COMMENTATIONES MATHEMATICAE Vol. 49, No. 1 (2009), 59-66

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# Existence results for asymptotic linear Hammerstein integral equations via Morse Theory

**Abstract.** In this paper we establish some results for asymptotic linear Hammerstein integral equations. Using Morse theory and in particular critical groups we prove a number of existence results.

*2000 Mathematics Subject Classification:* 47J20, 47J30.

*Key words and phrases:* Asymptotic linear, critical groups, existence.

**1. Introduction.** In this paper we use critical point theory to establish some existence results for the Hammerstein integral equation

$$u(x) = \lambda \int_\Omega k(x,y)\, u(y)\, dy - \int_\Omega k(x,y)\, f(y,u(y))\, dy \quad \text{for} \quad x \in \Omega; \tag{1.1}$$

here $\lambda > 0$ and $\Omega$ is a closed bounded subset of $\mathbf{R}^n$. We will discuss both the nonresonance and resonance problems. We look for solutions to (1.1) in $C(\Omega)$. Throughout this paper we will also use the usual Lebesgue space $L^2(\Omega)$ with norm $|\,.\,|_{L^2}$ and inner product $(\,.\,,\,.\,)$. A discussion of (1.1) using variational methods can be found in the books [3, 8]. For a more recent treatment using the ideas in [3, 8] we refer the reader to [4]. The results we present here are new and are based on critical groups and the multiplicity theory was motivated from ideas in [1, 7].

Throughout this paper we assume the kernel $k : \Omega \times \Omega \to \mathbf{R}$ satisfies the following:

$$k \in C(\Omega \times \Omega, \mathbf{R}) \tag{1.2}$$

$$k(x,y) = k(y,x) \quad \text{for} \quad x, y \in \Omega \tag{1.3}$$

and

$$\int_{\Omega\times\Omega} k(x,y)\, u(x)u(y)\, dx\, dy \geq 0 \;\forall u \in L^2(\Omega). \tag{1.4}$$

Let
$$K\,u(x) = \int_\Omega k(x,y)\,u(y)\,dy \quad \text{for} \quad x \in \Omega \quad \text{and} \quad u \in C(\Omega).$$
It is well known [3, 8] that $K : L^2(\Omega) \to L^2(\Omega)$ is a linear, completely continuous, self adjoint, nonnegative (i.e. $(Ky, y) \geq 0$ for all $y \in L^2(\Omega)$) operator. Also the square root operator of $K$, $K^{\frac{1}{2}} : L^2(\Omega) \to L^2(\Omega)$ exists. From the spectral theory of such operators [9] we know that $K$ has a countably infinite number of real eigenvalues $(\mu_i)$ (recall $\mu_i$ is an eigenvalue of $K$ if there exists a $\psi_i \in L^2(\Omega)$ with $\mu_i\,K\,\psi_i = \psi_i$) with $\mu_i > 0$ for all $i$. ASSUME throughout this paper that if $K\,\psi = 0$ for some $\psi \in L^2(\Omega)$ then $\psi = 0$. Then we can relabel the eigenvalues $(\mu_i)$ so that
$$\mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots\ldots$$
and note $\mu_1 > 0$.

Also we assume

$$f : \Omega \times \mathbf{R} \to \mathbf{R} \quad \text{is continuous.} \tag{1.5}$$

Let
$$N_f\,u(x) = f(x, u(x)) \quad \text{for} \quad x \in \Omega \quad \text{and} \quad u \in C(\Omega).$$
Our strategy in this paper is to show

$$v = \lambda\,K\,v \,-\, K^{\frac{1}{2}}\,N_f\,K^{\frac{1}{2}}\,v \tag{1.6}$$

has a solution in $L^2(\Omega)$. This will automatically guarantee that

$$u = \lambda\,K\,v \,-\, K\,N_f\,u \tag{1.7}$$

has a solution in $C(\Omega)$. To see this let $v \in L^2(\Omega)$ be a solution of (1.6). Then from linearity we have that
$$K^{\frac{1}{2}}\,v = \lambda\,K^{\frac{1}{2}}\,K\,v \,-\, K^{\frac{1}{2}}\,K^{\frac{1}{2}}\,N_f\,K^{\frac{1}{2}}\,v = \lambda\,K\,K^{\frac{1}{2}}\,v - K\,N_f\,K^{\frac{1}{2}}\,v,$$
so $K^{\frac{1}{2}}\,v$ is a solution of (1.7).

