.

Throughout the present paper, the space $(Y, \mathcal{V})$ will always mean the topological space (Y, τ), where $\tau = t\,(\mathcal{V})$. So, we say that a multifunction $F : (X, \pi) \to (Y, \mathcal{V})$ is $u.s.c.$ (resp. $l.s.c., u.\alpha.c., l.\alpha.c., u.p.c., l.p.c., u.q.c., l.q.c., u.\beta.c., l.\beta.c.$) if it has the corresponding property with respect to $t(\mathcal{V})$.

According to [21], [4] and [17], for a quasi-uniform space $(Y, \mathcal{V})$ we define

$$V^+ = \{(A, B) \in P(Y) \times P(Y) : B \subset V(A)\} \text{ and}$$

$$V^- = \big\{(A, B) \in P(Y) \times P(Y) : A \subset V^{-1}(B)\big\}$$

for all $V \in \mathcal{V}$, where $V(A) = \{y \in Y : (p, y) \in V \; for\ some\ p \in A\}$.

The family $\{V^+ : V \in \mathcal{V}\}$ (resp. $\{V^- : V \in \mathcal{V}\}$) is a base for the upper Hausdorff quasi-uniformity $\mathcal{V}_u$ (resp. lower Hausdorff quasi-uniformity $\mathcal{V}_l$) on $P(Y)$. The sets $V^+ \cap V^-, V \in \mathcal{V}$ form a base for the Hausdorff quasi-uniformity $\mathcal{V}_H$ on $P(Y)$.

So, every quasi-uniformity $\mathcal{V}$ on Y generates the following topologies on the family $P(Y)$ of all nonempty subsets of Y: $t(\mathcal{V}_u)$, $t((\mathcal{V}_u)^{-1})$, $t((\mathcal{V}_u)^s)$, $t(\mathcal{V}_1)$, $t((\mathcal{V}_1)^{-1})$, $t((\mathcal{V}_1)^s)$, $t(\mathcal{V}_H)$, $t((\mathcal{V}_H)^{-1})$, $t((\mathcal{V}_H)^s)$.

To simplify the description, we will use the following abbreviations:

$t(\mathcal{V}_u) = T_u$, $t((\mathcal{V}_u)^{-1}) = T_u^{-1}$, $t((\mathcal{V}_u)^s) = T_u^s$, $t(\mathcal{V}_l) = T_l$, $t((\mathcal{V}_l)^{-1}) = T_l^{-1}$, $t((\mathcal{V}_l)^s) = T_l^s$, $t(\mathcal{V}_H) = T_H$, $t((\mathcal{V}_H)^{-1}) = T_H^{-1}$, $t((\mathcal{V}_H)^s) = T_H^s$.

Given a topology T on P(Y), we denote by $C(F,T)$ (resp. $\alpha C(F,T)$, $pC(F,T)$, $qC(F,T)$, $\beta C(F,T)$) the set of all points at which the function $F : (X, \pi) \to (P(Y), T)$ is continuous (resp. $\alpha-continuous$, $pre-continuous$, $quasi-continuous$, $\beta - continuous$).

In particular, we have $C_l(F) = C(F, \tau_V)$, $\alpha C_l(F) = \alpha C(F, \tau_V)$, $pC_l(F) = pC(F, \tau_V)$, $qC_l(F) = qC(F, \tau_V)$, $\beta C_l(F) = \beta C(F, \tau_V)$ and, $C_u(F) = C(F, \tau_V)$, $\alpha C_u(F) = \alpha C(F, \tau_V)$, $pC_u(F) = pC(F, \tau_V)$, $qC_u(F) = qC(F, \tau_V)$, $\beta C_u(F) = \beta C(F, \tau_V)$.

We have the following connections between different types of continuity.

Remark 1.2 For any multifunction $F : (X, \pi) \to (Y, \mathcal{V})$, the following hold:

(a) $C(F, T_u) \cap C(F, T_u^{-1}) = C(F, T_u^s)$ and $C(F, T_l) \cap C(F, T_l^{-1}) = C(F, T_l^s)$, $\alpha C(F, T_u) \cap \alpha C(F, T_u^{-1}) = \alpha C(F, T_u^s)$ and $\alpha C(F, T_l) \cap \alpha C(F, T_l^{-1}) = \alpha C(F, T_l^s)$;

(b) $[\alpha C(F, T_u) \cap pC(F, T_u^{-1})] \cup [pC(F, T_u) \cap \alpha C(F, T_u^{-1})] \subset pC(F, T_u^s)$, $[\alpha C(F, T_u) \cap qC(F, T_u^{-1})] \cup [qC(F, T_u) \cap \alpha C(F, T_u^{-1})] \subset qC(F, T_u^s)$ and $[pC(F, T_u) \cap qC(F, T_u^{-1})] \cup [qC(F, T_u) \cap pC(F, T_u^{-1})] \subset \beta C(F, T_u^s)$;

(c) $C(F, T_u) \cap C(F, T_u^{-1}) = C(F, T_u^s)$ and $C(F, T_l) \cap C(F, T_l^{-1}) = C(F, T_l^s)$, $\alpha C(F, T_u) \cap \alpha C(F, T_u^{-1}) = \alpha C(F, T_u^s)$ and $\alpha C(F, T_l) \cap \alpha C(F, T_l^{-1}) = \alpha C(F, T_l^s)$;

(d) $C(F,T_H)\cap C(F,T_H^{-1})=C(F,T_H^s)$ and $\alpha C(F,T_H)\cap\alpha C(F,T_H^{-1})=\alpha C(F,T_H^s)$;

(e) $C(F,T_u)\cap C(F,T_l)=C(F,T_H)$ and $\alpha C(F,T_u)\cap\alpha C(F,T_l)=\alpha C(F,T_H)$;

(f) $[\alpha C(F,T_H)\cap pC(F,T_H^{-1})]\cup[pC(F,T_H)\cap\alpha C(F,T_H^{-1})]\subset pC(F,T_H^s), [\alpha C(F,T_H)$

$\cap qC(F,T_H^{-1})]\cup[qC(F,T_H)\cap\alpha C(F,T_H^{-1})]\subset qC(F,T_H^s)$ and $[pC(F,T_H)\cap qC(F,T_H^{-1})]\cup[qC(F,T_H)\cap pC(F,T_H^{-1})]\subset\beta C(F,T_H^s)$;

(g) $[\alpha C(F,T_H)\cap pC(F,T_H)]\cup[pC(F,T_H)\cap\alpha C(F,T_H)]\subset pC(F,T_H), [\alpha C(F,T_H)\cap qC(F,T_H)]\cup[qC(F,T_H)\cap\alpha C(F,T_H)]\subset qC(F,T_H)$ and $[pC(F,T_H)\cap qC(F,T_H)]\cup[qC(F,T_H)\cap pC(F,T_H)]\subset\beta C(F,T_H)$.

We remark that the continuity of multifunction $F:(X,\pi)\to(Y,\mathcal{V})$ with respect to T_l has been studied in [41] under the name H-lsc.

Throughout this paper we use the following notation:

$\mathcal{V}-C_l(F)=C(F,T_l)$, $\mathcal{V}-C_u(F)=C(F,T_u)$, $\mathcal{V}-\alpha C_l(F)=\alpha C(F,T_l)$, $\mathcal{V}-\alpha C_u(F)=\alpha C(F,T_u)$, $\mathcal{V}-pC_l(F)=pC(F,T_l)$, $\mathcal{V}-pC_u(F)=pC(F,T_u)$, $\mathcal{V}-qC_l(F)=qC(F,T_l)$, $\mathcal{V}-qC_u(F)=qC(F,T_u)$, $\mathcal{V}-\beta C_l(F)=\beta C(F,T_l)$, $\mathcal{V}-\beta C_u(F)=\beta C(F,T_u)$.

It is well known that $T_u\subset\tau^V$ and $\tau_V\subset T_l$ [4]. So we have

Remark 1.3 The following hold for any multifunction $F:(X,\pi)\to(Y,\mathcal{V})$:

(a) $C_u(F)\subset\mathcal{V}-C_u(F),\alpha C_u(F)\subset\mathcal{V}-\alpha C_u(F),pC_u(F)\subset\mathcal{V}-pC_u(F),qC_u(F)\subset\mathcal{V}-qC_u(F),\beta C_u(F)\subset\mathcal{V}-\beta C_u(F)$, and

(b) $\mathcal{V}-C_l(F)\subset C_l(F),\mathcal{V}-\alpha C_l(F)\subset\alpha C_l(F),\mathcal{V}-pC_l(F)\subset pC_l(F),\mathcal{V}-qC_l(F)\subset qC_l(F),\mathcal{V}-\beta C_l(F)\subset\beta C_l(F)$.

