COMMENTATIONES MATHEMATICAE Vol. 49, No. 1 (2009), 67-79

Bogdan Szal

A note on the $MRBSVS$ class and its application to the uniform convergence and boundedness of sine series

Abstract. A new class of γ rest bounded second variation sequences is defined. Some relationships between classes of considered sequences are proved. The results of Leindler [3] and author [8] are extended to our new class.

2000 Mathematics Subject Classification: 40A30, 42A10.

Key words and phrases: sine series, Fourier series, embedding relations, number sequences.

1. Introduction. Chaundy and Jolliffe [1] proved the following classical result (see also [10]).

Theorem 1.1 *Suppose that $b_n \geq b_{n+1}$ and $b_n \to 0$. Then a necessary and sufficient condition for the uniform convergence of the series*

$$\sum_{n=1}^{\infty} b_n \sin nk \tag{1}$$

is $nb_n \to 0$.

In [5] Leindler defined a new class of sequences in the following way:

Definition 1.2 Let $\gamma := (\gamma_n)$ be a positive sequence. A null sequence $c := (c_n)$ of real numbers satisfying the inequality

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c)\, \gamma_m, \quad m = 1, 2, \ldots$$

with a positive constant $K(c)$ is said to be a sequence of γ Rest Bounded Variation, in symbol: $c \in \gamma RBVS$.

If $\gamma_n \equiv c_n$ and $c_n > 0$, then we call the sequence c the Rest Bounded Variation Sequence; and briefly we write $c \in RBVS$. In [6] L. Leindler introduced the class of Mean Rest Bounded Variation Sequences ($MRBVS$), where γ is defined by a certain arithmetical mean of the sequence c, e.g.,

$$\gamma_m := \frac{1}{m} \sum_{n \geq m/2}^{m} c_n \tag{2}$$

It is easy to see that the class $MRBVS$ includes the class $RBVS$, consequently the classes of almost monotone and monotone sequences, too.

In [3] L. Leindler generalized above theorem to the class $RBVS$. Namely, he proved the following theorems.

THEOREM 1.3 *If a sequence $b = (b_n)$ belongs to the class $RBVS$, then the condition $nb_n \to 0$ as $n \to \infty$ is both necessary and sufficient for the uniform convergence of series (1).*

THEOREM 1.4 *If a sequence $b = (b_n)$ belongs to the class $RBVS$, then the condition $nb_n = O(1)$ is both necessary and sufficient for the uniform boundedness of the partial sums of series (1).*

THEOREM 1.5 *Suppose that $b \in RBVS$. Then a necessary and sufficient condition for the series (1) to be the Fourier series of a continuous function is $nb_n \to 0$.*

We generalized these results to the class $MRBVS$ and we proved that $RBVS \neq MRBVS$ ([8]). In [4] L. Leindler shown that Theorem 1.3 and Theorem 1.4 are true if (b_n) belongs to the class $\gamma RBVS$ but he proved that for the class $\gamma RBVS$ only a sufficient condition in those theorems is valid.

A nonnegative sequence c is said to be a sequence of Group Bounded Variation ($GBVS$) if there exists a natural number N such that

$$\sum_{n=m}^{2m} |c_n - c_{n+1}| \leq K(c) \max_{m \leq n \leq m+N} c_n$$

holds for all m. In [2] R. Le and S. Zhou proved that Theorem 1.1 is true if a sequence $b \in GBVS$.

Moreover, for a more general class

$$MVBV = \left\{ a_n \in \mathbb{C} : \sum_{n=m}^{2m} |a_n - a_{n+1}| \leq C \sum_{n=[m/c]}^{[cm]} \frac{|a_n|}{n} \text{ for some } c > 1 \right\}$$

a necessary and sufficient condition for the uniform convergence of the series were proved in [9] and [7].

It is clear that
$$MRBVS \subseteq MVBV.$$
Furthermore (see [9])
$$GBVS \subseteq MVBV$$

Let $b_n = \frac{2+(-1)^n}{n^2}$. It is clear that $nb_n \to 0$ and by Weierstrass's theorems the series (1) is uniform convergence but the sequence (b_n) is not monotonic and does not belong to none of considered classes above (see Theorem 2.1).