Let $\Phi : L^2(\Omega) \to \mathbf{R}$ be given by

$$\Phi(u) = \frac{1}{2}\,|u|^2_{L^2} - \frac{\lambda}{2}(K\,u\,,\,u) + \int_\Omega \int_0^{K^{\frac{1}{2}}\,u(x)} f(x,v)\,dv\,dx \tag{1.8}$$

for $u \in L^2(\Omega)$. It is well known [3, 4, 8] that if there exists a $v \in L^2(\Omega)$ with

$$\Phi'(v) = 0, \tag{1.9}$$

then $v$ is a solution of (1.6).

Remark 1.1 It is worth pointing out here that one could extend the results of this paper so that $f : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ above and indeed one could look at the more general problem

$$u(x) = \lambda \int_\Omega k(x,y)\, u(y)\, dy - \int_\Omega k_1(x,y)\, f(y, u(y))\, dy \quad \text{for a.e.} \quad x \in \Omega;$$

here we look for solutions in $L^p(\Omega)$. Also here the continuity of $f$ is replaced by $f$ is $(p,q)$ Carathéodory of potential type (see [8, Chapter 6 and 7]) and (1.2) is replaced by a condition to guarantee that $K : L^{q_0}(\Omega) \to L^{p_0}(\Omega)$; here $p, q, p_0, q_0$ are as in [8, Chapter 6 and 7].

We will obtain a variety of existence results for (1.6) in the next section using Morse theory. For convenience we recall here some results which we will need in Section 2. Let $\Phi$ be a real valued function on a real Banach space $W$ and assume $\Phi \in C^1(W, \mathbf{R})$. For every $c \in \mathbf{R}$ let

$$\Phi^c = \{x \in W : \ \Phi(x) \le c\} \ \ \text{(the sublevel sets at } c\text{)}$$

and

$$K = \{x \in W : \ \Phi'(x) = 0\} \ \ \text{(the set of critical points of } \Phi\text{)}.$$

In Morse theory the local behavior of $\Phi$ near an isolated critical point $u$ is described by the sequence of critical groups

$$C^q(\Phi, u) = H^q(\Phi^c \cap U, \Phi^c \cap U \backslash \{u\}), \ q \ge 0$$

where $c = \Phi(u)$ is the corresponding critical value and $U$ is a neighborhood of $u$ containing no other critical points of $\Phi$. When the critical values are bounded from below and $\Phi$ satisfies (C) the global behavior of $\Phi$ can be described by the critical groups at infinity

$$C^q(\Phi, \infty) = H^q(W, \Phi^a), \ q \ge 0$$

where $a$ is less than all critical values. A critical point $u$ of $\Phi$ with $C^1(\Phi, u) \ne 0$ is called a mountain pass point.

Next we discuss the variational eigenvalues from [7] in our situation here. Consider

$$u = \lambda\, K\, u$$

where $K : L^2(\Omega) \to L^2(\Omega)$. Let

$$M = \{u \in L^2(\Omega) : \ \frac{1}{2}\,(u,u) = 1\}, \ J(u) = \frac{1}{2}\,(K\,u, u)$$

with

$$\Psi(u) = \frac{1}{J(u)}, \ u \in L^2(\Omega)\backslash\{0\} \ \text{ and } \ \widetilde{\Psi} = \Psi|_M.$$

Let $F$ denote the class of symmetric subsets of $M$ and $i(M_0)$ the Fadell–Rabinowitz cohomological index of $M_0 \in F$. Then

$$\lambda_k = \inf_{M_0 \in F, \ i(M_0) \ge k} \ \sup_{u \in M_0} \tilde{\Psi}(u), \quad 1 \le k \le \infty.$$