For a given net $\{A_\sigma:\sigma\in\Sigma\}$ of subsets A_σ of a topological space X we denote by $\liminf A_\sigma$ and $\limsup A_\sigma$ the sets $\bigcup\{\bigcap\{A_\sigma:\sigma\geqslant\delta\}:\delta\in\Sigma\}$ and $\bigcap\{\bigcup\{A_\sigma:\sigma\geqslant\delta\}:\delta\in\Sigma\}$, respectively. We say that $\{A_\sigma:\sigma\in\Sigma\}$ converges to $A\subset X$, denoted by $\lim A_\sigma=A$, if $A=\liminf A_\sigma=\limsup A_\sigma$. Furthermore, a point $x\in X$ is called a limit point (resp. cluster point) of $\{A_\sigma:\sigma\in\Sigma\}$, denoted by $x\in LiA_\sigma$ (resp. $x\in LsA_\sigma$) if each neighbourhood of x meets $\{A_\sigma:\sigma\in\Sigma\}$ eventually (resp. frequently). We say that $\{A_\sigma:\sigma\in\Sigma\}$ topologically convergent to A, denoted by $A=LtA_\sigma$, if $A=LiA_\sigma=LsA_\sigma$, [28], [15].

A net $\{f_\sigma:\sigma\in\Sigma\}$ of functions f_σ from a topological space (X,π) to a metric space (X,d) is called quasi-uniformly convergent to $f:(X,\pi)\to(Y,d)$ at $x\in X$, if for each $\epsilon>0$ there exists $\sigma_0\in\Sigma$ such that for each $\sigma\geqslant\sigma_0$ there is a neighbourhood U of x such that $d(f_\sigma(p),f(p))<\epsilon$ for all $p\in U$, [37].

Recently, the following types of cluster sets for multifunctions were studied in [39]:

DEFINITION 1.4 Given a multifunction $F : (X, \pi) \to (Y, \tau)$ and $x \in X$, we define $u.\alpha.C(F, x)$ (resp. $u.q.C(F, x), u.p.C(F, x), u.\beta.C(F, x), u.C(F, x), l.\alpha.C(F, x)$, $l.q.C(F, x), l.p.C(F, x), l.\beta.C(F, x), l.C(F, x)$) to be the set of all points $y \in Y$ such that
$x \in Int(Cl(Int(F^+(W))))$ (resp. $x \in Cl(Int(F^+(W))), x \in Int(Cl(F^+(W)))$,
$x \in Cl(Int(Cl(F^+(W)))), x \in Cl(F^+(W)), x \in Int(Cl(Int(F^-(W))))$,
$x \in Cl(Int(F^-(W))), x \in Int(Cl(F^-(W))), x \in Cl(Int(Cl(F^-(W))))$,
$x \in Cl(F^-(W))$) for any open set $W \subset Y$ with $y \in W$.

It is easy to see that $l.C(F, x)$ is equal to the classical type of cluster set introduced by Hrycay [11], in the case of topologies which fulfill the T_1-separation axiom. Furthermore, if a single-valued function $f : (X, \pi) \to (Y, \tau)$ is treated as a multifunction $F : (X, \pi) \to (Y, \tau)$ given by $F(x) = \{f(x)\}$, then $u.C(F, x) = l.C(F, x) = C(f, x)$ and $u.q.C(F, x) = l.q.C(F, x) = q.C(f, x)$, where $C(f, x)$ (resp. $q.C(f, x)$ was investigated in [2] and [42] (resp. in [22] and [40]).

The following two results are known and illustrated the connection between convergence of a transfinite sequence $\{f_\xi : \xi < \omega_1\}$ of functions and convergence of the corresponding sequences $\{C(f_\xi) : \xi < \omega_1\}$ and $\{C(f_\xi, x) : \xi < \omega_1\}$ of the sets of the continuity points and cluster sets, respectively, where ω_1 denotes the smallest uncountable ordinal number.

THEOREM 1.5 ([8, THEOREM 3.4]) *For any transfinite sequence $\{f_\xi : \xi < \omega_1\}$ of functions from a topological space (X, π) to a metric space (Y, d) which is quasi-uniformly convregent to $f : (X, \pi) \to (Y, d)$ at any $x \in X$, the following hold:*

(a) $C(f) = \lim C(f_\xi)$;

(b) $C(f, x) = \lim C(f_\xi, x) = LtC(f_\xi, x)$ *for all* $x \in X$.

In this paper we will consider this type of connections in which $\{F_\xi : \xi < \omega_1\}$ is a transfinite sequence of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ with countable base.

Given a pair (F, H) of multifunctions from (X, π) to $(Y, \mathcal{V})$ and $V \in \mathcal{V}$, for notational convenience let us denote
$D_u(F, H, V) = \{p \in X : (F(p), H(p)) \in V^+\}$ and
$D_l(F, H, V) = \{p \in X : (F(p), H(p)) \in V^-\}$.

DEFINITION 1.6 A net $\{F_\sigma : \sigma \in \Sigma\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ is said to be

(a) upper (resp. lower) $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$, if $x \in \liminf D_u(F, F_\sigma, V)$ (resp. $x \in \liminf D_l(F, F_\sigma, V)$) for all $V \in \mathcal{V}$;

(b) inversely upper (resp. inversely lower) $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$, if $x \in \liminf D_u(F_\sigma, F, V)$ (resp. $x \in \liminf D_l(F_\sigma, F, V)$) for all $V \in \mathcal{V}$;

(c) Hausdorff (resp. inversely Hausdorff) $\mathcal{V}$-convergent to a multifunction F at a point $x \in X$, if it is both upper and lower (resp. inversely upper and inversely lower) $\mathcal{V}$-convergent to F at a point x;

(d) strictly upper (resp. lower, Hausdorff) $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$, if it is both upper and inversely upper (resp. lower and inversely lower, Hausdorff and inversely Hausdorff) $\mathcal{V}$-convergent to F at x.

REMARK 1.7 Let $\{F_\sigma : \sigma \in \Sigma\}$ be a net of multifunctions from X to Y and let T be a topology on P(Y). We say that $\{F_\sigma : \sigma \in \Sigma\}$ is T-convergent at $x \in X$ to a multifunction $F : X \to Y$ if the net $\{F_\sigma(x) : \sigma \in \Sigma\}$ is T-convergent to F(x), [10, p.364, 352]. It is easy to see that the upper (or lower) (resp. inversely upper (or lower), strictly upper (or lower), Hausdorff, inversely Hausdorff, strictly Hausdorff) $\mathcal{V}$-convergence is equivalent to the T_u-convergence (or T_l-convergence) (resp. T_u^{-1}-convergence, (or T_l^{-1}-convergence), T_u^s-convergence (or T_l^s-convergence), T_H-convergence, T_H^{-1}-convergence, T_H^s-convergence).

In [5], the Hausdorff $\mathcal{V}$-convergence was called quasiuniformly convergence. Let us remark also that in [19], the lower (resp. upper, Hausdorff) $\mathcal{V}$-convergence was treated as some type of the bornological convergence, namely S^--convergence (resp. S^+-convergence, $\hat{S}$-convergence) in the case when $S = \{X\}$.

REMARK 1.8 It is easy to prove that the lower (resp. inversely lower) $\mathcal{V}$-convergence of $\{F_\sigma : \sigma \in \Sigma\}$ to F at x implies that Cl(F(x))$\subset LiF_\sigma(x)$ (resp. $LsF_\sigma(x) \subset Cl(F(x))$). So, if F has closed values, then the strictly lower $\mathcal{V}$-convergence of $\{F_\sigma : \sigma \in \Sigma\}$ to F at a point x implies that $LtF_\sigma(x) = F(x)$, i.e. the pointwise convergence at x, [3], [12].

DEFINITION 1.9 A net $\{F_\sigma : \sigma \in \Sigma\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ is said to be

(a) upper (resp. lower) quasi-uniformly $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$, if $x \in \liminf Int(D_u(F, F_\sigma, V))$ (resp.$x \in \liminf Int(D_l(F, F_\sigma, V))$) for all $V \in \mathcal{V}$;

(b) inversely upper (resp. inversely lower) quasi-uniformly $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$, if $x \in \liminf Int(D_u(F_\sigma, F, V))$ (resp. $x \in \liminf Int(D_l(F_\sigma, F, V))$) for all $V \in \mathcal{V}$;

(c) strictly upper (resp. lower) quasi-uniformly $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$, if it is both upper and inversely upper (resp. lower and inversely lower) quasi-uniformly $\mathcal{V}$-convergent to F at x.