In order to formulate our new results we define another such class of sequences that the above sequence (b_n) belongs to it.

Definition 1.6 Let $\gamma := (\gamma_n)$ be a positive sequence. A null sequence $c := (c_n)$ of positive numbers is called γ Rest Bounded Second Variation, briefly $c \in \gamma RBSVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+2}| \leq K(c)\,\gamma_m \tag{3}$$

for all natural numbers m.

If $\gamma_n \equiv c_n$ and $c_n > 0$, then we call the sequence c the Rest Bounded Second Variation Sequence; and briefly we write $c \in RBSVS$. Consequently, if γ is defined by (2) we shall say that the sequence c is the Mean Rest Bounded Second Variation, briefly $c \in MRBSVS$.

It is clear that
$$RBSVS \subset MRBSVS.$$

In the present paper we show that $\gamma RBVS \subset \gamma RBSVS$ but $RBVS \neq RBSVS$ and $MRBVS \neq MRBSVS$. Moreover, we prove that Theorem 1.3, Theorem 1.4 and Theorem 1.5 are true if a sequence b belongs to $MRBSVS$.

2. Main results.

We have the following results:

Theorem 2.1 *The following properties are valid:*

(i) $\gamma RBVS \subset \gamma RBSVS$,

(ii) there exists a sequence $d =: (d_n)$, *with the property that* $nd_n \to 0$ *as* $n \to \infty$, *which belongs to the class* $MRBSVS$ *but it does not belong to the class* $MVBV$.

(iii) there exists a sequence $a =: (a_n)$, *with the property that* $na_n \to 0$ *as* $n \to \infty$, *which belongs to the class* $MRBSVS$ *but it does not belong to the class* $RBSVS$.

Theorem 2.2 *If a sequence* $b = (b_n)$ *belongs to the class* $MRBSVS$, *then the condition* $nb_n \to 0$ *as* $n \to \infty$ *is both necessary and sufficient for the uniform convergence of series (1).*

THEOREM 2.3 *If a sequence $b = (b_n)$ belongs to the class $MRBSVS$, then the condition $nb_n = O(1)$ is both necessary and sufficient for the uniform boundedness of the partial sums of series (1).*

THEOREM 2.4 *Suppose that $b \in MRBSVS$. Then a necessary and sufficient condition for the series (1) to be the Fourier series of a continuous function is $nb_n \to 0$.*

REMARK 2.5 By the embedding relation $RBVS \subset MRBSVS$ we can observe that Theorem 1.3, Theorem 1.4 and Theorem 1.5 are the corollaries of Theorem 2.2, Theorem 2.3 and Theorem 2.4, respectively.

REMARK 2.6 By Theorem 2.1 (i) we derive that the results from [8] are the corollaries of Theorem 2.2, Theorem 2.3 and Theorem 2.4, too.

3. Proofs of Theorems. In this section we shall prove our results.

3.1. Proof of Theorem 2.1.

(i) Let $(c_n) \in \gamma RBVS$. Then for all m

$$\sum_{n=m}^{\infty} |c_n - c_{n+2}| \le \sum_{n=m}^{\infty} \left(|c_n - c_{n+1}| + |c_{n+1} - c_{n+2}|\right)$$

$$\le 2 \sum_{n=m}^{\infty} |c_n - c_{n+1}| \le 2K(c)\,\gamma_m$$

and $(c_n) \in \gamma RBSVS$.

(ii) Let $d_n = \frac{2+(-1)^n}{n^2}$. It is clear that $nd_n \to 0$ as $n \to \infty$.

Now, we show that the sequence (d_n) does not belong to the class $MVBV$.