We know from Theorem 4.2.1 (ii) of [7] that $\lambda_1 = \mu_1$ where

$$\mu_1 = \min_{u \neq 0} \frac{(u,u)}{(Ku,u)}.$$

Remark 1.2 In fact in our situation (i.e. for the Hammerstein integral equation described above) $\lambda_k = \mu_k$ for all $k$ as was noted by Kanishka Perera. Suppose for simplicity that all the $\mu_i' s$ described above are simple with

$$\mu_1 < \mu_2 < \mu_3 < \dots\ .$$

Fix $k \geq 1$ so if $\mu_k < \mu < \mu_{k+1}$ the set $\widetilde{\Psi}^{\mu}$ deformation retracts (see the introduction of [7]) to the intersection of $M$ with the subspace spanned by the eigenvectors of $\mu_1, \dots, \mu_k$, which is a $(k-1)$-dimensional sphere, so (4.2.6) of [7] holds with all the $\lambda' s$ replaced by $\mu' s$. This together with (4.2.6) of [7] itself gives $\lambda_k = \mu_k$.

Now from Proposition 9.4.1 (ii) of [7] (note $K^{\frac{1}{2}}\, N_f\, K^{\frac{1}{2}} : L^2(\Omega) \to L^2(\Omega)$ is compact (completely continuous)) we have the following result in our situation.

Theorem 1.3 *Suppose*

$$\frac{\lambda}{2}\,(K\,u,u) - \int_{\Omega} \int_0^{K^{\frac{1}{2}}\,u(x)} f(x,w)\,dw\,dx = \frac{\lambda^\star}{2}\,(Ku,u) + o(|u|_{L^2}^2) \quad \textit{as} \quad u \to 0$$

*and zero is an isolated critical point.*
(i). *If* $\lambda^\star < \lambda_1$ *then* $C^q(\Phi,0) = \delta_{q,0}\,\mathbf{Z}_2$.
*(ii). If* $\lambda_k < \lambda^\star < \lambda_{k+1}$ *then* $C^k(\Phi,0) \neq 0$.

In our main multiplicity result of Section 2 we will use Theorem 1.3 together with the following result in [2, 5].

Theorem 1.4 *Let* $\Phi$ *be a* $C^1$ *functional defined on a Banach space. If* $\Phi$ *is bounded from below, satisfies (C) and* $C^k(\Phi,0) \neq 0$ *for some* $k \geq 1$*, then* $\Phi$ *has two nontrivial critical points.*

Remark 1.5 Theorem 1.4 was also proved in [7, Corollary 3.10.1] for the case $k \geq 2$.

**2. Main results.** For notational purposes for $u \in L^2(\Omega)$ let

$$F(u) = \frac{\lambda}{2}\,(K\,u,u) - \int_{\Omega} \int_0^{K^{\frac{1}{2}}\,u(x)} f(x,w)\,dw\,dx$$

and

$$H(u) = \int_{\Omega} \int_0^{K^{\frac{1}{2}}\,u(x)} f(x,w)\,dw\,dx - \frac{1}{2}\,(K^{\frac{1}{2}}\,N_f\,K^{\frac{1}{2}}\,u,u).$$

Throughout this section we assume

$$K^{\frac{1}{2}}\, N_f\, K^{\frac{1}{2}}\,(u) = o(|u|^2_{L^2}) \ \text{ as } \ u \to \infty. \tag{2.1}$$

Our first result (nonresonance case) follows from [7, Theorem 5.2.5] (the proof in [7] uses the notion of cohomological linking) since we have set up our problem (1.1) (see (1.6)) in the abstract setting of [7]. Again we note that $\lambda_k = \mu_k$.

Theorem 2.1 *Suppose* (2.1) *holds and* $\lambda \in (\lambda_k, \lambda_{k+1})$ *for some* $k \in \{1, 2, ...\}$. *Then* $\Phi$ *satisfies (PS). If in addition* $\Phi$ *has a finite number of critical points then* (1.6) *(so* (1.7)*) has a solution* $u$ *with* $C^k(\Phi, u) \neq 0$.

Next we discuss the resonance case. Let $\mathcal{N}$ denote the class of sequences $(u_j) \subseteq L^2(\Omega)$ such that $\rho_j = |u|_{L^2} \to \infty$ and $u^\star_j = \frac{u_j}{\rho_j}$ converges weakly to some $u^\star \neq 0$. For the resonance problem we will assume either

$(H_+)$. $H$ is bounded from below and every sequence $(u_j) \in \mathcal{N}$ has a subsequence such that $H(t\,u_j) \to \infty \ \forall t \geq 1$

or

$(H_-)$. $H$ is bounded from above and every sequence $(u_j) \in \mathcal{N}$ has a subsequence such that $H(t\,u_j) \to -\infty \ \forall t \geq 1$.