(d) Hausdorff (resp. inversely Hausdorff) quasi-uniformly $\mathcal{V}$-convergent to a multifunction F at a point $x \in X$, if it is both upper and lower (resp. inversely upper and inversely lower) quasi-uniformly $\mathcal{V}$-convergent to F at a point x;

(e) strictly Hausdorff quasi-uniformly $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$, if it is both Hausdorff and inversely Hausdorff quasi-uniformly $\mathcal{V}$-convergent to F at x;

(f) α-upper (resp. α-lower, inversely α-upper, inversely α-lower) quasi-uniformly $\mathcal{V}$-convergent to a multifunction $F : X \to Y$ at a point $x \in X$, if
$x \in \liminf Int(Cl(Int(D_u(F, F_\sigma, V))))$
(resp. $x \in \liminf Int(Cl(Int(D_l(F, F_\sigma, V))))$,
$x \in \liminf Int(Cl(Int(D_u(F_\sigma, F, V)))), x \in \liminf Int(Cl(Int(D_l(F_\sigma, F, V)))))$
for all $V \in \mathcal{V}$.

Analogously as in the parts (c), (d) and (e) we understand the notions of strictly α-upper, α-lower, α-Hausdorff, inversely α-Hausdorff and strictly α-Hausdorff quasi-uniformly $\mathcal{V}$-convergence.

Let us remark that in [6] and [7], the Hausdorff quasi-uniformly $\mathcal{V}$-convergence was called almost quasiuniformly convergence.

2. Main results.

THEOREM 2.1 *If a net of multifunctions $F_\sigma : X \to Y, \sigma \in \Sigma$, is inversely lower (resp. inversely upper) quasi-uniformly $\mathcal{V}-$convergent to F at $x \in X$, then*

(a) *$Lsl.C(F_\sigma, x) \subset l.C(F, x)$ (resp. $Lsu.C(F_\sigma, x) \subset u.C(F, x)$). If, additionally, this net is lower (resp. upper) $\mathcal{V}-$convergent to F and has the above convergence property at any point $x \in X$, then*

(b) *$\limsup \mathcal{V} - C_l(F_\sigma) \subset \mathcal{V} - C_l(F)$ (resp. $\limsup \mathcal{V} - C_u(F_\sigma) \subset \mathcal{V} - C_u(F)$);*

(c) *$\limsup C_l(F_\sigma) \subset C_l(F)$.*

PROOF (a) : Assume that $y \in Lsl.C(F_\sigma, x)$ (resp. $y \in Lsu.C(F_\sigma, x)$). Let O and U be open sets containing y and x respectively, and let us take $W, V \in \mathcal{V}$ such that $W \circ W \subset V$ and $V(y) \subset O$. The type of convergence implies the existence of a $\gamma \in \Sigma$ such that $x \in Int(D_l(F_\sigma, F, W)$ (resp. $x \in Int(D_u(F_\sigma, F, W)$ for every $\sigma \in \Sigma$ with $\sigma \geqslant \gamma$. Furthermore, bacause of our assumption we have $y \in Cl(\bigcup\{l.C(F_\sigma, x) : \sigma \geqslant \gamma\})$ (resp. $y \in Cl(\bigcup\{u.C(F_\sigma, x) : \sigma \geqslant \gamma\})$, therefore $W(y) \cap l.C(F_\delta, x) \neq \emptyset$ (resp. $W(y) \cap u.C(F_\delta, x) \neq \emptyset$) for some $\delta \in \Sigma$ with $\delta \geqslant \gamma$. So, $x \in Cl(F_\delta^-(W(y)))$ (resp. $x \in Cl(F_\delta^+(W(y))))$ and consequently, there exists a point $p \in U$ such that $F_\delta(p) \cap W(y) \neq \emptyset$) and $(F_\delta(p), F(p)) \in W^-$ (resp. $F_\delta(p) \subset W(y)$ and $(F_\delta(p), F(p)) \in W^+$). As a result, there exist points $a \in F_\delta(p)$ and $b \in F(p)$ such that $(y, a) \in W$ and $(a, b) \in W$ (resp. for every $a \in F(p)$ there exists $b \in F_\delta(p)$ such that $(b, a) \in W$ and $(y, b) \in W$). Thus $F(p) \cap V(y) \neq \emptyset$ (resp. $F(p) \subset V(y)$) which proves that $x \in Cl(F^-(O))$ (resp. $x \in Cl(F^+(O))$, so that $y \in l.C(F, x)$ (resp. $y \in u.C(F, x)$) and completes the proof of (a).

(b) : Assume that $x \in \limsup \mathcal{V} - C_l(F_\sigma)$ (resp. $x \in \limsup \mathcal{V} - C_u(F_\sigma)$) and let $\mathcal{M} \in T_l$ (resp. $\mathcal{M} \in T_u$) be such that $F(x) \in \mathcal{M}$. Then, we can find $V \in \mathcal{V}$ satisfying $V^-(F(x)) \subset \mathcal{M}$ resp. $V^+(F(x)) \subset \mathcal{M}$). Let us take $W \in \mathcal{V}$ such that $W \circ W \circ W \subset V$. From the inversely lower (resp. upper) quasi-uniformly $\mathcal{V}$-convergence and from the lower (resp. upper) $\mathcal{V}-$convergence at x there exists $\gamma \in \Sigma$ such that

$x \in Int(D_l(F_\sigma, F, W)$ (resp. $x \in Int(D_u(F_\sigma, F, W)))$ (1)

and $F(x) \subset W^{-1}(F_\sigma(x))$ (resp. $F_\sigma(x) \subset W(F(x)))$

for each $\sigma \in \Sigma$ with $\sigma \geqslant \gamma$. (2)

Furthermore, by the assumption there exists $\delta \in \Sigma$ with $\delta \geqslant \gamma$ such that $x \in \mathcal{V}-C_l(F_\delta)$ (resp. $x \in \mathcal{V}-C_u(F_\delta)$). This implies the existence of an open neighbourhood U od x such that $F_\delta(p) \in W^-(F_\delta(x))$ (resp. $F_\delta(p) \in W^+(F_\delta(x))$ for each $p \in U$. Note that (1) implies that $x \in U_1 = U \cap Int(D_l(F_\delta, F, W)$ (resp. $x \in U_2 = U \cap Int(D_u(F_\delta, F, W))$. It will suffice to prove that $F(z) \in V^-(F(x))$ (resp. $F(z) \in V^+(F(x)))$ for each $z \in U_1$ (resp. $z \in U_2$). To do it let us take $z \in U_1$ (resp. $z \in U_2$). Then $(F_\delta(x), F_\delta(z)) \in W^-$ and $(F_\delta(z), F(z)) \in W^-$ (resp. $(F_\delta(x), F_\delta(z)) \in W^+$ and $(F_\delta(z), F(z)) \in W^+$) or equivalently $F_\delta(x) \subset W^{-1}(F_\delta(z))$ and $F_\delta(z) \subset W^{-1}(F(z))$ (resp. $F_\delta(z) \subset W(F_\delta(x))$ and $F(z) \subset W(F_\delta(z)))$. So, for each $a \in F(x)$ (resp. $a \in F(z)$) there exist points $s \in F_\delta(x)$ by (2), $r \in F_\delta(z)$ and $c \in F(z)$ such that $(a, s) \in W$, $(s, r) \in W$ and $(r, c) \in W$ (resp. $s \in F_\delta(z)$, $r \in F_\delta(x)$ and $c \in F(x)$ by (2) such that $(s, a) \in W$, $(r, s) \in W$ and $(c, r) \in W$). This gives $a \in V^{-1}(F(z))$ (resp.$a \in V(F(x)))$ and proves that $F(z) \in V^-(F(x))$ (resp. $F(z) \in V^+(F(x)))$.

(c) : Assume that $x \in \limsup C_l(F_\sigma)$ and let G be an open subset of Y such that $F(x) \cap G \neq \emptyset$. Let us take $V \in \mathcal{V}$ and $W \in \mathcal{V}$ such that $W \circ W \subset V$ and $V(y) \subset G$ for some $y \in F(x)$. In the same way as in the above part of our proof we choose an $\gamma \in \Sigma$ such that the statements (1) and (2) hold and, by the assumption we have $x \in U = Int(D_l(F_\delta, F, W) \cap Int(F_\delta^-(W(y))$ for some $\delta \in \Sigma$ with $\delta \geqslant \gamma$. We will prove that $U \subset F^-(G)$. If $z \in U$, then $F_\delta(z) \subset W^{-1}(F(z))$ and $F_\delta(z) \cap W(y) \neq \emptyset$. Therefore there exist points $s \in F_\delta(z)$ and $r \in F(z)$ such that $(y, s) \in W$ and $(s, r) \in W$. So $r \in V(y)$ and consequently, $z \in F^-(G)$. This proves that $x \in C_l(F)$. ∎

In [31] Njastad introduced the notion of α-sets in a topological space (X, π), and proved that the collection π^α of all α-sets in (X, π) is a topology on X finer than π. The interior of a subset A of X with respect to π^α can be expressed as $\alpha - Int(A) = A \cap Int(Cl(Int(A)))$ [1]. Therefore we have the following remark.