Let $A_m = \{n,\ m \le n \le 2m \text{ and } n \text{ is even}\}$

$$\sum_{n=m}^{2m} |d_n - d_{n+1}| = \sum_{n=m}^{2m} \left| \frac{2+(-1)^n}{n^2} - \frac{2+(-1)^{n+1}}{(n+1)^2} \right|$$

$$= \sum_{n=m}^{2m} \left| \frac{4n+4+(-1)^n\left(2n^2+2n+1\right)}{n^2 (n+1)^2} \right|$$

$$\ge \sum_{n \in A_m} \frac{2n^2+6n+5}{n^2 (n+1)^2} \ge \sum_{n \in A_m} \frac{1}{n^2} \ge \frac{1}{4m}$$

and since

$$\sum_{n=[m/c]}^{[cm]} \frac{|d_n|}{n} \le \frac{c}{m} \sum_{n=[m/c]}^{[cm]} \frac{2+(-1)^n}{n^2} \le \frac{c}{m} \sum_{n=[m/c]}^{[cm]} \frac{3}{n^2} \ll \frac{1}{m^2},$$

the inequality

$$\sum_{n=m}^{2m} |d_n - d_{n+1}| \leq K(d) \sum_{n=[m/c]}^{[cm]} \frac{|d_n|}{n}$$

does not hold, that is, (d_n) does not belong to $MVBV$ and consequently to $RBVS$, $MRBVS$ and $GBVS$.

Finally, we show that the sequence (d_n) belongs to $MRBSVS$.

For all m we have

$$\sum_{n=m}^{\infty} |d_n - d_{n+2}| = \sum_{n=m}^{\infty} \left| \frac{2+(-1)^n}{n^2} - \frac{2+(-1)^{n+2}}{(n+2)^2} \right|$$

$$= \sum_{n=m}^{\infty} \frac{4\left(2+(-1)^n\right)(n+1)}{n^2(n+2)^2} \leq 12 \sum_{n=m}^{\infty} \frac{1}{n^3} \ll \frac{1}{m^2} \leq \frac{2+(-1)^m}{m^2} = d_m$$

and $(d_n) \in RBSVS$. Since $RBSVS \subset MRBSVS$ we get that $(d_n) \in MRBSVS$.

(iii) Denote by $\mu_m := 2^m$ for $m = 1, 2, 3, ...$ and define a sequence (a_n) by the following formulas $a_1 = 1$ and

$$a_n := \frac{1+m+(-1)^n m}{m^2 \mu_m} \quad \text{if } \mu_m \leq n < \mu_{m+1}.$$

It is clear that $nd_n \to 0$ as $n \to \infty$. Namely, for any $n > 2$ there exists a natural number m such that $\mu_m \leq n < \mu_{m+1}$. Hence

$$na_n \leq \mu_{m+1} \frac{1+m+(-1)^{n+1} m}{m^2 \mu_m} \leq \frac{2(1+2m)}{m^2} \leq \frac{6}{m} \leq \frac{6}{\ln n/2}.$$

Since the sequence $\frac{6}{\ln n/2} \to 0$ as $n \to \infty$, we obtain that $na_n \to 0$ as $n \to \infty$.

Now, we show that the sequence (a_n) does not belong to the class $RBSVS$. Namely, for $m \geq 2$ we have

$$\sum_{k=\mu_m+1}^{\infty} |a_k - a_{k+2}| \geq \sum_{k=\mu_m+1}^{\mu_{m+1}-2} |a_k - a_{k+2}| = \sum_{k=\mu_m+1}^{\mu_{m+1}-3} |a_k - a_{k+2}|$$

$$+ \left|a_{\mu_{m+1}-2} - a_{\mu_{m+1}}\right| = \left| \frac{1+2m}{m^2 \mu_m} - \frac{1+2(m+1)}{(m+1)\mu_{m+1}} \right|$$

$$= \frac{2m^3 + 7m^2 + 8m + 2}{2m^2 (m+1)^2 \mu_m} \geq \frac{1}{m\mu_m}$$

and since $a_{\mu_m+1} = \frac{1}{m^2 \mu_m}$, the inequality

$$\sum_{k=n}^{\infty} |a_k - a_{k+2}| \leq K(a)\, a_n$$

does not hold, that is, (a_n) does not belong to $RBSVS$.