The following result (resonance case) follows from [7, Theorem 5.2.7].

Theorem 2.2 *Suppose* (2.1) *holds. Then*

(i) $\Phi$ *satisfies (C) if* $(H_+)$ *or* $(H_-)$ *holds;*

(ii) $\Phi$ *is bounded from below if* $\lambda = \lambda_1$ *and* $(H_+)$ *holds (also note if* $\Phi$ *has a finite number of critical points then* $\Phi$ *has a global minimizer* $u$ *with* $C^q(\Phi, u) = \delta_{q,0}\, \mathbf{Z}_2$*);*

(iii) (1.6) *(so* (1.7)*) has a solution* $u$ *with* $C^k(\Phi, u) \neq 0$ *if* $\lambda \in [\lambda_k, \lambda_{k+1})$ *and* $(H_-)$ *holds (again we are assuming* $\Phi$ *has a finite number of critical points);*

(iv) (1.6) *(so* (1.7)*) has a solution* $u$ *with* $C^k(\Phi, u) \neq 0$ *if* $\lambda \in (\lambda_k, \lambda_{k+1}]$ *and* $(H_+)$ *holds (again we are assuming* $\Phi$ *has a finite number of critical points).*

Our main result is a multiplicity result when $\lambda = \lambda_1$.

Theorem 2.3 *Suppose* (2.1) *and* $(H_+)$ *hold. In addition assume there exists a* $\lambda^\star > \mu_1$ *with* $\lambda^\star \neq \lambda_i$ *for* $i \in \{2, 3, ....\}$ *and with*

$$\int_\Omega \int_0^{K^{\frac{1}{2}}\, u(x)} f(x, w)\, dw\, dx = \frac{\lambda^\star}{2}\,(Ku, u) + o(|u|^2_{L^2}) \quad \textit{as} \ \ u \to 0 \tag{2.2}$$

*and also assume* $f(x, 0) = 0$ *for* $x \in \Omega$. *Then* $\Phi$ *(with* $\lambda = \lambda_1$*) has two nontrivial critical points (so* (1.6) *and* (1.7) *with* $\lambda = \lambda_1$ *have two nontrivial solutions).*

Proof Theorem 2.2 (i) and (ii) guarantee that $\Phi$ is bounded from below and satisfies (C). Note $\Phi'(0) = 0$ since $f(x,0) = 0$ for $x \in \Omega$. We may assume the origin is an isolated critical point (otherwise we have a sequence of nontrivial critical points of $\Phi$ and we are finished). Now Theorem 1.3 guarantees that $C^k(\Phi, 0) \neq 0$ for some $k \geq 1$. Then $\Phi$ has two nontrivial critical points by Theorem 1.4. □

We can improve this result if we use the ideas in Theorem 2.3 above with a result from [7]. ■

Theorem 2.4 *Suppose* (2.1) *and* $(H_+)$ *hold. In addition assume* $f(x,0) = 0$ *for* $x \in \Omega$ *and either*

$$\frac{\lambda_k}{2}(Ku, u) \leq F(u) < \frac{\lambda_{k+1}}{2}(Ku, u) \;\; \forall\, u \in B_\rho(0) \backslash \{0\}$$

*for some* $k$ *such that* $\lambda_k < \lambda_{k+1}$ *and* $\rho > 0$*, or*

$$\frac{\lambda_\star}{2}(Ku, u) + o(|u|_{L^2}^2) \leq F(u) \leq \frac{\lambda_{\star\star}}{2}(Ku, u) + o(|u|_{L^2}^2) \quad \text{as} \;\; u \to 0$$

*for some* $\lambda_k < \lambda_\star \leq \lambda_{\star\star} < \lambda_{k+1}$*. Then* $\Phi$ *(with* $\lambda = \lambda_1$*) has two nontrivial critical points (so* (1.6) *and* (1.7) *with* $\lambda = \lambda_1$ *have two nontrivial solutions).*

Proof We may assume the origin is an isolated critical point (otherwise we have a sequence of nontrivial critical points of $\Phi$ and we are finished). Now [7, Lemma 7.2.1] guarantees that $C^k(\Phi, 0) \neq 0$ for some $k \geq 1$. Then $\Phi$ has two nontrivial critical points by Theorem 1.4. ■

Our next result follows immediately from [7, Theorem 7.1.2].