Remark 2.2 For every multifunctions $F, F_\sigma : X \to Y, \sigma \in \Sigma$, the following hold:

(a) the upper (resp. lower) semi continuity of $F : (X, \pi^\alpha) \to (Y, \tau)$ is equivalent to the upper (resp. lower) $\alpha-$continuity of $F : (X, \pi) \to (Y, \tau)$;

(b) the set $\mathcal{V} - C_u(F)$ (resp. $\mathcal{V} - C_l(F)$) for $F : (X, \pi^\alpha) \to (Y, \mathcal{V})$ is equal to the set $\mathcal{V} - \alpha C_u(F)$ (resp. $\mathcal{V} - \alpha C_l(F)$) for $F : (X, \pi) \to (Y, \mathcal{V})$;

(c) the net of multifunctions $F_\sigma : (X, \pi^\alpha) \to (Y, \mathcal{V})$ is strictly lower (resp. upper) quasi-uniformly $\mathcal{V}-$convergent to $F : (X, \pi^\alpha) \to (Y, \mathcal{V})$ at $x \in X$ if and only if the net of multifunctions $F_\sigma : (X,\pi) \to (Y,\mathcal{V})$ is both strictly upper (resp. lower) $\mathcal{V}-$convergent and strictly $\alpha-$upper (resp. $\alpha-$lower) quasi-uniformly $\mathcal{V}-$convergent to $F : (X,\pi) \to (Y,\mathcal{V})$ at $x \in X$.

It is easy to see that Theorem 2.1 together with the above remark immediately imply the following result.

COROLLARY 2.3 *If a net of multifunctions $F_\sigma : (X,\pi) \to (Y,\mathcal{V}), \sigma \in \Sigma$, is both inversely $\alpha-$lower (resp. $\alpha-$upper) quasi-uniformly $\mathcal{V}-$convergent and lower (resp. upper) $\mathcal{V}-$convergent to F at any point $x \in X$, then*

(a) $\limsup \mathcal{V} - \alpha C_l(F_\sigma) \subset \mathcal{V} - \alpha C_l(F)$ *(resp.* $\limsup \mathcal{V} - \alpha C_u(F_\sigma) \subset \mathcal{V} - \alpha C_u(F)$) *and*

(b) $\limsup \alpha C_l(F_\sigma) \subset \alpha C_l(F)$.

THEOREM 2.4 *If a net of multifunctions $F_\sigma : (X,\pi) \to (Y,\mathcal{V}), \sigma \in \Sigma$, is inversely $\alpha-$lower quasi-uniformly $\mathcal{V}-$convergent to F at a point $x \in X$, then*

(a) $Lsl.\alpha.C(F_\sigma, x) \subset l.\alpha.C(F, x)$;

(b) $Lsl.p.C(F_\sigma, x) \subset l.p.C(F, x)$;

(c) $Lsl.q.C(F_\sigma, x) \subset l.q.C(F, x)$;

(d) $Lsl.\beta.C(F_\sigma, x) \subset l.\beta.C(F, x)$.

If, additionally, this net is lower $\mathcal{V}-$convergent to F and has the above convergence property at any point $x \in X$, then:

(e) $\limsup \mathcal{V} - pC_l(F_\sigma) \subset \mathcal{V} - pC_l(F)$ *and* $\limsup pC_l(F_\sigma) \subset pC_l(F)$;

(f) $\limsup \mathcal{V} - qC_l(F_\sigma) \subset \mathcal{V} - qC_l(F)$ *and* $\limsup qC_l(F_\sigma) \subset qC_l(F)$;

(g) $\limsup \mathcal{V} - \beta C_l(F_\sigma) \subset \mathcal{V} - \beta C_l(F)$ *and* $\limsup \beta C_l(F_\sigma) \subset \beta C_l(F)$.

PROOF We first assume that $y \in Lsl.\alpha.C(F_\sigma, x)$ (resp. $y \in Lsl.p.C(F_\sigma, x)$) and we will show that $y \in l.\alpha.C(F, x)$ (resp. $y \in l.p.C(F, x)$). Let O be an open set containing y and let $W, V \in \mathcal{V}$ be such that $W \circ W \subset V$ and $V(y) \subset O$. The type of convergence implies the existence of a $\gamma \in \Sigma$ such that $y \in Int(Cl(Int(D_l(F_\sigma, F, W))))$ for every $\sigma \in \Sigma$ with $\sigma \geqslant \gamma$. From our assumption we know that $y \in Cl(\bigcup\{l.\alpha.C(F_\sigma, x) : \sigma \geqslant \gamma\})$ (resp. $y \in Cl(\bigcup\{l.p.C(F_\sigma, x) : \sigma \geqslant \gamma\})$), and therefore $W(y) \cap l.\alpha.C(F_\delta, x) \neq \emptyset$ (resp. $W(y) \cap l.p.C(F_\delta, x) \neq \emptyset$) for some $\delta \in \Sigma$ with $\sigma \geqslant \gamma$. So, $x \in Int(Cl(Int(F_\delta^-(W(y)))))$ (resp. $x \in Int(Cl(F_\delta^-(W(y)))))$. It is suffices to show that $A \subset Cl(Int(F^-(O)))$ (resp. $A \subset Cl(F^-(O))$), where
$A = Int(Cl(Int(D_l(F_\delta, F, W)))) \cap Int(Cl(Int(F_\delta^-(W(y)))))$ and
$B = Int(Cl(Int(D_l(F_\delta, F, W)))) \cap Int(Cl(F_\delta^-(W(y))))$. Let $z \in A$ (resp. $z \in B$) and let U be an open subset containing z. Then $P_A \subset Int(F^-(O))$ and $P_B \subset F^-(O)$, where $P_A = U \cap Int(D_l(F_\delta, F, W)) \cap Int(F_\delta^-(W(y)))$ and $P_B = U \cap$

$Int(D_l(F_\delta, F, W)) \cap F_\delta^-(W(y))$. Indeed, for each $p \in P_B$, we have $F_\delta(p) \subset W(F(p))$ and $F_\delta(p) \cap W(y) \neq \emptyset$. Then, $(y, a) \in W$ and $(a, b) \in W$ for some $a \in F_\delta(p)$ and $b \in F(p)$. Thus $p \in F^-(V(y))$ and consequently, $p \in F^-(O)$. Of cours, since $P_A \subset P_B$, we have $P_A \subset Int(F^-(O))$.

Now suppose that $y \in Lsl.q.C(F_\sigma, x)$ (resp. $y \in Lsl.\beta.C(F_\sigma, x)$). Let O and U be open sets containing y and x, respectively, and let us take $W, V \in \mathcal{V}$ such that $W \circ W \subset V$ and $V(y) \subset O$. Analogously as in the previous part of the proof, by the type of convergence we have $x \in \liminf Int(Cl(Int(D_l(F_\sigma, F, W))))$ and consequently, there exists $\gamma \in \Sigma$ such that $x \in Int(Cl(Int(D_l(F_\sigma, F, W))))$ for every $\sigma \in \Sigma$ with $\sigma \geqslant \gamma$. Then, under our assumption, $y \in Cl(\bigcup\{l.q.C(F_\sigma, x) : \sigma \geqslant \gamma\})$ (resp. $y \in Cl(\bigcup\{l.\beta.C(F_\sigma, x) : \sigma \geqslant \gamma\})$), and therefore
$x \in C = U \cap Int(Cl(Int(D_l(F_\delta, F, W)))) \cap Cl(Int(F_\delta^-(W(y))))$ (resp. $x \in D = U \cap Int(Cl(Int(D_l(F_\delta, F, W)))) \cap Cl(Int(Cl(F_\delta^-(W(y)))))$ for some $\delta \in \Sigma$ with $\delta \geqslant \gamma$. It suffices to show that $C \subset Cl(Int(F^-(O))$ (resp. $D \subset Cl(Int(Cl(F^-(O))))$. Let $p \in C$ (resp. $p \in D$) and let Z be an open subset containing p. It is easy to verify that $P_C \subset F^-(O)$ and $P_D \subset Cl(F^-(O))$ where $P_C = Z \cap U \cap Int(D_l(F_\delta, F, W)) \cap Int(F_\delta^-(W(y)))$ and $P_D = Z \cap U \cap Int(D_l(F_\delta, F, W)) \cap Int(Cl(F_\delta^-(W(y))))$. So, $p \in Cl(Int(F^-(O)))$ (resp. $p \in Cl(Int(Cl(F^-(O)))))$ and we have completed the proof of the first part of our theorem.