Next, we prove that the sequence (a_n) belongs to the class $MRBSVS$.
Let $n \geq 2$. Then for any n there exist $m \geq 1$ and $r = 0, 1, 2..., \mu_m - 1$ such that

$$n = \mu_m + r.$$

If $r < \mu_m - 1$, then

$$\sum_{k=2n}^{\infty} |a_k - a_{k+2}| = \sum_{k=\mu_{m+1}+2r}^{\infty} |a_k - a_{k+2}|$$

$$= \sum_{k=\mu_{m+1}+2r}^{\mu_{m+2}-3} |a_k - a_{k+2}| + \left| a_{\mu_{m+2}-2} - a_{\mu_{m+2}} \right| + \left| a_{\mu_{m+2}-1} - a_{\mu_{m+2}+1} \right|$$

$$\sum_{s=0}^{\infty} \left(\sum_{k=\mu_{m+2+s}+2r}^{\mu_{m+3+s}-3} |a_k - a_{k+2}| \right.$$

$$\left. + \left| a_{\mu_{m+3+s}-2} - a_{\mu_{m+3+s}} \right| + \left| a_{\mu_{m++3+s}-1} - a_{\mu_{m+3+s}+1} \right| \right)$$

and if $r = \mu_m - 1$, then

$$\sum_{k=2n}^{\infty} |a_k - a_{k+2}| = \sum_{k=\mu_{m+2}-2}^{\infty} |a_k - a_{k+2}|$$

$$= \left| a_{\mu_{m+2}-2} - a_{\mu_{m+2}} \right| + \left| a_{\mu_{m+2}-1} - a_{\mu_{m+2}+1} \right|$$

$$\sum_{s=0}^{\infty} \left(\sum_{k=\mu_{m+2+s}+2r}^{\mu_{m+3+s}-3} |a_k - a_{k+2}| \right.$$

$$\left. + \left| a_{\mu_{m+3+s}-2} - a_{\mu_{m+3+s}} \right| + \left| a_{\mu_{m++3+s}-1} - a_{\mu_{m+3+s}+1} \right| \right).$$

Therefore

$$\sum_{k=2n}^{\infty} |a_k - a_{k+2}| = \left| \frac{1 + 2(m+1)}{(m+1)^2 \mu_{m+1}} - \frac{1 + 2(m+2)}{(m+2)^2 \mu_{m+2}} \right|$$

$$+ \left| \frac{1}{(m+1)^2 \mu_{m+1}} - \frac{1}{(m+2)^2 \mu_{m+2}} \right|$$

$$+ \sum_{s=0}^{\infty} \left(\left| \frac{1 + 2(m+2+s)}{(m+2+s)^2 \mu_{m+2+s}} - \frac{1 + 2(m+3+s)}{(m+3+s)^2 \mu_{m+3+s}} \right| \right.$$

$$\left. \left| \frac{1}{(m+2+s)^2 \mu_{m+2+s}} - \frac{1}{(m+3+s)^2 \mu_{m+3+s}} \right| \right)$$

$$\leq \frac{m+2}{(m+1)^2 \mu_m} + \sum_{s=0}^{\infty} \frac{m+s+3}{(m+2+s)^2 \mu_{m+1+s}}$$

$$\le \frac{2}{m\mu_m} + \frac{1}{m\mu_m}\sum_{s=0}^{\infty}\frac{1}{2^s} = \frac{4}{m\mu_m}.$$

Let

$$\begin{aligned} A_{m,r} &: = \{k; \mu_m + r \le k \le \mu_{m+1} + 2r \text{ and } k \text{ is even}\},\\ B_{m,r} &: = \{k; \mu_m + r \le k < \mu_{m+1} \text{ and } k \text{ is even}\},\\ C_{m,r} &: = \{k; \mu_{m+1} \le k \le \mu_{m+1} + 2r \text{ and } k \text{ is even}\}. \end{aligned}$$