Theorem 2.5 *Suppose* (2.1) *and* $f(x,0) = 0$ *for* $x \in \Omega$*. In addition assume* $\Phi$ *has a finite number of critical points and either*

$$F(u) \leq \frac{\lambda_1}{2}(Ku, u) \;\; \forall\, u \in B_\rho(0)$$

*for some* $\rho > 0$*, or*

$$F(u) \leq \frac{\lambda^\star}{2}(Ku, u) + o(|u|_{L^2}^2) \quad \text{as} \;\; u \to 0$$

*for some* $0 \leq \lambda^\star < \mu_1$*. Then* (1.6) *has a mountain pass solution* $v \neq 0$ *(with* $\Phi(v) > 0$*) in the following cases:*

(i) $\lambda \in (\lambda_k, \lambda_{k+1})$;

(ii) $\lambda \in [\lambda_k, \lambda_{k+1})$ *and* $(H_-)$ *holds;*

(iii) $\lambda \in (\lambda_k, \lambda_{k+1}]$ *and* $(H_+)$ *holds.*

Remark 2.6 It is also easy to write an analogue of [7, Theorem 7.3.1] for (1.6); we leave this to the reader.

Our final result is a multiplicity result which follows from [7, Theorem 7.4.2].

Theorem 2.7 *Suppose* (2.1) *holds. In addition assume*

$$f(x,-u) = -f(x,u) \quad \textit{for all} \quad (x,u) \in \Omega \times \mathbf{R}.$$

(i) *If* $\lambda < \lambda_k$ *and*

$$F(u) \geq \frac{\lambda_\star}{2}(Ku,u) + o(|u|^2_{L^2}) \quad \textit{as} \quad u \to 0$$

*for some* $\lambda_\star > \lambda_{k+m-1}$ *then* (1.6) *has* $m$ *distinct pairs (note* $\Phi$ *is even) of solutions;*

(ii) *If* $\lambda > \lambda_{k+m-1}$ *and*

$$F(u) \leq \frac{\lambda_{\star\star}}{2}(Ku,u) + o(|u|^2_{L^2}) \quad \textit{as} \quad u \to 0$$

*for some* $\lambda_{\star\star} < \lambda_k$ *then* (1.6) *has* $m$ *distinct pairs (note* $\Phi$ *is even) of solutions.*

## References

[1] R.P. Agarwal, K. Perera and D. O'Regan, *Multiplicity results for p-sublinear p-Laplacian problems involving indefinite eigenvalue problems via Morse theory*, to appear.

[2] K.C. Chang, *Infinite-dimensional Morse theory and multiple solution problems*, Progress in Nonlinear Differential Equations and their Applications, Vol. **6**, Birkhäuser, Boston, 1993.

[3] M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon, Oxford, 1964.

[4] F. Li, Y. Li and Z. Liang, *Existence of solutions to nonlinear Hammerstein integral equations and applications*, Jour. Math. Anal. Appl. **323** (2006), 209–227.

[5] J.Q. Liu and S.H. Li, *An existence theorem for multiple critical points and its application*, Kexue Tongbao (Chinese) **29** (1984), 1025–1027.

[6] D. O'Regan, *Multiplicity results for sublinear and superlinear Hammerstein integral equations via Morse theory*, Communications in Applied Analysis, to appear.

[7] K. Perera, R.P. Agarwal and D. O'Regan, *Morse-theoretic aspects of p-Laplacian type operators*, *Progress in Nonlinear Differential Equations and their Applications*, Birkhäuser, Boston, to appear.

[8] R. Precup, *Methods in nonlinear integral equations*, Kluwer, Dordrecht, 2002.

[9] N. Young, *An introduction to Hilbert spaces*, Cambridge University Press, Cambridge, 1989.

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(Received: 22.05.2008)