To prove the second part, suppose that $x \in \limsup \mathcal{V} - pC_l(F_\sigma)$ (resp. $x \in \limsup \mathcal{V} - qC_l(F_\sigma)$, $x \in \limsup \mathcal{V} - \beta C_l(F_\sigma)$) and let $\mathcal{M} \in T_l$ be such that $F(x) \in \mathcal{M}$. Let us take $V, W \in \mathcal{V}$ such that $V^-(F(x)) \subset \mathcal{M}$ and $W \circ W \circ W \subset V$. Under the assumptions concerning the type of convergence there exists $\gamma \in \Sigma$ such that

$F(x) \subset W^{-1}(F_\sigma(x))$ and

$$x \in Int(Cl(Int(D_l(F_\sigma, F, W)) \text{ for each } \sigma \in \Sigma \text{ with } \sigma \geqslant \gamma. \tag{3}$$

From the assumption there exists $\delta \in \Sigma$ with $\delta \geqslant \gamma$ such that $x \in \mathcal{V} - pC_l(F_\delta)$ (resp. $x \in \mathcal{V} - qC_l(F_\delta)$, $x \in \mathcal{V} - \beta C_l(F_\delta)$). Therefore, $x \in Int(Cl(F_\delta^{-1}(W^-(F_\delta(x)))))$ (resp. $x \in Cl(Int(F_\delta^{-1}(W^-(F_\delta(x)))))$, $x \in Cl(Int(Cl(F_\delta^{-1}(W^-(F_\delta(x))))$). To proof that $x \in \mathcal{V} - pC_l(F)$ (resp. $x \in \mathcal{V} - qC_l(F)$, $x \in \mathcal{V} - \beta C_l(F)$) we need only to show that $x \in Int(Cl(F^{-1}(V^-(F(x)))))$ (resp. $x \in Cl(Int(F^{-1}(V^-(F(x)))))$, $x \in Cl(Int(Cl(F^{-1}(V^-(F(x))))))))$). Consider the following sets:
$A = Int(Cl(Int(D_l(F_\delta, F, W)))) \cap Int(Cl(F_\delta^{-1}(W^-(F_\delta(x)))))$,
(resp. $B = Int(Cl(Int(D_l(F_\delta, F, W)))) \cap Cl(Int(F_\delta^{-1}(W^-(F_\delta(x))))$,
$C = Int(Cl(Int(D_l(F_\delta, F, W)))) \cap Cl(Int(Cl(F_\delta^{-1}(W^-(F_\delta(x))))))$.
It will suffice to prove that

$$A \subset Cl(F^{-1}(V^-(F(x)))) \tag{4}$$

$$(\text{resp. } B \subset Cl(Int(F^{-1}(V^-(F(x))))), \tag{5}$$

$$C \subset Cl(Int(Cl(F^{-1}(V^-(F(x)))))) \tag{6}$$

So let us take $z \in A$ (resp. $z \in B, z \in C$) and let O be an open subset containing z. It is easy to show that the following nonempty subset $P_A = O \cap Int(D_l(F_\delta, F, W)) \cap F_\delta^{-1}(W^-(F_\delta(x)))$ (resp. $P_B = O \cap Int(D_l(F_\delta, F, W)) \cap Int(F_\delta^{-1}(W^-(F_\delta(x))))$, $P_C = O \cap Int(D_l(F_\delta, F, W)) \cap Int(Cl(F_\delta^{-1}(W^-(F_\delta(x))))))$, is contained in $F^{-1}(V^-(F(x)))$ (resp. $Int(F^{-1}(V^-(F(x))))$, $Int(Cl(F^{-1}(V^-(F(x))))))$. Indeed, for any point $p \in P_A$ we have $F_\delta(p) \subset W^{-1}(F(p))$ and $F_\delta(p) \in W^-(F_\delta(x))$ or equivalently $F_\delta(x) \subset W^{-1}(F_\delta(p))$. Because of (3), for every $a \in F(x)$ we have $(a, b) \in W$ for some $b \in F_\delta(x)$. Consequently, $(b, c) \in W$ and $(c, d) \in W$ for some $c \in F_\delta(p)$ and

$d \in F(p)$. This gives $F(x) \subset V^{-1}(F(p))$ or equivalently, $F(p) \in V^{-}(F(x)))$. So $p \in F^{-1}(V^{-}(F(x)))$. It is clear that $P_B \subset Int(P_A)$, thus $P_B \subset Int(F^{-1}(V^{-}(F(x))))$. It is also evident that $P_C \subset Cl(P_A)$, hence $P_C \subset Int(Cl(F^{-1}(V^{-}(F(x)))))$. So, the inclusions (4), (5) and (6) are proved.

Because the relationship between the inclusions in the cases (b) and (c) of Theorem 2.1 is analogous to the relationship between the inclusions in the cases (e), (f) and (g) of the present theorem, we omit this part of the proof. ■

THEOREM 2.5 *If a net of multifunctions $F_\sigma : (X, \pi) \to (Y, \mathcal{V}), \sigma \in \Sigma$, is inversely $\alpha-$upper quasi-uniformly $\mathcal{V}-$convergent to F at a point $x \in X$, then*

(a) $Lsu.\alpha.C(F_\sigma, x) \subset u.\alpha.C(F, x)$;

(b) $Lsu.p.C(F_\sigma, x) \subset u.p.C(F, x)$;

(c) $Lsu.q.C(F_\sigma, x) \subset u.q.C(F, x)$;

(d) $Lsu.\beta.C(F_\sigma, x) \subset u.\beta.C(F, x)$.

If, additionally, this net is upper $\mathcal{V}-$convergent to F and has the above convergence property at any point $x \in X$, then:

(e) $\limsup \mathcal{V} - pC_u(F_\sigma) \subset \mathcal{V} - pC_u(F)$;

(f) $\limsup \mathcal{V} - qC_u(F_\sigma) \subset \mathcal{V} - qC_u(F)$;

(g) $\limsup \mathcal{V} - \beta C_u(F_\sigma) \subset \mathcal{V} - \beta C_u(F)$.

The relationship between the inclusions (a) in Theorem 2.1 is analogous to the relationship between the inclusions (a), (b), (c) and (d) in Theorem 2.4 and the inclusions (a), (b), (c) and (d) of the present theorem. Also, the relationship between the inclusions (b) in Theorem 2.1 is analogous to the relationship between the first inclusions of (e), (f) and (g) in Theorem 2.4 and the inclusions (e), (f) and (g) of the present theorem. So we may omit the proof.

THEOREM 2.6 *If a transfinite sequence $\{F_\xi : \xi < \omega_1\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ with countable base is lower (resp. upper) quasi-uniformly $\mathcal{V}-$convergent to F at a point $x \in X$, then*

(a) $\liminf l.C(F_\xi, x) \supset l.C(F, x)$ *(resp.*$\liminf u.C(F_\xi, x) \subset u.C(F, x)$).

If, additionally, this net is inversely lower (resp. upper) $\mathcal{V}-$convergent to F and has the above convergence property at any point $x \in X$, then:

(b) $\liminf \mathcal{V} - C_l(F_\xi) \subset \mathcal{V} - C_l(F)$; *(resp.* $\liminf \mathcal{V} - C_u(F_\xi) \supset \mathcal{V} - C_u(F)$) *and*

(c) $\liminf C_l(F_\xi) \supset C_l(F)$.

PROOF (a) : Assume that $y \in l.C(F,x)$ (resp. $y \in u.C(F,x)$) and let $\mathcal{B}(\mathcal{V}) = \{V_n : n \in N\}$ be a countable base of $\mathcal{V}$. By the assumption about the type of convergence, $x \in \liminf Int(D_l(F,F_\xi,V_n))$ (resp. $x \in \liminf Int(D_l(F,F_\xi, V_n)))$ for all $n \in N$, that is, for every $n \in N$, there exists an ordinar $\alpha(n) < \omega_1$ such that

$$x \in Int(D_l(F,F_\xi,V_n)) \quad (7)$$

$$\text{(resp. } x \in Int(D_u(F,F_\xi,V_n))) \text{ for all countable } \xi\, \alpha(n). \quad (8)$$

From the properties of ordinal numbers we can choose an $\alpha(x) < \omega_1$ such that $\alpha(x) > \alpha(n)$ for all $n \in N$. Consequently, $x \in Int(D_l(F,F_\xi,V_n)$ (resp. $x \in Int(D_u(F,F_\xi,V_n))$ for every $n \in N$ and every countable ordinar $\xi > \alpha(x)$. It suffices to show that $y \in l.C(F_\xi,x)$ (resp. $y \in u.C(F_\xi,x)$ for all countable ordinar $\xi > \alpha(x)$. Let O and U be a pair of open sets with $y \in O$ and $x \in U$. Furthermore, let $W,V \in \mathcal{V}$ and $V_k \in \mathcal{B}(\mathcal{V})$ be such that $V(y) \subset O$, $W \circ W \subset V$ and $V_k \subset W$, and let ξ be established with $\alpha(x) \leqslant \xi < \omega_1$. Then $x \in U \cap Cl(F^-(V_k(y))) \cap Int(D_l(F,F_\xi,V_k))$ (resp. $x \in U \cap Cl(F^+(V_k(y))) \cap Int(D_u(F,F_\xi,V_k))$. As a result, there exists a point $p \in U$ such that $F(p) \cap V_k(y) \neq \emptyset$ and $F(p) \subset V_k^{-1}(F_\xi(p))$ (resp. $F(p) \subset V_k(y)$ and $F_\xi(p) \subset V_k(F(p)))$. Consequently, there exist $a \in F(p)$ and $b \in F_\xi(p)$ such that $(y,a),(a,b) \in W$ (resp. for every $a \in F_\xi(p)$ there exists $b \in F(p)$ such that $(b,a),(y,b) \in W$). So $F_\xi(p) \cap O \neq \emptyset$ (resp. $F_\xi(p) \subset O$) which implies that $x \in Cl(F_\xi^-(O))$ (resp. $x \in Cl(F_\xi^+(O)))$ and consequently, proves that $y \in l.C(F_\xi,x)$ (resp. $y \in u.C(F_\xi,x)$).