Then

$$\sum_{k=2n}^{\infty}|a_k - a_{k+2}| \le \frac{8}{\mu_m + r + 1}\sum_{k\in A_{m,r}}\frac{1}{m\mu_m}$$

$$\le \frac{8}{\mu_m + r + 1}\left(\sum_{k\in B_{m,r}}\frac{1}{m\mu_m} + 4\sum_{k\in C_{m,r}}\frac{1}{(m+1)\,\mu_{m+1}}\right)$$

$$\le \frac{8}{\mu_m + r + 1}\left(\frac{1}{2}\sum_{k\in B_{m,r}}\frac{1+2m}{m^2\mu_m} + 2\sum_{k\in C_{m,r}}\frac{1+2\,(m+1)}{(m+1)\,\mu_{m+1}}\right)$$

$$\le \frac{8}{\mu_m + r + 1}\left(\frac{1}{2}\sum_{k=\mu_m+r}^{\mu_{m+1}-1}a_k + 2\sum_{k=\mu_{m+1}}^{\mu_{m+1}+2r}a_k\right) \le \frac{16}{n+1}\sum_{k=n}^{2n}a_k. \tag{4}$$

If $n = 1$, then by (4)

$$\sum_{k=2}^{\infty}|a_k - a_{k+2}| = \sum_{k=2}^{3}|a_k - a_{k+2}| + \sum_{k=4}^{\infty}|a_k - a_{k+2}|$$

$$\le \sum_{k=2}^{3}|a_k - a_{k+2}| + \frac{16}{3}\sum_{k=2}^{4}a_k \ll \frac{1}{2}\sum_{k=1}^{2}a_k. \tag{5}$$

(4) and (5) imply that $(a_n) \in MRBSVS$.
This complete the proof. □

3.2. Proof of Theorem 2.2.
First we prove the necessity. Setting $x = \frac{\pi}{4m}$, we get

$$\sum_{n=m}^{2m}b_n\sin nx = \sum_{n=m}^{2m}b_n\sin\frac{n\pi}{4m} \ge \sin\frac{\pi}{4}\sum_{n=m}^{2m}b_n. \tag{6}$$

If $b \in MRBSVS$, by (3) with $\frac{1}{m+1}\sum\limits_{n=m}^{2m}b_n$ in place of γ_m, we have

$$b_{2m} \le b_{2m} + b_{2m+1} = \sum_{n=2m}^{\infty}(b_n - b_{n+2}) \le \sum_{n=2m}^{\infty}|b_n - b_{n+2}|$$

$$\leq K(b)\frac{1}{m+1}\sum_{n=m}^{2m} b_n \tag{7}$$

and

$$b_{2m-1} \leq \sum_{n=2m-1}^{\infty}(b_n - b_{n+2}) \leq \sum_{n=2m-1}^{\infty}|b_n - b_{n+2}| \leq$$

$$\sum_{n=2m}^{\infty}|b_n - b_{n+2}| \leq K(b)\frac{1}{m+1}\sum_{n=m}^{2m} b_n. \tag{8}$$

Hence, by (6), (7) and (8), and taking into account that the series (1) converges uniformly, we obtain that $mb_{2m} \to 0$ and $mb_{2m-1} \to 0$ as $m \to \infty$, and these verify the necessity of the condition $nb_n \to 0$ as $n \to \infty$.

Now, we prove the sufficiency. Denote

$$\varepsilon_n := \sup_{k\geq n} kb_k \quad \text{and} \quad r_n(x) := \sum_{k=2n}^{\infty} b_k \sin kx.$$

In view of the assumptions, we have that $\varepsilon_n \to 0$ as $n \to \infty$. We will show that

$$|r_n(x)| \ll \varepsilon_n \tag{9}$$

also holds. Since $r_n(k\pi) = 0$, it suffices to prove (9) for $0 < x < \pi$.

First we show that for $x \neq k\pi$

$$\sum_{k=n}^{\infty} b_k \sin kx = \frac{1}{4\sin\frac{x}{2}\cos\frac{x}{2}}\left\{\sum_{k=n}^{\infty}(b_k - b_{k+2})(1-\cos(k+1)x)\right.$$

$$\left. -(b_n + b_{n+1})(1-\cos nx)\right\} + \frac{1}{2\cos\frac{x}{2}} b_n \sin\left(n-\frac{1}{2}\right)x. \tag{10}$$