(b) : Assume that $x \in \mathcal{V} - C_l(F)$ (resp. $x \in \mathcal{V} - C_u(F)$) and let $\mathcal{B}(\mathcal{V})$ be as above. Then, from the type of convergence, for every $n \in N$ there is an ordinar $\alpha(n) < \omega_1$ such that

$$F_\xi(x) \subset V_n^{-1}(F(x)) \text{ (resp. } F(x) \subset V_n(F_\xi(x))) \quad (9)$$

and the property (7) (resp. (8)) hold for all countable ordinar $\xi > \alpha(n)$. From the properties of ordinar numbers we can choose an ordinar $\alpha < \omega_1$ such that $\alpha > \alpha(n)$ for all $n \in N$. We will show that $x \in \bigcap\{\mathcal{V} - C_l(F_\xi) : \alpha \leqslant \xi < \omega_1\}$ (resp. $x \in \bigcap\{\mathcal{V} - C_u(F_\xi) : \alpha \leqslant \xi < \omega_1\}$). Let ξ be established with $\alpha \leqslant \xi < \omega_1$ and let $\mathcal{M} \in T_l$ (resp. $\mathcal{M} \in T_u$) be such that $F_\xi(x) \in \mathcal{M}$. We choose $V,W \in \mathcal{V}$ and $V_k \in \mathcal{B}(\mathcal{V})$ such that $V^-(F_\xi(x)) \subset \mathcal{M}$ (resp. $V^+(F_\xi(x)) \subset \mathcal{M}$), $W \circ W \circ W \subset V$ and $V_k \subset W$. By the assumption there exists an open neighborhood U of x such that $F(p) \in V_k^-(F(x))$ (resp. $F(p) \in V_k^+(F(x))$ for each $p \in U$. We will show that $F_\xi(z) \in V^-(F_\xi(x))$ (resp. $F_\xi(z) \in V^+(F_\xi(x))$ for each $z \in U^* = U \cap Int(D_l(F,F_\xi,V_k))$ (resp. $z \in U^* = U \cap Int(D_u(F,F_\xi,V_k)))$. If $z \in U^*$, then $(F(x),F(z)) \in V_k^-$ and $(F(z),F_\xi(z)) \in V_k^-$ (resp. $(F(x),F(z)) \in V_k^+$ and $(F(z),F_\xi(z)) \in V_k^+$), or equivalently $F(x) \subset V_k^{-1}(F(z))$ and $F(z) \subset V_k^{-1}(F_\xi(z))$ (resp. $F(z) \subset V_k(F(x))$ and $F_\xi(z) \subset V_k(F(z)))$. If $a \in F_\xi(x)$ (resp. $a \in F_\xi(z)$), then there exist $b \in F(x)$ by (9), $c \in F(z)$ and $d \in F_\xi(z))$ (resp. $r \in F(z)$, $s \in F(x)$ and $c \in F_\xi(x)$ by (9)) such that $(a,b),(b,c),(c,d) \in V_k$ (resp. $(r,a),(s,r),(c,s) \in V_k$). Therefore $(a,d) \in V$ (resp. $(c,a) \in V$) and consequently, $a \in V^{-1}(F_\xi(z))$ (resp. $a \in V(F_\xi(x)))$ which proves that $(F_\xi(x),F_\xi(z)) \in V^-$ (resp.$(F_\xi(x),F_\xi(z)) \in V^+$) and the proof of (b) is complete.

Now assume that $x \in C_l(F)$. Invoking (7) and (9) we will prove that $x \in \bigcap\{C_l(F_\xi) : \alpha \leqslant \xi < \omega_1\}$, where α is the same as above. Let ξ be such that $\alpha \leqslant \xi < \omega_1$ and let G be an open subset of Y such that $F_\xi(x) \cap G \neq \emptyset$. Choose $V, W \in \mathcal{V}$ and $V_k \in \mathcal{V}$ such that $W \circ W \subset V$, $V_k \subset W$ and $V(y) \subset G$ for some $y \in F_\xi(x)$. It is clear from (9), that $F(x) \cap V_k(y) \neq \emptyset$ so, by the assumption and (7) we have $x \in U = Int(D_l(F, F_\xi, V_k)) \cap Int(F^{-1}(V_k(y)))$. To finish the proof it will suffice to show that $U \subset F_\xi^-(V(y))$. If $z \in U$, then $F(z) \subset V_k^-(F_\xi(z))$ and $F(z) \cap V_k(y) \neq \emptyset$. Consequently, there exist points $r \in F(z)$ and $s \in F_\xi(z)$ such that $(y, r) \in V_k$ and $(r, s) \in V_k$. So $s \in V(y)$, which gives $z \in F_\xi^-(V(y))$ and finishes the proof of (c). ■

Analogously to Corollary 2.3, from the above result we have

Corollary 2.7 *If a transfinite sequence $\{F_\xi : \xi < \omega_1\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ with countable base is both α-lower (resp. α-upper) quasi-uniformly $\mathcal{V}$-convergent and inversely lower (resp. upper) $\mathcal{V}$-convergent to F at any point $x \in X$, then*

(a) $\liminf \mathcal{V} - \alpha C_l(F_\xi) \supset \mathcal{V} - \alpha C_l(F)$ *(resp.* $\liminf \mathcal{V} - \alpha C_u(F_\xi) \supset \mathcal{V} - \alpha C_u(F)$*;*

(b) $\liminf \alpha C_l(F_\xi) \supset \alpha C_l(F)$*.*

Theorem 2.8 *If a transfinite sequence $\{F_\xi : \xi < \omega_1\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ with countable base is α-lower quasi-uniformly $\mathcal{V}$-convergent to F at a point $x \in X$, then*

(a) $\liminf l.\alpha.C(F_\xi, x) \supset l.\alpha.C(F, x)$*;*

(b) $\liminf l.p.C(F_\xi, x) \supset l.p.C(F, x)$*;*

(c) $\liminf l.q.C(F_\xi, x) \supset l.q.C(F, x)$*;*

(d) $\liminf l.\beta.C(F_\xi, x) \supset l.\beta.C(F, x)$*;*

If, additionally, this sequence is inversely lower $\mathcal{V}$-convergent to F and has the above convergence property at any point $x \in X$, then:

(e) $\liminf \mathcal{V} - pC_l(F_\xi) \supset \mathcal{V} - pC_l(F)$ *and* $\liminf pC_l(F_\xi) \supset pC_l(F)$*;*

(f) $\liminf \mathcal{V} - qC_l(F_\xi) \supset \mathcal{V} - qC_l(F)$ *and* $\liminf qC_l(F_\xi) \supset qC_l(F)$*;*

(g) $\liminf \mathcal{V} - \beta C_l(F_\xi) \supset \mathcal{V} - \beta C_l(F)$ *and* $\liminf \beta C_l(F_\xi) \supset \beta C_l(F)$*.*

Proof First we will prove (a) and (b) simultaneously. Suppose that $y \in l.\alpha.C(F, x)$ (resp. $y \in l.p.C(F, x)$). By the α-lower quasi-uniformly $\mathcal{V}$-convergence, $x \in \liminf Int(Cl(Int(D_l(F, F_\xi, V_n))))$ for all $V_n \in \mathcal{B}(\mathcal{V})$ and consequently, as in the above proof, there exists $\alpha(x) < \omega_1$ such that $x \in Int(Cl(Int(D_l(F, F_\xi, V_n))))$ for every