An elementary calculation gives

$$\sum_{k=n}^{\infty} b_k \cos kx = \frac{1}{2}\sum_{k=n}^{\infty}(b_k + b_{k+1})\cos kx + \frac{1}{2}\sum_{k=n}^{\infty}(b_k - b_{k+1})\cos kx,$$

whence

$$\frac{1}{2}\sum_{k=n}^{\infty} b_k \cos kx = \frac{1}{2}\sum_{k=n}^{\infty}(b_k + b_{k+1})\cos kx - \frac{1}{2}\sum_{k=n+1}^{\infty} b_k \cos(k-1)x$$

$$= \frac{1}{2}\sum_{k=n}^{\infty}(b_k + b_{k+1})\cos kx - \frac{1}{2}\cos x\sum_{k=n+1}^{\infty} b_k \cos kx - \frac{1}{2}\sin x\sum_{k=n+1}^{\infty} b_k \sin kx.$$

Thus

$$\frac{1}{2}(1+\cos x)\sum_{k=n+1}^{\infty} b_k \cos kx = \frac{1}{2}\sum_{k=n}^{\infty}(b_k+b_{k+1})\cos kx$$

$$-\frac{1}{2}\sin x\sum_{k=n+1}^{\infty} b_k \sin kx - \frac{1}{2}b_n \cos nx$$

and if $x \neq (2l+1)\pi$, then

$$\sum_{k=n+1}^{\infty} b_k \cos kx$$

$$= \frac{1}{2\cos\frac{x}{2}}\left\{\sum_{k=n}^{\infty}(b_k+b_{k+1})\cos kx - \sin x\sum_{k=n+1}^{\infty} b_k \sin kx - b_n\cos nx\right\}. \tag{11}$$

Further

$$\sum_{k=n}^{\infty} b_k \sin kx = \frac{1}{2}\sum_{k=n}^{\infty}(b_k+b_{k+1})\sin kx + \frac{1}{2}\sum_{k=n}^{\infty}(b_k-b_{k+1})\sin kx,$$

whence

$$\frac{1}{2}\sum_{k=n}^{\infty} b_k \sin kx = \frac{1}{2}\sum_{k=n}^{\infty}(b_k+b_{k+1})\sin kx$$

$$-\frac{1}{2}\cos x\sum_{k=n+1}^{\infty} b_k \sin kx + \frac{1}{2}\sin x\sum_{k=n+1}^{\infty} b_k\cos kx.$$

Using (11) we get

$$\frac{1}{2}\sum_{k=n}^{\infty} b_k \sin kx = \frac{1}{2}\sum_{k=n}^{\infty}(b_k+b_{k+1})\sin kx + \frac{\sin\frac{x}{2}}{2\cos\frac{x}{2}}\sum_{k=n}^{\infty}(b_k+b_{k+1})\cos kx$$

$$-\frac{1}{2}\sum_{k=n+1}^{\infty} b_k\sin kx - \frac{\sin\frac{x}{2}}{2\cos\frac{x}{2}}b_n\cos nx$$

$$= \frac{1}{2\cos\frac{x}{2}}\sum_{k=n}^{\infty}(b_k+b_{k+1})\sin\left(k+\frac{1}{2}\right)x$$

$$-\frac{1}{2}\sum_{k=n+1}^{\infty} b_k\sin kx - \frac{\sin\frac{x}{2}}{2\cos\frac{x}{2}}b_n\cos nx$$

and therefore

$$\sum_{k=n}^{\infty} b_k\sin kx = \frac{1}{2\cos\frac{x}{2}}\sum_{k=n}^{\infty}(b_k+b_{k+1})\sin\left(k+\frac{1}{2}\right)x$$

$$+\frac{1}{2}b_n\sin nx-\frac{\sin\frac{x}{2}}{2\cos\frac{x}{2}}b_n\cos nx$$

$$=\frac{1}{2\cos\frac{x}{2}}\left\{\sum_{k=n}^{\infty}(b_k+b_{k+1})\sin\left(k+\frac{1}{2}\right)x+a_n\sin\left(n-\frac{1}{2}\right)x\right\}.$$