$n \in N$ and every countable ordinal $\xi > \alpha(x)$. We will show that $y \in l.\alpha.C(F,x)$ (resp. $y \in l.p.C(F,x)$) for all countable ordinal $\xi > \alpha(x)$. Let O be an open set containing y, let $W, V \in \mathcal{V}$ and $V_k \in \mathcal{B}(\mathcal{V})$ be such that $V(y) \subset O$, $W \circ W \subset V$ and $V_k \subset W$, and let ξ be established with $\alpha \leqslant \xi < \omega_1$. Then, under our assumption, $x \in U = Int(Cl(Int(F^-(V_k(y))))) \cap Int(Cl(Int(D_l(F, F_\xi, V_k))))$ (resp. $x \in U = Int(Cl(F^-(V_k(y)))) \cap Int(Cl(Int(D_l(F, F_\xi, V_k)))))$. It suffices to show that $U \subset Cl(Int(F_\xi^-(O)))$ (resp. $U \subset Cl(F_\xi^-(O)))$. Let $z \in U$ and let Z be an open subset containing z. Then $G = Z \cap Int(F^-(V_k(y))) \cap Int(D_l(F, F_\xi, V_k))$ (resp. $K = Z \cap F^-(V_k(y)) \cap Int(D_l(F, F_\xi, V_k)))$ is an open non-empty (resp. non-empty) subset of $F_\xi^-(O)$. In fact, for any $p \in K$ we have $F(p) \cap V_k(y) \neq \emptyset$ and $(F(p), F_\xi(p)) \in V_k^{-1}$ that is, $(y, a) \in V_k$ for some $a \in F(p)$ and $F(p) \subset V_k^{-1}(F_\xi(p))$. So, $(a, b) \in V_k$ for some $b \in F_\xi(p)$ and consequently, $b \in V(y)$ which proves that $p \in F_\xi^-(O)$. Thus $K \subset Z \cap F_\xi^-(O)$ and, since $G \subset K$, we also have $G \subset Z \cap Int(F_\xi^-(O))$.

Now suppose that $y \in l.q.C(F,x)$ (resp. $y \in l.\beta.C(F,x)$) and, analogously as above, we will ahow that $y \in l.q.C(F_\xi, x)$ (resp. $y \in l.\beta.C(F_\xi, x)$) for all countable ordinal $\xi > \alpha(x)$. Let U and O be open sets such that $x \in U$ and $y \in O$. If $W, V \in \mathcal{V}$ and $V_k \in \mathcal{B}(\mathcal{V})$ are chosen as above, then for an established ξ with $\alpha \leqslant \xi < \omega_1$, under our assumption, $x \in U \cap Cl(Int(F^-(V_k(y)))) \cap Int(Cl(Int(D_l(F, F_\xi, V_k)))$ (resp. $x \in U \cap Cl(Int(Cl(F^-(V_k(y))))) \cap Int(Cl(Int(D_l(F, F_\xi, V_k))))$. To prove that $x \in Cl(Int(F_\xi^-(O)$ (resp. $x \in Cl(Int(Cl(F_\xi^-(O))))$ it suffices to show that $A \subset F_\xi^-(O)$ (resp. $B \subset Cl(F_\xi^-(O))$, where $A = U \cap Int(F^-(V_k(y))) \cap Int(D_l(F, F_\xi, V_k))$ and $B = U \cap Int(Cl(F^-(V_k(y))) \cap Int(D_l(F, F_\xi, V_k)))$. First let $z \in A$, then $F(z) \cap V_k(y) \neq \emptyset$ and $F(z) \subset V_k^{-1}(F_\xi(z))$. So, $(y, a), (a, b) \in V_k$ for some $a \in F(z)$ and $b \in F_\xi(z)$ and consequently, $b \in V(y)$ which gives $z \in F_\xi^-(O)$. Now, if $s \in B$ and P is an open subset containing s, then $L = P \cap U \cap F^-(V_k(y)) \cap Int(D_l(F, F_\xi, V_k))$ is a non-empty subset of $F_\xi^-(O)$. Indeed, for every $s \in L$ we have $F(s) \cap V_k(y) \neq \emptyset$ and $F(s) \subset V_k^{-1}(F_\xi(s))$ and, analogously as in the previous part we have $s \in F_\xi^-(O)$. So, the first part of the proof is complete.

Next, we will prove (e), (f) anf (g). Let $x \in \mathcal{V} - pC_l(F)$ (resp. $x \in \mathcal{V} - qC_l(F)$, $x \in \mathcal{V} - \beta C_l(F)$). Analogously as in the proof of Theorem 2.6 (b), by the assumptions about the type of convergence, for every $V_n \in \mathcal{B}(\mathcal{V})$ there is an ordinal $\alpha(n) < \omega_1$ such that $x \in Int(Cl(Int(D_l(F, F_\xi, V_n))))$ and the property (9) hold for all countable ordinal $\xi > \alpha(n)$. Also, we can choose an ordinal $\alpha < \omega_1$ such that $\alpha > \alpha(n)$ for all $n \in N$ and, we will show that $x \in \bigcap\{\mathcal{V} - pC_l(F_\xi) : \alpha \leqslant \xi < \omega_1\}$ (resp. $x \in \bigcap\{\mathcal{V} - qC_l(F_\xi) : \alpha \leqslant \xi < \omega_1\}, x \in \bigcap\{\mathcal{V} - \beta C_l(F_\xi) : \alpha \leqslant \xi < \omega_1\}$). Let ξ be established with $\alpha \leqslant \xi < \omega_1$ and let $\mathcal{M} \in T_l$ be such that $F_\xi(x) \in \mathcal{M}$. We choose $W, V \in \mathcal{V}$ and $V_k \in \mathcal{B}(\mathcal{V})$ such that $V^-(F_\xi(x)) \subset \mathcal{M}$, $W \circ W \circ W \subset V$ and $V_k \subset W$. By the assumption we have $x \in Int(Cl(F^{-1}(V_k^-(F(x)))))$ (resp. $x \in Cl(Int(F^{-1}(V_k^-(F(x))))), x \in Cl(Int(Cl(F^{-1}(V_k^-(F(x)))))))$. Let us put $D = Int(Cl(Int(D_l(F, F_\xi, V_k)))) \cap Int(Cl(F^{-1}(V_k^-(F(x)))))$, (resp. $E = Int(Cl(Int(D_l (F, F_\xi, V_k))) \cap Cl(Int(F^{-1}(V_k^-(F(x)))))$,
$H = Int(Cl(Int(D_l(F, F_\xi, V_k)))) \cap Cl(Int(Cl(F^{-1}(V_k^-(F(x)))))))$. Let $z \in D$ (resp. $z \in E, z \in H$) and let O be an open subset containing z. We will show that the subset P_D (resp. P_E, P_H) given by $P_D = O \cap Int(D_l(F, F_\xi, V_k)) \cap F^{-1}(V_k^-(F(x)))$ (resp. $P_E = O \cap Int(D_l(F, F_\xi, V_k)) \cap Int(F^{-1}(V_k^-(F(x))))$, $P_H = O \cap Int(D_l(F, F_\xi, V_k)) \cap$

$Int(Cl(F^{-1}(V_k^-(F(x))))))$ is contained in $F_\xi^{-1}(W^-(F_\xi(x)))$ (resp. $Int(F_\xi^{-1}(W^-(F_\xi(x)))), Int(Cl(F_\xi^{-1}(W^-(F_\xi(x))))))$. Let $p \in P_D$, then $F(p) \subset V_k^{-1}(F_\xi(p))$ and $F(x) \subset V_k^{-1}(F(p))$. By (9), for every $a \in F_\xi(x)$ there exists $b \in F(x)$ such that $(a,b) \in V_k$ and consequently, $(b,c) \in V_k$ for some $c \in F(p)$ and $(c,d) \in V_k$ for some $d \in F_\xi(p)$. So, $a \in V^{-1}(F_\xi(p))$ and we get $F_\xi(x) \subset V^{-1}(F(p))$ or equivalently $(F_\xi(x), F_\xi(p)) \in V^-$ which proves that $F_\xi(p) \in V^-(F_\xi(x))$ i.e., $p \in F_\xi^{-1}(V^-(F_\xi(x)))$. So, we obtain that $D \subset Int(Cl(F_\xi^{-1}(V^-(F_\xi(x)))))$ and consequently, that $x \in \mathcal{V} - pC_l(F_\xi)$. Since $P_E \subset Int(P_D)$, we have $E \subset Cl(Int(F_\xi^{-1}(V^-(F_\xi(x)))))$ which implies that $x \in \mathcal{V} - qC_l(F_\xi)$. Analogously, we get $H \subset Cl(Int(Cl(F_\xi^{-1}(V^-(F_\xi(x)))))))$ because $P_H \subset Cl(P_D)$ and consequently, $\mathcal{V} - \beta C_l(F_\xi)$.

The relationship between the inclusions in the cases (b) and (c) of Theorem 2.6 is analogous to the reationship between the inclusions in the cases (e), (f) and (g) of the present theorem, so we omit this part of the proof. ■

For reasons similar to those given for Theorem 2.5, we omit the proof of the following result.