By Abel's transformation we get

$$\sum_{k=n}^{\infty}b_k\sin kx=\frac{1}{2\cos\frac{x}{2}}\left\{\sum_{k=n}^{\infty}(b_k-b_{k+2})\sum_{i=0}^{k}\sin\left(i+\frac{1}{2}\right)x\right.$$

$$\left.-(b_n+b_{n+1})\sum_{i=0}^{n-1}\sin\left(i+\frac{1}{2}\right)x+b_n\sin\left(n-\frac{1}{2}\right)x\right\}.$$

Since for $x\neq 2l\pi$ and $k=0,1,2,...$

$$\sum_{i=0}^{k}\sin\left(i+\frac{1}{2}\right)x=\frac{1-\cos(k+1)x}{2\sin\frac{x}{2}}$$

we get (10).

First we show that (9) is valid for $0<x\leq\frac{\pi}{2}$.

Let $N=N(x)\geq 2$ be the integer such that

$$\frac{\pi}{N+1}<x\leq\frac{\pi}{N}. \tag{12}$$

Then

$$r_n(x)=\sum_{k=2n}^{2(n+N)-1}b_k\sin kx+\sum_{k=2(n+N)}^{\infty}b_k\sin kx=r_n^{(1)}(x)+r_n^{(2)}(x).$$

Hence, by (12),

$$\left|r_n^{(1)}(x)\right|\leq x\sum_{k=2n}^{2(n+N)-1}kb_k\leq 2xN\varepsilon_n\leq 2\pi\varepsilon_n. \tag{13}$$

If $(b_n)\in MRBSVS$, then using (10), the inequality $\frac{1}{\pi}x\leq\sin\frac{x}{2}$ $(x\in(0,\pi))$ and (12) we obtain

$$\left|r_n^{(2)}(x)\right|\leq\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\left\{\sum_{k=2(n+N)}^{\infty}|b_k-b_{k+2}|+b_{2(n+N)}+b_{2(n+N)+1}\right\}$$

$$+\frac{1}{2\cos\frac{x}{2}}b_{2(n+N)}\leq\frac{1}{\sin\frac{x}{2}\cos\frac{x}{2}}\sum_{k=2(n+N)}^{\infty}|b_k-b_{k+2}|+\frac{1}{2\cos\frac{x}{2}}b_{2(n+N)}$$

$$\leq \frac{\pi}{x\cos\frac{\pi}{4}} \sum_{k=2(n+N)}^{\infty} |b_k - b_{k+2}| + \frac{1}{2\cos\frac{\pi}{4}} b_{2(n+N)}$$

$$\ll (N+1)\left(\sum_{k=2(n+N)}^{\infty} |b_k - b_{k+2}| + b_{2(n+N)}\right).$$

By (7) we get

$$\left|r_n^{(2)}(x)\right| \ll 2K(b)\frac{N+1}{n+N+1}\sum_{k=n+N}^{2(n+N)} b_k$$

(14) $$\leq 2K(b)\frac{1}{n+N+1}\sum_{k=n+N}^{2(n+N)} kb_k \leq 2K(b)\,\varepsilon_n.$$

Now, we prove (9) for $\frac{\pi}{2} \leq x < \pi$.

Let $M := M(x) \geq 2$ be the integer such that

(15) $$\pi - \frac{\pi}{M} \leq x < \pi - \frac{\pi}{M+1}.$$

Then

$$r_n(x) = \sum_{k=2n}^{2(n+M)-1} b_k \sin kx + \sum_{k=2(n+M)}^{\infty} b_k \sin kx = r_n^{(3)}(x) + r_n^{(4)}(x).$$

Using the inequality $\sin x \leq \pi - x$ $(x \in (0,\pi))$ and (15) we get

(16) $$\left|r_n^{(3)}(x)\right| \leq \left|r_n^{(1)}(x)\right| \leq (\pi - x)\sum_{k=2n}^{2(n+M)-1} kb_k \leq 2(\pi - x)M\varepsilon_n \leq 2\pi\varepsilon_n.$$

If $(b_n) \in MRBSVS$, then using (10), the inequality $1 - \frac{1}{\pi}x \leq \cos\frac{x}{2}$ $(x \in (0,\pi))$ and (15) we obtain

$$\left|r_n^{(4)}(x)\right| \leq \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\left\{\sum_{k=2(n+M)}^{\infty} |b_k - b_{k+2}| + b_{2(n+M)} + b_{2(n+M)+1}\right\}$$

$$+\frac{1}{2\cos\frac{x}{2}} b_{2(n+M)} \leq \frac{1}{\sin\frac{x}{2}\cos\frac{x}{2}} \sum_{k=2(n+M)}^{\infty} |b_k - b_{k+2}| + \frac{1}{2\cos\frac{x}{2}} b_{2(n+M)}$$

$$\leq \frac{1}{\left(1-\frac{1}{\pi}x\right)\sin\frac{\pi}{4}} \sum_{k=2(n+M)}^{\infty} |b_k - b_{k+2}| + \frac{1}{2\left(1-\frac{1}{\pi}x\right)} b_{2(n+M)}$$

$$\ll (M+1)\left(\sum_{k=2(n+M)}^{\infty} |b_k - b_{k+2}| + b_{2(n+M)}\right).$$

Thus, by (7) we get

$$\left|r_n^{(4)}\left(x\right)\right| \ll 2K\left(b\right)\frac{M+1}{n+M+1}\sum_{k=n+M}^{2(n+M)} b_k$$

$$\leq 2K\left(b\right)\frac{1}{n+M+1}\sum_{k=n+M}^{2(n+M)} kb_k \leq 2K\left(b\right)\varepsilon_n. \tag{17}$$

From the estimations (13), (14), (16) and (17)we obtain the uniform convergence of series (1) and thus the proof is complete.

3.3. Proof of Theorem 2.3.

The proof of Theorem 2.3 goes analogously as the proof of Theorem 2.2. Now, we have

$$\sum_{n=m}^{2m} b_n \leq K.$$

Hence, applying (7) and (8) we obtain that $mb_{2m} \leq K$ and $mb_{2m-1} \leq K$.

In the proof of sufficiency, the only difference is that ε_n should be replaced by a positive constant.

3.4. Proof of Theorem 2.4.

If $nb_n \to 0$ as $n \to \infty$, by Theorem 2.2, we obtain that series (1) is uniformly convergent. From this and by the Fejér's theorem we obtain that the series (1) is the Fourier series of a continuous function.

Now, we prove the necessity of the condition $nb_n \to 0$. If series (1) is the Fourier series of a continuous function, then the $(C,1)$ −means

$$\sigma_n\left(x\right) = \sum_{k=1}^{n} b_k\left(1-\frac{k}{n+1}\right)\sin kx$$

of this series converges uniformly. In particular

$$\sigma_{4m}\left(\frac{\pi}{8m}\right) \to 0 \text{ as } m \to \infty. \tag{18}$$

Using the inequality $\sin x \geq \frac{2}{\pi}x$ in $\left[0, \frac{\pi}{2}\right]$ we obtain that

$$\sigma_{4m}\left(x\right) = \sum_{k=1}^{4m} b_k\left(1-\frac{k}{4m+1}\right)\sin kx \geq \sum_{k=1}^{4m} b_k\left(1-\frac{k}{4m+1}\right)\frac{2kx}{\pi} \tag{19}$$

for $x \in \left[0, \frac{\pi}{8m}\right]$. Hence, by (19) and (7),

$$\sigma_{4m}\left(\frac{\pi}{8m}\right) \geq \sum_{k=1}^{4m} b_k\left(1-\frac{k}{4m+1}\right)\frac{k}{4m}$$

$$\geq \frac{1}{4m}\sum_{k=m}^{2m} b_k\left(1-\frac{k}{4m+1}\right)k \geq \frac{1}{8}\sum_{k=m}^{2m} b_k \geq \frac{1}{8K\left(b\right)}mb_{2m}$$

and by (18), $nb_n \to 0$ as $n \to \infty$. Thus the proof is complete.

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Bogdan Szal
University of Zielona Góra, Faculty of Mathematics, Computer Science and Econometrics
65-516 Zielona Góra, ul. Szafrana 4a, Poland
E-mail: B.Szal@wmie.uz.zgora.pl

(Received: 8.05.2009)