THEOREM 2.9 *If a transfinite sequence $\{F_\xi : \xi < \omega_1\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ with countable base is α-upper quasi-uniformly $\mathcal{V}$-convergent to F at a point $x \in X$, then*

(a) $\liminf u.\alpha.C(F_\xi, x) \supset u.\alpha.C(F, x)$;

(b) $\liminf u.p.C(F_\xi, x) \supset u.p.C(F, x)$;

(c) $\liminf u.q.C(F_\xi, x) \supset u.q.C(F, x)$;

(d) $\liminf u.\beta.C(F_\xi, x) \supset u.\beta.C(F, x)$.

If, additionally, this sequence is inversely upper $\mathcal{V}$-convergent to F and has the above convergence property at any point $x \in X$, then:

(e) $\liminf \mathcal{V} - pC_u(F_\xi) \supset \mathcal{V} - pC_u(F)$;

(f) $\liminf \mathcal{V} - qC_u(F_\xi) \supset \mathcal{V} - qC_u(F)$;

(g) $\liminf \mathcal{V} - \beta C_u(F_\xi) \supset \mathcal{V} - \beta C_u(F)$.

As an immediate consequence of Theorems 2.1 and 2.6 we obtain the following result

COROLLARY 2.10 *If a transfinite sequence $\{F_\xi : \xi < \omega_1\}$ of multifunctions from a topological space (X, π) to a quasi-uniform space $(Y, \mathcal{V})$ with countable base is strictly lower (resp. upper) quasi-uniformly $\mathcal{V}$-convergent to F at a point $x \in X$, then*

(a) $l.C(F, x) = \lim l.C(F_\xi, x) = Ltl.C(F_\xi, x)$ *(resp.* $u.C(F, x) = \lim u.C(F_\xi, x) = Ltu.C(F_\xi, x)$*).*

If this sequence has the above convergence property at any point $x \in X$, then:

(b) $\mathcal{V} - C_l(F) = \lim \mathcal{V} - C_l(F_\xi)$; *(resp.* $\mathcal{V} - C_u(F) = \lim \mathcal{V} - C_u(F_\xi)$) *and,*

(c) $C_l(F) = \lim C_l(F_\xi)$.

Immediately from Corollaries 2.3 and 2.7 we obtain:

Corollary 2.11 *If a transfinite sequence* $\{F_\xi : \xi < \omega_1\}$ *of multifunctions from a topological space* (X, π) *to a quasi-uniform space* $(Y, \mathcal{V})$ *with countable base is both strictly* α*-lower (resp.* α*-upper) quasi-uniformly* $\mathcal{V}$*-convergent and strictly lower (resp. upper)* $\mathcal{V}$*-convergent to* F *at any point* $x \in X$*, then*

(a) $\mathcal{V} - \alpha C_l(F) = \lim \mathcal{V} - \alpha C_l(F_\xi)$ *(resp.* $\mathcal{V} - \alpha C_u(F) = \lim \mathcal{V} - \alpha C_u(F_\xi)$);

(b) $\alpha C_l(F) = \lim \alpha C_l(F_\xi)$.

Combining Theorems 2.4 and 2.8, we have

Corollary 2.12 *If a transfinite sequence* $\{F_\xi : \xi < \omega_1\}$ *of multifunctions from a topological space* (X, π) *to a quasi-uniform space* $(Y, \mathcal{V})$ *with countable base is strictly* α*-lower quasi-uniformly* $\mathcal{V}$*-convergent to* F *at a point* $x \in X$*, then*

(a) $l.\alpha.C(F, x) = \lim l.\alpha.C(F_\xi, x) = Ltl.\alpha.C(F_\xi, x)$;

(b) $l.p.C(F, x) = \lim l.p.C(F_\xi, x) = Ltl.p.C(F_\xi, x)$;

(c) $l.q.C(F, x) = \lim l.q.C(F_\xi, x) = Ltl.q.C(F_\xi, x)$;

(d) $l.\beta.C(F, x) = \lim l.\beta.C(F_\xi, x) = Ltl.\beta.C(F_\xi, x)$.

If, additionally, this sequence is strictly lower $\mathcal{V}$*-convergent to* F *and has the above convergence property at any point* $x \in X$*, then:*

(e) $\mathcal{V} - pC_l(F) = \lim \mathcal{V} - pC_l(F_\xi)$ *and* $pC_l(F) = \lim pC_l(F_\xi)$;

(f) $\mathcal{V} - qC_l(F) = \lim \mathcal{V} - qC_l(F_\xi)$ *and* $qC_l(F) = \lim qC_l(F_\xi)$;

(g) $\mathcal{V} - \beta C_l(F) = \lim \mathcal{V} - \beta C_l(F_\xi)$ *and* $\beta C_l(F) = \lim \beta C_l(F_\xi)$.

Finally, using Theorems 2.5 and 2.9 we obtain the following result

Corollary 2.13 *If a transfinite sequence* $\{F_\xi : \xi < \omega_1\}$ *of multifunctions from a topological space* (X, π) *to a quasi-uniform space* $(Y, \mathcal{V})$ *with countable base is strictly* α*-upper quasi-uniformly* $\mathcal{V}$*-convergent to* F *at a point* $x \in X$*, then*

(a) $u.\alpha.C(F, x) = \lim u.\alpha.C(F_\xi, x) = Ltu.\alpha.C(F_\xi, x)$;

(b) $u.p.C(F, x) = \lim u.p.C(F_\xi, x) = Ltu.p.C(F_\xi, x)$;

(c) $u.q.C(F, x) = \lim u.q.C(F_\xi, x) = Ltu.q.C(F_\xi, x)$;

(d) $u.\beta.C(F, x) = \lim u.\beta.C(F_\xi, x) = Ltu.\beta.C(F_\xi, x)$.

If, additionally, this sequence is strictly upper $\mathcal{V}$*-convergent to* F *and has the above convergence property at any point* $x \in X$*, then:*

(e) $\mathcal{V}-pC_u(F)=\lim\mathcal{V}-pC_u(F_\xi)$;

(f) $\mathcal{V}-qC_u(F)=\lim\mathcal{V}-qC_u(F_\xi)$;

(g) $\mathcal{V}-\beta C_u(F)=\lim\mathcal{V}-\beta C_u(F_\xi)$.

It is known (see[11, Theorem 3.3]) that a multifunction $F:(X,\pi)\to(Y,\tau)$ has closed graph with respect to the product topology $\pi\times\tau$ if and only if $F(x)=l.C(F,x)$ for any $x\in X$, where the graph of F, denoted by $Gr(F)$, is the set $\{(x,y)\in X:y\in F(x)\}$. Thus, applying Corollary 2.10 together with Remark 1.8 we obtain the followingconclusion.

Corollary 2.14 *Let $\{F_\xi:\xi<\omega_1\}$ be a transfinite sequence of multifunctions with closed graphs from a topological space (X,π) to a quasi-uniform space $(Y,\mathcal{V})$ with countable base. If this sequence is strictly lower quasi-uniformly $\mathcal{V}$-convergent to F at any point $x\in X$ and F has closed values, then the graph of F is closed.*

A multifunction $F:(X,\pi)\to(Y,\tau)$ is said to be minimal [18], [27] (resp. α-minimal, p-minimal, β-minimal [38]), if for any $x\in X$, $x\in Cl(Int(F^+(W)))$ (resp. $x\in Int(Cl(Int(F^+(W))))$, $x\in Int(Cl(F^+(W)))$, $x\in Cl(Int(Cl(F^+(W)))))$ for each open set $W\in Y$ such that $x\in F^-(W)$.

It is easy to see that for any pair (H,F) of multifunctions $H,F:(X,\pi)\to(Y,\tau)$, the inclusion $H\subset F$ implies that $u.\alpha.C(F,x)\subset u.\alpha.C(H,x), u.q.C(F,x)\subset u.q.C(H,x), u.p.C(F,x)\subset u.p.C(H,x)$ and $u.\beta.C(F,x)\subset u.\beta.C(H,x)$ for any $x\in X$, where $H\subset F$ means that $Gr(H)\subset Gr(F)$. Hence, by Theorem 7 [39], it follows that the property $F(x)\subset u.\alpha.C(F,x)$ (resp.$F(x)\subset u.q.C(F,x), F(x)\subset u.p.C(F,x), F(x)\subset u.\beta.C(F,x))$ for any $x\in X$, is equivalent to the α-minimality (resp. minimality, p-minimality, β-minimality) of F. So, by Corollary 2.13 and Remark 1.8 we have the following result

Corollary 2.15 *Let $\{F_\xi:\xi<\omega_1\}$ be a transfinite sequence of α-minimal (resp. minimal, p-minimal, β-minimal) multifunctions from a topological space (X,π) to a quasi-uniform space $(Y,\mathcal{V})$ with countable base. If this sequence is both strictly lower and strictly upper quasi-uniformly $\mathcal{V}$-convergent to a multifunction $F:(X,\pi)\to(Y,\tau)$ at any point $x\in X$ and F has closed values, then F is α-minimal (resp. minimal, p-minimal, β-minimal).*

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